Physics 8.07 **Practice Exam First Midterm** **Fall 2011**

DETACH THESE SHEETS & USE AT YOUR CONVENIENCE

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_o}$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times\mathbf{B}=\mu_o\mathbf{J}+\mu_o\varepsilon_o\frac{\partial\mathbf{E}}{\partial t}$$

$$\nabla\cdot\mathbf{B}=0$$

$$\int_{\substack{\text{closed}\\\text{surface}}}\mathbf{E}\cdot\hat{\mathbf{n}}\,da=\frac{q_{\text{enclosed}}}{\varepsilon_o}$$

$$\int_{\substack{\text{bounding}\\\text{contour}}}\mathbf{E}\cdot d\mathbf{l}=-\frac{d}{dt}\int_{\substack{\text{open}\\\text{surface}}}\mathbf{B}\cdot\hat{\mathbf{n}}\,da$$

$$\int_{\substack{\text{bounding}\\\text{contour}}}\mathbf{B}\cdot d\mathbf{l}=\mu_o\varepsilon_o\left[I_{\text{through}}+\frac{d}{dt}\int_{\substack{\text{open}\\\text{surface}}}\mathbf{E}\cdot\hat{\mathbf{n}}\,da\right]$$

$$\int_{\substack{\text{closed}\\\text{surface}}}\mathbf{B}\cdot\hat{\mathbf{n}}\,da=0$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$$\mathbf{E}=-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A}=-\mu_o\mathbf{J}$$

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi=-\frac{\rho}{\varepsilon_o}$$

$$\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}=0$$

$$\phi(\mathbf{r},t)=\frac{1}{4\pi\,\varepsilon_o}\int\frac{\rho(\mathbf{r}',t'_{ret})}{\left|\mathbf{r}-\mathbf{r}'\right|}d^3x'$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_o}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t'_{ret})}{\left|\mathbf{r}-\mathbf{r}'\right|}d^3x'$$

$$t'_{ret}=t-\left|\mathbf{r}-\mathbf{r}'\right|/c$$

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{r^3}\frac{\left[3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}})-\mathbf{p}\right]}{4\pi\,\varepsilon_o}+\frac{1}{c\,r^2}\frac{\left[3\hat{\mathbf{n}}(\dot{\mathbf{p}}\cdot\hat{\mathbf{n}})-\dot{\mathbf{p}}\right]}{4\pi\,\varepsilon_o}+\frac{1}{rc^2}\frac{(\ddot{\mathbf{p}}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{4\pi\,\varepsilon_o}$$

$$\mathbf{B}(\mathbf{r},t)=\frac{\mu_o}{4\pi}\left[\frac{\dot{\mathbf{p}}}{r^2}+\frac{\ddot{\mathbf{p}}}{c\,r}\right]\times\hat{\mathbf{n}}$$

all evaluated at $t'=t-r/c$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon_o E^2+\frac{B^2}{2\mu_o}\right]+\nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right)=-\mathbf{E}\cdot\mathbf{J}$$

Energy density of field $=\frac{1}{2}\varepsilon_o E^2+\frac{B^2}{2\mu_o}$

Energy flux density = $\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}$

$$\frac{\partial}{\partial t}\left[\varepsilon_o\mathbf{E}\times\mathbf{B}\right]+\nabla\cdot\left(-\overleftrightarrow{\mathbf{T}}\right)=-\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right]$$

Momentum density = $\varepsilon_o\mathbf{E}\times\mathbf{B}$

Momentum flux density = $-\overleftrightarrow{\mathbf{T}}$

$$\frac{\partial}{\partial t}\left[\varepsilon_o\mathbf{r}\times\left(\mathbf{E}\times\mathbf{B}\right)\right]+\nabla\cdot\left(-\mathbf{r}\times\overleftrightarrow{\mathbf{T}}\right)=-\mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right]$$

Angular momentum density = $\varepsilon_o\mathbf{r}\times\left(\mathbf{E}\times\mathbf{B}\right)$

Angular momentum flux density = $-\mathbf{r}\times\overleftrightarrow{\mathbf{T}}$

$$\overleftrightarrow{\mathbf{T}}=\varepsilon_o\left[\mathbf{E}\mathbf{E}-\frac{1}{2}\overleftrightarrow{\mathbf{I}}\,E^2\right]+\frac{1}{\mu_o}\left[\mathbf{B}\mathbf{B}-\frac{1}{2}\overleftrightarrow{\mathbf{I}}\,B^2\right]$$

$\overleftrightarrow{\mathbf{T}}_{\mathbf{E}}\cdot\hat{\mathbf{n}}$ lies in the plane defined by $\mathbf{E}$ and $\hat{\mathbf{n}}$.

The magnitude of $\overleftrightarrow{\mathbf{T}}_{\mathbf{E}}\cdot\hat{\mathbf{n}}$ is always $\frac{1}{2}\varepsilon_o E^2$

If you go an angle θ to get to $\mathbf{E}$ from $\hat{\mathbf{n}}$, go an angle 2θ in the same sense to get to $\overleftrightarrow{\mathbf{T}}_{\mathbf{E}}\cdot\hat{\mathbf{n}}$

All of the above apply to $\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\cdot\hat{\mathbf{n}}$ except for the fact that $\left|\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\cdot\hat{\mathbf{n}}\right|$ is always $B^2/2\mu_o$

$$\mathbf{p}(t') = \int \mathbf{r}'\rho(\mathbf{r}',t')d^3x'$$

$$\frac{dW}{d\Omega\ dt} = \frac{c}{\mu_o}(\mathbf{E}\times\mathbf{B})\cdot(r^2\ \hat{\mathbf{r}})$$

$$\frac{dW_{el\ dip}}{d\Omega\ dt} = \frac{c\ r^2}{\mu_o}\left[\frac{\mu_o}{4\pi}\frac{\ddot{\mathbf{p}}\,\mathbf{x}\,\hat{\mathbf{n}}}{c\ r}\right]^2 = \frac{\mu_o\ \ddot{p}^{\ 2}}{(4\pi)^2 c}\sin^2\theta$$

$$\frac{dW_{el\ dip}}{dt} = \frac{1}{4\pi\,\varepsilon_o}\frac{2}{3}\frac{|\ddot{\mathbf{p}}|^2}{c^3}$$

$$\frac{dW_{el\ dip}}{dt} = \frac{1}{4\pi\,\varepsilon_o}\frac{2}{3}\frac{q^2a^2}{c^3}$$

$$\mathbf{m}(t') = \frac{1}{2}\int \mathbf{r}'\times\mathbf{J}(\mathbf{r}',t')d^3x'$$

$$\frac{dW_{mag\ dip}}{dt} = \frac{\mu_o}{4\pi}\frac{2|\ddot{\mathbf{m}}|^2}{3c^3}$$

$$\rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{X}(t))$$

$$\mathbf{J}(\mathbf{r},t) = q\mathbf{u}(t)\delta^3(\mathbf{r}-\mathbf{X}(t))$$

$$\mathbf{u}(t) = \frac{d}{dt}\mathbf{X}(t)$$

$$\mathbf{p}(t') = \int \mathbf{r}'\rho(\mathbf{r}',t')d^3x' = q\mathbf{X}(t')$$

$$\mathbf{m}(t') = \frac{1}{2}\int \mathbf{r}'\times\mathbf{J}(\mathbf{r}',t')d^3x' = \frac{q}{2}\mathbf{X}(t')\times\mathbf{u}(t')$$

For a localized distribution of current,

$$\int_{\text{all space}} \mathbf{J}(\mathbf{r}',t')d^3x' = -\int_{\text{all space}} \left(\mathbf{r}'\left[\nabla'\cdot\mathbf{J}(\mathbf{r}',t')\right]\right)d^3x'$$

$$\frac{\partial\rho(\mathbf{r}',t')}{\partial t'} + \nabla'\cdot\mathbf{J}(\mathbf{r}',t') = 0$$

$$|\,\mathbf{r}-\mathbf{r}'\,| = r - \hat{\mathbf{n}}\cdot\mathbf{r}' + ... \qquad \hat{\mathbf{n}} = \mathbf{r}/|\mathbf{r}|$$

$$\frac{1}{|\,\mathbf{r}-\mathbf{r}'\,|} = \frac{1}{r} + \frac{\hat{\mathbf{n}}\cdot\mathbf{r}'}{r^2} + ...$$

$$E_{radiation} = cB_{radiation}$$

$$\rho(\mathbf{r}',t'_{ret}) = \rho(\mathbf{r}',t-r/c+\hat{\mathbf{n}}\cdot\mathbf{r}'/c+...) \ \text{ Taylor series}$$

$$\rho(\mathbf{r}',t'_{ret}) \cong \rho(\mathbf{r}',t-r/c) + \frac{\hat{\mathbf{n}}\cdot\mathbf{r}'}{c}\frac{\partial}{\partial t'}\rho(\mathbf{r}',t-r/c) + ...$$

Problem 1

The general solution for the potentials given the sources everywhere in space and time is

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\,\varepsilon_o}\int \frac{\rho(\mathbf{r}',t'_{ret})}{|\mathbf{r}-\mathbf{r}'|}d^3x' \quad \mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t'_{ret})}{|\mathbf{r}-\mathbf{r}'|}d^3x' \quad t'_{ret} = t - |\mathbf{r}-\mathbf{r}'|/c$$

Suppose we are far away from a localized distribution of charge and current that is varying slowly in time.

a. From the equations above, with suitable approximations, derive the *radiation part* of the magnetic field associated with the electric dipole moment $\mathbf{p}(t)$.

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t'_{ret})}{|\mathbf{r}-\mathbf{r}'|}d^3x' \approx \frac{\mu_o}{4\pi r}\int \mathbf{J}(\mathbf{r}',t'-r/c)\,d^3x' + ...$$

Using the vector identity given on the equation sheet, we have that

$$\mathbf{A}(\mathbf{r},t) \approx \frac{\mu_o}{4\pi r}\int \mathbf{J}(\mathbf{r}',t-r/c)\,d^3x' + ... = -\frac{\mu_o}{4\pi r}\int \left(\mathbf{r}'\left[\nabla'\cdot\mathbf{J}(\mathbf{r}',t-r/c)\right]\right)d^3x + ...$$

$$= \frac{\mu_o}{4\pi r}\int\left(\mathbf{r}'\frac{\partial\rho(\mathbf{r}',t-r/c)}{\partial t'}\right)d^3x + ... = \frac{\mu_o}{4\pi r}\frac{d\mathbf{p}(t-r/c)}{dt'} + ...$$

where we have used conservation of charge from the equation sheet and the definition of the electric dipole moment from the equation sheet, on the second line of the equation above.

To find the magnetic field we must take the curl of **A**. Since we are only interested in the radiation (*1/r*) terms, we have

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t) \approx \frac{\mu_o}{4\pi r}\nabla\times\frac{d\mathbf{p}(t-r/c)}{dt'}$$

But

$$\frac{\partial}{\partial x_i}\frac{d\mathbf{p}(t-r/c)}{dt'} = \frac{d^2\mathbf{p}(t-r/c)}{(dt')^2}\frac{\partial}{\partial x_i}\left(-\frac{r}{c}\right) = -\frac{1}{c}\frac{d^2\mathbf{p}(t-r/c)}{(dt')^2}\frac{x_i}{r} = -\frac{1}{c}\frac{d^2\mathbf{p}(t-r/c)}{(dt')^2}n_i$$

So $\frac{\mu_o}{4\pi r}\nabla\times\frac{d\mathbf{p}(t-r/c)}{dt'} = -\frac{1}{c}\frac{\mu_o}{4\pi r}\hat{\mathbf{n}}\times\frac{d^2\mathbf{p}(t-r/c)}{(dt')^2}$, and thus $\mathbf{B}_{radiation}(\mathbf{r},t) \approx \frac{\mu_o}{4\pi rc}\ddot{\mathbf{p}}(t-r/c)\times\hat{\mathbf{n}}$

b. Given your answer in (a), which must be transverse to the radial direction $\hat{\mathbf{n}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$, what is the associated radiation electric field?

The wave is propagating radial outward so the magnetic field must be perpendicular to that direction and the radiation electric field must both be perpendicular to the radial direction and to the radiation magnetic field. A little thought shows us that we must have

$$\mathbf{E}_{radiation} = c\mathbf{B}_{radiation} \times \hat{\mathbf{n}} = \frac{\mu_o}{4\pi r}\left(\ddot{\mathbf{p}} \times \hat{\mathbf{n}}\right) \times \hat{\mathbf{n}}$$

c. What is the rate at which energy is flowing outward per unit area in the radial direction?

$$\frac{\mathbf{E}_{radiation} \times \mathbf{B}_{radiation}}{\mu_o} = \frac{1}{\mu_o}\left(c\mathbf{B}_{radiation} \times \hat{\mathbf{n}}\right) \times \mathbf{B}_{radiation} = \frac{cB^2_{radiation}}{\mu_o}\hat{\mathbf{n}} = \frac{\mu_o}{16\pi^2 r^2 c}\left|\ddot{\mathbf{p}} \times \hat{\mathbf{n}}\right|^2$$

Problem 2

A capacitor with capacitance C is in a circuit with an inductor with inductance L. There is no resistance as we normally think of it in this circuit. The capacitor has maximum energy and charge Q_o at $t = 0$, and the energy then sloshes back and forth between the inductor and capacitor at a frequency $\omega = 1/\sqrt{LC}$. We assume that the dimensions of this circuit are such that the speed of light transit time across the circuit is much shorter than $2\pi/\omega$. The distance between the plates of the capacitor is d, and the area of the plates is A_C, so that $C = \frac{\varepsilon_o A_C}{d}$. Take the electric dipole moment of the capacitor to be the charge times the distance between the plates. The inductance is a solenoid, with N turns, cross-sectional area A_L, and length h, so that $L = \frac{\mu_o N^2 A_L}{h}$. Assume that the dipole moment of the capacitor is Qd.

(a) What is the time-averaged rate at which this system radiates electric dipole radiation, in terms of d, ω, c, ε_o, and Q_o?

$$\left\langle \frac{dW_{rad}}{dt} \right\rangle = \frac{1}{4\pi\varepsilon_o} \frac{2\left\langle (\ddot{p})^2 \right\rangle}{c^3} = \frac{d^2 Q_o^2 \omega^4}{4\pi\varepsilon_o c^3}$$

(b) Take the total energy radiated in one period of the oscillation (your answer in (a) times $\frac{2\pi}{\omega}$ and divide it by the average energy in the capacitor, $\frac{Q_o^{\ 2}}{4C}$. Show this ratio is small if the speed of light transit time across the capacitor is small.

$$\left\langle \frac{dW_{rad}}{dt} \right\rangle \frac{2\pi}{\omega} = \frac{2\pi}{\omega} \frac{d^2 Q_o^2 \omega^4}{4\pi\varepsilon_o c^3} = \frac{d^2 Q_o^2 \omega^3}{2\varepsilon_o c^3}$$

$$\frac{4C}{Q_o^2} \left\langle \frac{dW_{rad}}{dt} \right\rangle \frac{2\pi}{\omega} = \frac{4C}{Q_o^2} \frac{d^2 Q_o^2 \omega^3}{2\varepsilon_o c^3} = \frac{2d^2\omega^3}{\varepsilon_o c^3} C = \frac{2d^2\omega^3}{\varepsilon_o c^3} \frac{\varepsilon_o A_C}{d} = \frac{2d\omega^3}{c^3} A_C$$

$$\frac{4C}{Q_o^2} \left\langle \frac{dW_{rad}}{dt} \right\rangle \frac{2\pi}{\omega} = 2(2\pi)^3 \frac{A_C d}{c^3 T^3} << 1$$ as long as the transit time across any dimension of the capacitor is small compared to T

(c) The current $I(t)$ in this circuit is given by $I(t) = \frac{d}{dt} Q(t)$, so that it is clear that the time-averaged value of I^2 is $\left\langle I^2 \right\rangle = \frac{\omega^2 Q_o^{\ 2}}{2}$. Use this relation to write your answer in (a) for the energy radiated as $\left\langle I^2 \right\rangle R_{radiation}$, where $R_{radiation}$ is the "radiation resistance", and has units of ohms. Give an expression for $R_{radiation}$ in terms of d, ω, c, and ε_o.

$$\left\langle \frac{dW_{rad}}{dt} \right\rangle = \frac{d^2 Q_o^2 \omega^4}{4\pi\varepsilon_o c^3} = \left\langle I^2 \right\rangle \frac{d^2\omega^2}{2\pi\varepsilon_o c^3} = \left\langle I^2 \right\rangle R_{radiation} \qquad R_{radiation} = \frac{d^2\omega^2}{2\pi\varepsilon_o c^3}$$

(d) Using $\omega = \frac{1}{\sqrt{LC}}$, and the equations for L and C given above, to show that the radiation resistance you have from (b) can be written in the form $c\mu_o$ times a dimensionless expression which involves the geometry of the capacitor and inductor, and N. The constant $c\mu_o$ has dimensions of ohms ($c\mu_o$ = 377 ohms), and is sometimes called the radiation resistance of free space. This "radiation resistance" has the same effect as a true resistance--the energy in the circuit slowly decreases as it is irreversibly lost to the system through radiation.

$$R_{radiation} = \frac{d^2\omega^2}{2\pi\varepsilon_o c^3} = \frac{d^2}{2\pi\varepsilon_o c^3}\frac{1}{LC} = \frac{d^2}{2\pi\varepsilon_o c^3}\frac{hd}{\left(\mu_o N^2 A_L\right)\left(\varepsilon_o A_C\right)}$$

$$= \frac{1}{\varepsilon_o c^3}\frac{1}{\varepsilon_o \mu_o}\left[\frac{1}{2\pi}\frac{hd^3}{N^2 A_L A_C}\right] = \frac{1}{\varepsilon_o c}\left[\frac{1}{2\pi}\frac{hd^3}{N^2 A_L A_C}\right] = \mu_o c\left[\frac{1}{2\pi}\frac{hd^3}{N^2 A_L A_C}\right]$$