Physics 8.07 **Fall 2011**

SOLUTIONS 2ND MIDTERM 11/18/2011

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_o}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_o \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\phi(\mathbf{r},t) = \frac{1}{\left[1 - \hat{\mathbf{n}}(t'_{ret}) \cdot \boldsymbol{\beta}(t'_{ret})\right]} \frac{q}{4\pi\,\varepsilon_o} \frac{1}{\left|\mathbf{r} - \mathbf{X}(t'_{ret})\right|}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mathbf{u}(t'_{ret})}{c^2}\,\phi(\mathbf{r},t)$$

$$t'_{ret} = t - \left|\mathbf{r} - \mathbf{X}(t'_{ret})\right|/c$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{E}(\mathbf{r},t) = \left[\frac{q}{4\pi\,\varepsilon_o} \frac{\hat{\mathbf{n}} - \boldsymbol{\beta}}{\gamma_u^2 (1 - \hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3 R^2}\right]_{ret} + \left[\frac{q}{4\pi\,\varepsilon_o}\frac{1}{c}\frac{\hat{\mathbf{n}} \mathbf{\ x} \left\{(\hat{\mathbf{n}} - \boldsymbol{\beta}) \mathbf{\ x}\, \dot{\boldsymbol{\beta}}\right\}}{(1 - \hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3 R}\right]_{ret}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\left[\hat{\mathbf{n}} \mathbf{\ x\ E}\right]_{ret}$$

$$\frac{dW_{rad}}{d\Omega\, dt'} = \frac{q^2}{(4\pi)^2 c\varepsilon_o} \frac{\left|\hat{\mathbf{n}} \times \left[(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]\right|^2}{(1 - \hat{\mathbf{n}}\cdot\boldsymbol{\beta})^5}$$

$$\frac{dW_{rad}}{dt'} = \frac{1}{4\pi\,\varepsilon_o}\frac{2q^2}{3c}\gamma^6\left[\left|\dot{\boldsymbol{\beta}}\right|^2 - \left|\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}\right|^2\right]$$

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix}$$

$$J^\mu = (c\rho, J_x, J_y, J_z)$$

$$A^\mu = (\frac{\phi}{c}, A_x, A_y, A_z)$$

$$\bar{E}_x = E_x$$

$$\bar{E}_y = \gamma\left(E_y - \mathrm{v}\,B_z\right)$$

$$\bar{E}_z = \gamma\left(E_z + \mathrm{v}B_y\right)$$

$$\bar{B}_x = B_x$$

$$\bar{B}_y = \gamma\left(B_y + \frac{\mathrm{v}}{c^2}E_z\right)$$

$$\bar{B}_z = \gamma(B_z - \frac{\mathrm{v}}{c^2}E_y)$$

Contra-variant 4 vector transforms as

$$\begin{pmatrix} \bar{S}^0 \\ \bar{S}^1 \\ \bar{S}^2 \\ \bar{S}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S^0 \\ S^1 \\ S^2 \\ S^3 \end{pmatrix}$$

Co-variant 4 vector transforms as

$$\begin{pmatrix} \bar{S}_0 \\ \bar{S}_1 \\ \bar{S}_2 \\ \bar{S}_3 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

$$\begin{pmatrix} S^0 \\ S^1 \\ S^2 \\ S^3 \end{pmatrix} = \begin{pmatrix} -S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = (-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

$$\partial_\mu F^{\mu\nu} = -\mu_o J^\nu \qquad \partial_\mu G^{\mu\nu} = 0$$

$$X^\mu(t) = \begin{pmatrix} c\,t \\ \mathbf{X}(t) \end{pmatrix}$$

$$\mathbf{u}(t) = \frac{d\,\mathbf{X}(t)}{dt}$$

$$d\tau = d\,t\sqrt{\left(1 - \frac{\mathbf{u}^2(t)}{c^2}\right)}$$

$$\eta^\mu = \frac{d}{d\tau} X^\mu(\tau) = \begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix}$$

$$m\frac{d}{d\tau}\eta^\mu = qF^{\mu\sigma}\eta_\sigma$$

$$m\frac{d}{d\tau}\begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix} = m\gamma_u \frac{d}{dt}\begin{pmatrix} \gamma_u c \\ \gamma_u \mathbf{u}(t) \end{pmatrix}$$

$$= q\begin{pmatrix} \gamma_u \dfrac{\mathbf{E}\cdot\mathbf{u}}{c} \\ \gamma_u\left(\mathbf{E} + \mathbf{u}\,\mathbf{x}\,\mathbf{B}\right) \end{pmatrix}$$

$$Energy\ of\ \ particle = m\gamma_u c^2 = mc^2 / \sqrt{1 - \frac{u^2}{c^2}}$$

$$\mathbf{F}_{\text{radiation reaction}} = \frac{1}{4\pi\varepsilon_o}\frac{2}{3}\frac{q^2}{c^3}\frac{d^2\mathbf{V}}{dt^2}$$

Electrostatics:

$$\mathbf{E} = -\nabla\phi \qquad \phi(\mathbf{r}) = -\int_{\mathbf{r}_o}^{\mathbf{r}} \mathbf{E}(\mathbf{r}')\cdot d\,\mathbf{l}'$$

$$\phi(\mathbf{r}) = \int_{all\,space} \frac{\rho(\mathbf{r}')\,d^3x'}{4\pi\varepsilon_o\left|\mathbf{r} - \mathbf{r}'\right|}$$

$$W_N = \sum_{i=1}^{N}\sum_{j<i} \frac{q_i q_j}{4\pi\varepsilon_o\left|\mathbf{r}_i - \mathbf{r}_j\right|}$$

$$W = \int \frac{1}{2}\rho(\mathbf{r})\phi(\mathbf{r})\,d^3x = \int \frac{1}{2}\varepsilon_o E^2(\mathbf{r})d^3x$$

$$\nabla^2\phi(\mathbf{r}) = -\rho(\mathbf{r}) / \varepsilon_o$$

$$\int_{\substack{\text{closed}\\ \text{surface}}} \mathbf{E}\cdot\hat{\mathbf{n}}\,da = \frac{q_{\text{enclosed}}}{\varepsilon_o}$$

In general in spherical coordinates:

$$\phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^l + B_{lm}r^{-l-1}\right]Y_{lm}(\theta,\phi)$$

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{+im\phi}$$

For ϕ symmetry in spherical coordinates:

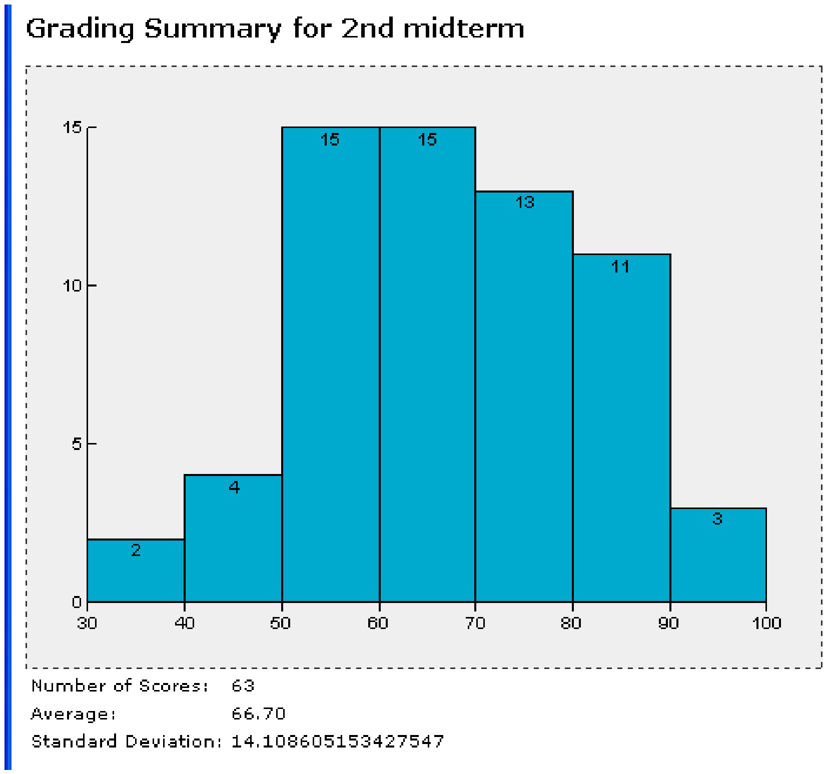
$$\phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

$$\int_{-1}^{1} P_l(x)P_m(x)dx = \frac{2}{2l+1}\delta_{lm}$$

Boundary conditions on E:

$$E_{2n} - E_{1n} = \hat{\mathbf{n}}\cdot\left(\mathbf{E}_2 - \mathbf{E}_1\right) = \sigma / \varepsilon_o$$

$$\mathbf{E}_{2t} = \mathbf{E}_{1t}$$



Problem 1: Moving Magnetic Dipoles

A classical point magnetic dipole moment $\mathbf{m} = m\hat{\mathbf{y}}$ at rest has a vector potential $\mathbf{A}$ given by

$$\mathbf{A} = \frac{\mu_o}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3} = \frac{\mu_o m}{4\pi}\frac{\hat{\mathbf{y}}\times\mathbf{r}}{r^3}$$

Suppose that this magnetic dipole $\mathbf{m} = m\hat{\mathbf{y}}$ moves with velocity $\mathbf{v} = \mathrm{v}\hat{\mathbf{x}}$, and assume $\mathrm{v} \ll c$.

(a) Show ***using the transformation properties of the four vector potential*** A^μ that in a frame where this magnetic dipole moves with velocity $\mathbf{v} = \mathrm{v}\hat{\mathbf{x}}$, with $\mathrm{v} \ll c$, it has an associated electric dipole moment $\mathbf{p}$. Give the magnitude and direction of that $\mathbf{p}$. [Hint: You may want to look at the first formula given in Problem 4 below].

We know the potential in the barred frame so to get back to the laboratory frame we use

$$\begin{pmatrix} \phi/c \\ A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\phi}/c \\ \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix}$$

We are only interested in the scalar potential in the lab frame, and that is given by

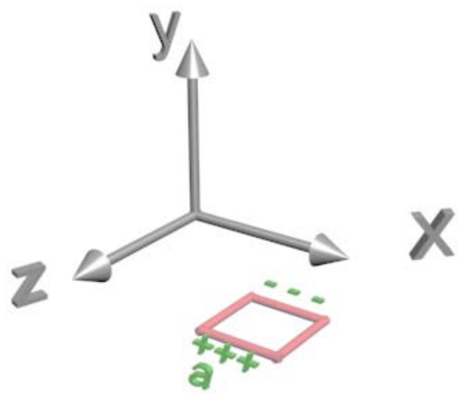
$$\frac{\phi}{c} = \gamma\beta\bar{A}_x \quad \text{or} \quad \phi \approx \mathrm{v}\bar{A}_x = \mathrm{v}\left[\frac{\mu_o m}{4\pi}\frac{\hat{\mathbf{y}}\times\bar{\mathbf{r}}}{\bar{r}^3}\right]_x \approx \frac{\mu_o m\mathrm{v}\bar{z}}{4\pi\bar{r}^3} \approx \frac{\mu_o m\mathrm{v}\bar{z}}{4\pi\bar{r}^3}$$

$$\bar{x} \approx x - \mathrm{v}t \quad \bar{\mathrm{y}} = \mathrm{y} \quad \bar{\mathrm{z}} = \mathrm{z} \quad \bar{\mathrm{t}} \approx \mathrm{t}$$

$$\phi \approx \frac{\mu_o m\mathrm{v}z}{4\pi\left[(x-\mathrm{v}t)^2 + y^2 + z^2\right]^{3/2}} \approx \frac{1}{4\pi\varepsilon_o}\frac{\left[\varepsilon_o\mu_o m\mathrm{v}z\right]}{\left[(x-\mathrm{v}t)^2 + y^2 + z^2\right]^{3/2}}.$$

This is the scalar potential due to a moving electric dipole with $\mathbf{p} = \hat{\mathbf{z}}\frac{m\mathrm{v}}{c^2}$

(b) Model the magnetic dipole $\mathbf{m} = m\hat{\mathbf{y}}$ above as a current "square" lying in the xz plane, with sides *a*, as shown in the sketch. In its rest frame, the square has a current *I* flowing around the square, and no net charge. Show ***using the transformation properties of the four vector*** J^μ that in a frame where this magnetic dipole moves with velocity $\mathbf{v} = \mathrm{v}\hat{\mathbf{x}}$, with $\mathrm{v} \ll \mathrm{c}$, it has an associated electric dipole moment **p**. Give the magnitude and direction of that **p**.



We know the current and the charge density in the moving frame, so to be back to the lab frame we use

$$\begin{pmatrix} c\lambda \\ I_x \\ I_y \\ I_z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c\bar{\lambda} \\ \bar{I}_x \\ \bar{I}_y \\ \bar{I}_z \end{pmatrix}$$

The current must be moving counterclockwise as viewed from above (on the y-axis) to give a magnetic moment in the +y direction, so we see from the above transformation, that the further side of the loop charges up negative with a charge per unit length given by $\lambda \approx -\beta I / c = -\mathrm{v}I / c^2$, for a total charge of $\lambda a \approx -Ia\mathrm{v} / c^2$. See sketch. Similarly the near side of the square will charge up positive to $\lambda a \approx +Ia\mathrm{v} / c^2$. We will therefore have an electric dipole with electric dipole moment $p \approx qa = Ia^2\mathrm{v} / c^2 = m\mathrm{v} / c^2$, in the positive z direction, just as above.in part (a).

Problem 2: Relativistic Particle Motion

A particle of mass *m* and charge *q* is at rest at the origin at time $t = \tau = 0$ in a constant electric field $\mathbf{E} = E\hat{\mathbf{x}}$. At some time later its velocity is $\mathbf{u} = u\hat{\mathbf{x}}$ and its gamma is $\gamma_u = 1/\sqrt{1-u^2/c^2}$.

(a) Find and solve a second order differential equation for $\gamma_u(\tau)$ (i.e. an equation involving $\frac{d^2\gamma_u(\tau)}{d\tau^2}$) given these initial conditions. Neglect any radiation losses.

From the equation sheet for **B** = 0, we have

$$m\frac{d}{d\tau}\gamma_u c = q\gamma_u \frac{\mathbf{E}\cdot\mathbf{u}}{c}$$
$$m\frac{d}{d\tau}\gamma_u \mathbf{u}(t) = q\gamma_u \mathbf{E}$$

So in this case we have

$$\frac{d}{d\tau}m\gamma_u c^2 = q\gamma_u Eu$$
$$m\frac{d}{d\tau}\gamma_u u(t) = q\gamma_u E$$

$$\frac{d^2}{d\tau^2}m\gamma_u c^2 = qE\frac{d}{d\tau}\gamma_u u \qquad \Rightarrow \frac{d^2}{d\tau^2}m\gamma_u c^2 = \left(\frac{qE}{mc}\right)^2 m\gamma_u c^2$$
$$\frac{d}{d\tau}\gamma_u u(t) = \frac{qE}{m}\gamma_u$$

The solution to the differential equation is $\gamma_u = A\sinh\Omega_E\tau + B\cosh\Omega_E\tau$ where $\Omega_E = qE/mc$.

We can determine *A* and *B* from the initial conditions, and in particular we have

$\gamma_u = 1$ at t = τ=0, since the speed is zero then,

$\frac{d}{d\tau}\gamma_u \propto Eu = 0$ at t = τ=0, since the speed is zero then, so $\gamma_u = \cosh\Omega_E\tau$

(b) What is the relation between coordinate time t and proper time τ for this charge ? [Hint: $dt = \gamma_u d\tau$]. Show that your relationship between t and τ reduces to the one you expect for "small" times. How do you define "small" time?

$dt = \gamma_u d\tau = \cosh\Omega_E\tau d\tau \quad \Rightarrow t = \frac{\sinh\Omega_E\tau}{\Omega_E}$ where we have chosen the constant of integration so that *t = 0* at $\tau = 0$. Small times are times for which $\Omega_E t \approx \Omega_E\tau << 1$. For small times in this sense we have

$$t = \frac{\sinh\Omega_E\tau}{\Omega_E} \approx \frac{\Omega_E\tau + ...}{\Omega_E} \approx \tau \quad \text{as we expect}$$

Problem 3: Electric field of a charge moving at constant speed

The electric field for a change in arbitrary motion is given by

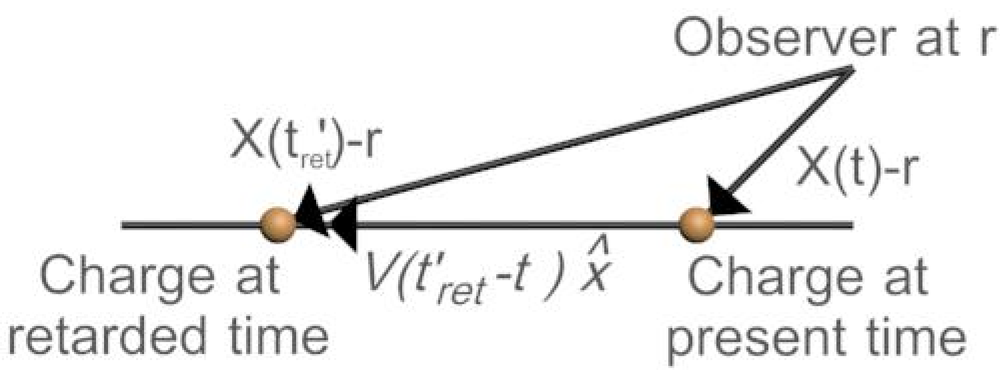
$$\mathbf{E}(\mathbf{r},t) = \left[\frac{q}{4\pi\,\varepsilon_o}\frac{\hat{\mathbf{n}}-\boldsymbol{\beta}}{\gamma_u^2(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3 R^2}\right]_{ret} + \left[\frac{q}{4\pi\,\varepsilon_o}\frac{1}{c}\frac{\hat{\mathbf{n}}\ \mathbf{x}\left\{(\hat{\mathbf{n}}-\boldsymbol{\beta})\ \mathbf{x}\ \dot{\boldsymbol{\beta}}\right\}}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3 R}\right]_{ret}$$

$$t'_{ret} = t - \left|\,\mathbf{r}-\mathbf{X}(t'_{ret})\,\right| / c$$

(a) Show that for a charge moving at constant velocity, the electric field the observer sees at time *t* points to the present position of the charge (that is the position of the charge at the observer's time *t*), and not to its position at the retarded time.

We know that $t'_{ret} = t - \left|\,\mathbf{r}-\mathbf{X}(t'_{ret})\,\right|/c$, which is the same as $c\left(t-t'_{ret}\right) = \left|\,\mathbf{r}-\mathbf{X}(t'_{ret})\,\right|$. Let us assume that the charge is moving at constant velocity $\mathbf{V}=V\hat{\mathbf{x}}$, and thus $\mathbf{X}(t')=Vt'\,\hat{\mathbf{x}}$, where we have assumed that the charge is at the origin at t' = 0. If we look at the diagram, we see geometrically and algebraically that

$$\mathbf{X}(t'_{ret})-\mathbf{r} = \mathbf{X}(t'_{ret})-\mathbf{X}(t)+\mathbf{X}(t)-\mathbf{r} = \left(\mathbf{X}(t)-\mathbf{r}\right)+V\left(t'_{ret}-t\right)\hat{\mathbf{x}}$$



Or $$\mathbf{X}(t)-\mathbf{r} = \left(\mathbf{X}(t'_{ret})-\mathbf{r}\right)-V\left(t'_{ret}-t\right)\hat{\mathbf{x}} = -\hat{\mathbf{n}}_{retarded}\left|\mathbf{X}(t'_{ret})-\mathbf{r}\right| - V\left(t'_{ret}-t\right)\hat{\mathbf{x}}$$

where we have used the fact that

$$\hat{\mathbf{n}}_{retarded} = \left(\mathbf{r}-\mathbf{X}(t'_{ret})\right)/\left|\mathbf{X}(t'_{ret})-\mathbf{r}\right| \quad \textit{from}\text{ the source }\textit{to}\text{ the observer}$$

thus

$$\mathbf{X}(t)-\mathbf{r} = \left(-\hat{\mathbf{n}}_{retarded} - \frac{V\left(t'_{ret}-t\right)}{\left|\mathbf{X}(t'_{ret})-\mathbf{r}\right|}\hat{\mathbf{x}}\right)\left|\mathbf{X}(t'_{ret})-\mathbf{r}\right|$$

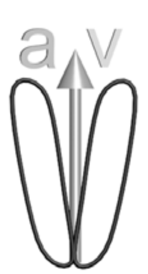
But we know that $c\left(t-t'_{ret}\right) = \left|\,\mathbf{r}-\mathbf{X}(t'_{ret})\,\right|$ or $\dfrac{\left(t-t'_{ret}\right)}{\left|\,\mathbf{r}-\mathbf{X}(t'_{ret})\,\right|} = \dfrac{1}{c}$

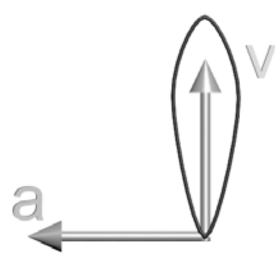
So

$$(\mathbf{r}-\mathbf{X}(t)) = |\mathbf{X}(t'_{ret})-\mathbf{r}|\left(\hat{\mathbf{n}}_{retarded} - \frac{V}{c}\hat{\mathbf{x}}\right) = |\mathbf{X}(t'_{ret})-\mathbf{r}|\left(\hat{\mathbf{n}}_{retarded} - \frac{\mathbf{V}}{c}\right)$$

Which is what we wanted to show, e.g. that the direction from the present position of the particle to the observer $(\mathbf{r}-\mathbf{X}(t))$, is parallel to $\left(\hat{\mathbf{n}}_{retarded} - \frac{\mathbf{V}}{c}\right) = \left(\hat{\mathbf{n}}_{retarded} - \boldsymbol{\beta}\right)$.

(b) Now assume that the particle is ultra-relativistic and accelerating, that is $\gamma \gg 1$, where $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-(\mathrm{v}/c)^2}$. Sketch on the diagrams below the angular distribution of the radiated energy for the two different orientations shown of the particle's acceleration **a** with respect to its velocity **v**.





(c) Derive an approximate expression for the angular width of the angular distribution of radiated energy for this ultra-relativistic charge. [Hint: for small ψ, $\cos\psi \approx 1-\psi^2/2+....$]. Express your answer in terms of γ.

$$\frac{dW_{rad}}{d\Omega\, dt'} = \frac{q^2}{(4\pi)^2 c\varepsilon_o}\frac{\left|\hat{\mathbf{n}}\times\left[(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|^2}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^5} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow \beta = \sqrt{1-\frac{1}{\gamma^2}} \approx 1-\frac{1}{2\gamma^2}+..$$

$$\frac{1}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^5} = \frac{1}{(1-\beta\cos\theta)^5} \cong \frac{1}{\left(1-\left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{\theta^2}{2}\right)\right)^5} \qquad \frac{1}{(1-\beta\cos\theta)^5} \cong \frac{1}{\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right)^5}$$

$$\frac{1}{(1-\beta\cos\theta)^5} \propto \frac{\gamma^{10}}{\left(1+\gamma^2\theta^2\right)^5}$$

So the angular width of the peak is proportional to $\frac{1}{\gamma}$

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Problem 4: Electrostatics

The potential due to a point dipole $\mathbf{p} = p\hat{\mathbf{z}}$ located at the origin in cylindrical coordinates is

$$\phi = \frac{1}{4\pi\varepsilon_o}\frac{\mathbf{p}\cdot\mathbf{n}}{r^2} = \frac{p}{4\pi\varepsilon_o}\frac{z}{\left(\rho^2+z^2\right)^{3/2}}$$

The electric field is given by $\mathbf{E} = -\nabla\phi = -\hat{\boldsymbol{\rho}}\frac{\partial\phi}{\partial\rho} - \hat{\mathbf{z}}\frac{\partial\phi}{\partial z}$.

Use the above information to solve the following problem. We have a semi-infinite conductor which occupies all of the space below $z = 0$, and which is held at zero potential. In the space above $z = 0$ we have only an electric dipole $\mathbf{p} = p\hat{\mathbf{z}}$ located a distance D up the z-axis, held rigidly in place.

(a) What is the electric potential everywhere?

Potential is due to the dipole at $z = D$ and an image dipole with the same direction as p located a distance $z = -D$. The potential is zero for z < 0 and

$$\phi = \frac{p}{4\pi\varepsilon_o}\left\{\frac{z-D}{\left(\rho^2+(z-D)^2\right)^{3/2}} + \frac{z+D}{\left(\rho^2+(z+D)^2\right)^{3/2}}\right\} \quad \text{for } z > 0$$

This satisfied our boundary condition at $z = 0$ (it is zero) and the right $\nabla^2\phi$ for $z > 0$. By uniqueness it must be the correct solution.

(b) What is the electric field everywhere?

$$\mathbf{E} = 0 \quad \text{for } z < 0$$

$$\mathbf{E} = \left[\begin{array}{l} \hat{\boldsymbol{\rho}}\dfrac{3\rho p}{4\pi\varepsilon_o}\left\{\dfrac{(z-D)}{\left(\rho^2+(z-D)^2\right)^{5/2}} + \dfrac{(z+D)}{\left(\rho^2+(z+D)^2\right)^{5/2}}\right\} \\ +\hat{\mathbf{z}}\dfrac{p}{4\pi\varepsilon_o}\left\{\dfrac{2(z-D)^2-\rho^2}{\left(\rho^2+(z-D)^2\right)^{5/2}} + \dfrac{2(z+D)^2-\rho^2}{\left(\rho^2+(z+D)^2\right)^{5/2}}\right\} \end{array}\right] \quad \text{for } z > 0$$

(c) What is the induced surface charge density σ on the conductor at $z = 0$?

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$$E_{2n} - E_{1n} = \sigma/\varepsilon_o = E_{z=0^+} - E_{z=0^-} = \frac{p}{4\pi\varepsilon_o}\left\{\frac{2(D)^2-\rho^2}{\left(\rho^2+(D)^2\right)^{5/2}} + \frac{2(D)^2-\rho^2}{\left(\rho^2+(D)^2\right)^{5/2}}\right\}$$

$$\sigma = \frac{p}{2\pi}\frac{\left(2D^2-\rho^2\right)}{\left(\rho^2+(D)^2\right)^{5/2}}$$

This is positive for $\rho < \sqrt{2}D$ and negative for $\rho > \sqrt{2}D$, integrates to zero net charge induced on the plane at z = 0 (you were not asked to show this last statement).

(d) What is the total force exerted on the conducting plane? You need only give an integral that represents that force. You do not need to evaluate that integral. The area element in the *z = 0* plane is $\rho d\rho d\phi$

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$$\mathbf{F} = \iint \rho d\rho d\phi\left[\frac{1}{2}\sigma E_{z=0+}\hat{\mathbf{z}}\right] = \hat{\mathbf{z}}\iint \rho d\rho d\phi\left[\frac{\sigma^2}{2\varepsilon_o}\right] = \hat{\mathbf{z}}\int_0^\infty \rho d\rho\frac{\pi}{\varepsilon_o}\left[\frac{p}{2\pi}\frac{\left(2D^2-\rho^2\right)}{\left(\rho^2+(D)^2\right)^{5/2}}\right]^2$$

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The factor of ½ above was universally missed. One way to think about this is that the electric field is going from non-zero just above to zero just below, so the average field is $E_{z=0+}/2$. Another way to say this is you do not want include the self field due to the surface charge density itself, which contributes half the electric field for z >0. A third way of seeing this is looking at the stress tensor, which will give you a pull upward with force per unit area $\varepsilon_o E_{z=0+}^2/2 = \sigma^2/2\varepsilon_o$

(d) How does the force you found in (d) depend on the distance *D*? You can answer this question without explicitly evaluating the integral in (d).

Introduce the dimensionless variable $\eta = \rho / D$. Then

$$\mathbf{F} = \hat{\mathbf{z}}\int_0^\infty \rho d\rho\frac{\pi}{\varepsilon_o}\left[\frac{p}{2\pi}\frac{\left(2D^2-\rho^2\right)}{\left(\rho^2+(D)^2\right)^{5/2}}\right]^2 = \hat{\mathbf{z}}\frac{p^2}{4\pi\varepsilon_o}\frac{D^6}{D^{10}}\int_0^\infty \eta d\eta\left[\frac{\left(2-\eta^2\right)}{\left(\eta^2+1\right)^{5/2}}\right]^2$$

The integral with respect to the dimensionless variable is just a number, with no dimensions, so we see the force goes as inverse fourth power of *D*.

Note (you were not asked to show this): we will show in a little bit that the force of an electric dipole in an eternal electric field with a gradient is given by $\mathbf{p}\cdot\nabla\mathbf{E}$. In this case the force on the dipole for z > 0 is

$$p\frac{d}{dz}\mathbf{E}_{\text{due to the image dipole at z = D}} = \hat{\mathbf{z}}p\frac{d}{dz}\frac{p}{4\pi\varepsilon_o}\left\{\frac{2\left(z+D\right)^2}{\left(\left(z+D\right)^2\right)^{5/2}}\right\}\Bigg|_{z=D} = \hat{\mathbf{z}}p\frac{d}{dz}\frac{p}{4\pi\varepsilon_o}\left\{\frac{2}{\left(z+D\right)^3}\right\}\Bigg|_{z=D}$$

$$p\frac{d}{dz}\mathbf{E}_{\text{due to the image dipole at z = D}} = -\hat{\mathbf{z}}p\frac{p}{4\pi\varepsilon_o}\left\{\frac{6}{\left(z+D\right)^4}\right\}\Bigg|_{z=D} = -\hat{\mathbf{z}}\frac{3}{8}\frac{p^2}{4\pi\varepsilon_o}\frac{1}{D^4}$$

And by action and reaction the force on the surface charge is up with this same magnitude. For this to be true and the answer above to be true we much have

$$\int_0^\infty \eta\, d\eta \left[\frac{\left(2-\eta^2\right)}{\left(\eta^2+1\right)^{5/2}}\right]^2 = \frac{3}{8}$$ which is in fact the case.