8.07 Final Exam December 17, 2009

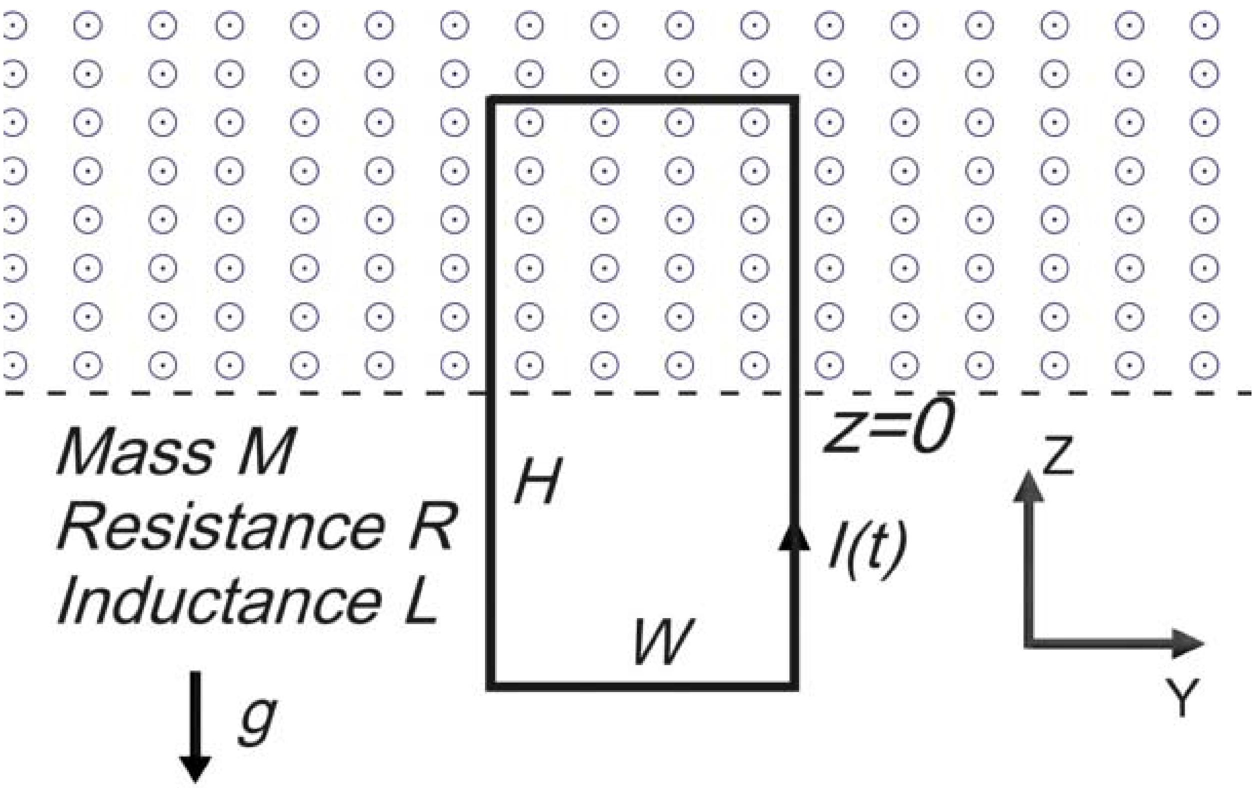
Name: ____

There are three problems. Problem 2 carries more weight.

Problem	Weight	Grade	Grader
1	30		
2	40		
3	30		
Total			

Problem 1 (30 points): Faraday's Law

A loop of mass M, resistance R, inductance L, height H, and width W sits in a magnetic field given by $\mathbf{B} = \hat{\mathbf{x}} \begin{cases} B_o & z \geq 0 \\ 0 & z < 0 \end{cases}$. At $t = 0$ the loop is at rest and its mid-point is at $z = 0$, as shown in the figure, and the current in the loop is zero at $t = 0$. The acceleration of gravity is downward at g. In answering the next few questions below, you may assume that the loop ***NEVER*** falls out of the magnetic field.



(a) What two differential equations determine the subsequent behavior of the loop? Write these two equations down in terms of the parameters given, as well as the location of the center of the loop, $z(t)$, and its speed, $\text{v}(t)=dz(t)/dt$, and the current in the loop at time t, $I(t)$. ***Take the direction of positive current to be counterclockwise, as shown in the figure.***

$$IR = -L\frac{dI}{dt} - \frac{d}{dt}W\left[B_o(H/2 + z(t))\right] \Rightarrow IR = -L\frac{dI}{dt} - WB_o\text{v}(t)$$

$$M\frac{d}{dt}\text{v}(t) = -Mg + IWB_o$$

(b) Manipulate your equations in (a) to find an equation for the conservation of total energy.

Multiplying the first equation by *I* gives

$$I^2R = -LI\frac{dI}{dt} - WIB_o\mathrm{v}(t) = -\frac{d}{dt}\frac{1}{2}LI^2 - WIB_o\mathrm{v}(t)$$

Multiplying the second equation by v gives

$$M\mathrm{v}\frac{d}{dt}\mathrm{v}(t) = -Mg\mathrm{v} + IWB_o\mathrm{v} = \frac{d}{dt}\frac{1}{2}M\mathrm{v}^2$$

Putting these two together gives conservation of energy

$$\frac{d}{dt}\left[\frac{1}{2}M\mathrm{v}^2 + Mgz + \frac{1}{2}LI^2\right] = -I^2R$$

(c) Now suppose the resistance of the loop is zero. In this situation, give the solution for *z(t)* that satisfies the boundary conditions, in terms of the parameters given. Again, assume that the loop never falls out of the magnetic field.

$$M\frac{d}{dt}\mathrm{v}(t) = -Mg + IWB_o \Rightarrow M\frac{d^2}{dt^2}\mathrm{v}(t) = WB_o\frac{dI}{dt}$$

$$0 = -L\frac{dI}{dt} - WB_o\mathrm{v}(t) \Rightarrow \frac{dI}{dt} = -\frac{WB_o\mathrm{v}(t)}{L}$$

so

$$M\frac{d^2}{dt^2}\mathrm{v}(t) = WB\frac{dI}{dt} = -\frac{W^2B_o^2}{L}\mathrm{v}(t) \Rightarrow \frac{d^2}{dt^2}\mathrm{v}(t) + \frac{W^2B_o^2}{ML}\mathrm{v}(t) = 0$$

so

$\mathrm{v}(t) = A\sin(\omega t)$ where $\omega^2 = \dfrac{W^2B_o^2}{ML}$ and we have picked the sin so that v is 0 at $t = 0$

$\Rightarrow z(t) = -\dfrac{A}{\omega}\left(\cos(\omega t) - 1\right)$ were we have picked the integration constant so that *z* is 0 at t =0

Finally, we determine the constant *A* from the fact that we have to satisfy at $t = 0$ the equation

$$M\frac{d}{dt}\mathrm{v}(t) = -Mg + IWB_o$$

This means that we must have $M\dfrac{d}{dt}\mathrm{v}(0) = MA\omega\cos(0) = -Mg \Rightarrow A = -\dfrac{g}{\omega}$

so

$$\mathrm{v}(t) = -\frac{g}{\omega}\sin(\omega t) \quad \text{where} \quad \omega^2 = \frac{W^2B_o^2}{ML} \quad \text{and} \quad z(t) = \frac{g}{\omega^2}\left(\cos(\omega t) - 1\right)$$

(d) Our assumption that the loop never falls out of the magnetic field imposes a condition on our parameters. What is that condition?

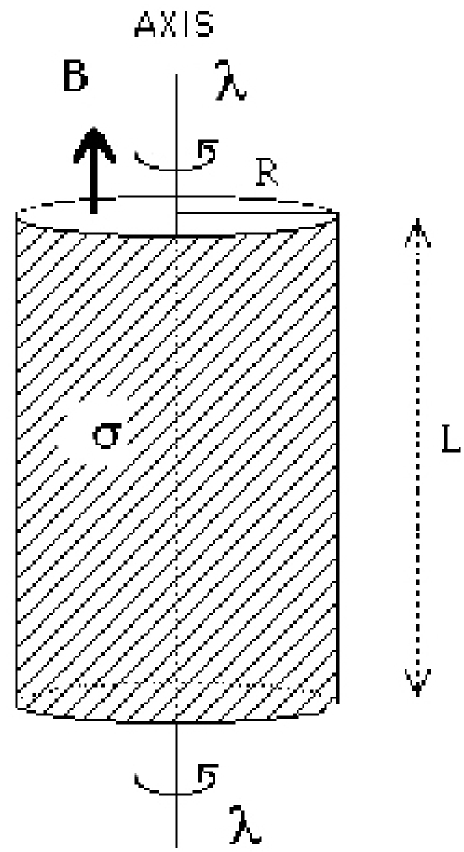
The maximum negative excursion of $z(t)$ given our formula above is $\frac{2g}{\omega^2}$

$$So\ \frac{2g}{\omega^2} < \frac{H}{2} \text{ or } \frac{4g}{\omega^2 H} < 1 \Rightarrow \frac{4MLg}{W^2 B_o^2 H} < 1$$

This makes sense since higher field strength and lighter rings will be less likely to fall out.

Problem 2 (40 points): Rotating Cylinder

A long cylinder has length L and radius R, with $L >> R$. The cylinder is suspended so that it can rotate freely about its axis with no friction. The cylinder carries a charge per unit area σ which is glued onto its surface, where $\sigma > 0$. Along the axis of the cylinder is a line charge with charge per unit length $\lambda = -2\pi R\sigma < 0$. Thus the electric field when the cylinder is not moving is given by



$$\mathbf{E}_{\text{static}}(r) = \begin{cases} -\dfrac{R}{r}\dfrac{\sigma}{\varepsilon_o}\hat{\mathbf{r}} & r < R \\ 0 & r > R \end{cases}$$

Now, we begin spinning the cylinder at a angular velocity $\omega(t)$ with $\omega R << c$. The motion of the charge glued onto the surface of the spinning sphere results in a surface current

$$\boldsymbol{\kappa}(t) = \sigma\,\omega(t)\,R\,\hat{\boldsymbol{\phi}}$$

We assume that we can use the quasi-static approximation to get a good approximation to the time dependent solution for **B** (good for variations in $\kappa(t)$ with time scales $T \approx \dfrac{\kappa}{d\kappa/dt} \gg \dfrac{R}{c}$)

(a) What is our quasi-static solution for **B**? Neglect fringing fields.

$$\mathbf{B}(\mathbf{r},t) = \begin{cases} \mu_o \sigma\,\omega(t) R\hat{\mathbf{z}} & (r < R) \\ 0 & (r > R) \end{cases}$$

(b) Given this quasi-static solution for **B**, what is the electric field everywhere in space? Neglect fringing fields.

$$\mathbf{E}(\mathbf{r},t) = \begin{cases} -\dfrac{R}{r}\dfrac{\sigma}{\varepsilon_o}\hat{\mathbf{r}} - \dfrac{r}{2}\mu_o R\sigma\dfrac{d\omega}{dt}\hat{\boldsymbol{\phi}} & r < R \\ -\dfrac{R^3}{2r}\mu_o\sigma\dfrac{d\omega}{dt}\hat{\boldsymbol{\phi}} & r > R \end{cases}$$

(c) What is the total magnetic energy?

$$W_M = \int \frac{B^2}{2\mu_o} d^3x = L\pi R^2 \frac{B^2}{2\mu_o} = \frac{1}{2} L\pi R^4 \mu_o \sigma^2 \omega^2$$

(d) Show that the total rate at which electromagnetic energy is being created as the cylinder is being spun up, $\int_{\text{all space}} -\mathbf{J}\cdot\mathbf{E}\, d^3x$, is equal to the rate at which the total magnetic energy is increasing.

$$\int_{\text{all space}} -\mathbf{J}\cdot\mathbf{E}\, d^3x = \int_{surface} -\boldsymbol{\kappa}\cdot\mathbf{E}\, da = \int_{surface} -\sigma\omega R E_\phi\, da = +L2\pi R^2\sigma\omega\left(\frac{R^2}{2}\mu_o\sigma\frac{d\omega}{dt}\right)$$

$$\int_{\text{all space}} -\mathbf{J}\cdot\mathbf{E}\, d^3x = \frac{d}{dt}\frac{1}{2}\mu_o L\pi R^4\sigma^2\omega^2 = \frac{dW_M}{dt}$$

(e) Calculate the flux of electromagnetic energy $\int_{surface}\left[\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right]\cdot\hat{\mathbf{r}}\, da$ through a cylindrical surface of radius *r* for *r* a little greater than *R* and also for *r* a little smaller than *R*. Do your results agree with what you expect from (c) and (d)?

$\int_{surface}\left[\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right]\cdot\hat{\mathbf{r}}\, da = 0$ for *r* a little greater than *R* since there is no magnetic field there. For *r* a little smaller than *R*, we have

$$\int_{surface}\left[\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right]\cdot\hat{\mathbf{r}}\, da = 2\pi RL\left\{\left[-\frac{\sigma}{\varepsilon_o}\hat{\mathbf{r}} - \frac{R^2}{2}\mu_o\sigma\frac{d\omega}{dt}\hat{\boldsymbol{\phi}}\right]\times R\sigma\omega\hat{\mathbf{z}}\right\}\cdot\hat{\mathbf{r}} = -\pi R^4 L\mu_o\sigma^2\omega\frac{d\omega}{dt} = -\frac{dW_M}{dt}$$

Where the negative sign indicates energy flow inward. This agrees with our results above.

(f) What is the total electromagnetic angular momentum $\int \mathbf{r}\times\left[\varepsilon_o\mathbf{E}\times\mathbf{B}\right]d^3x$ in terms of the parameters given? Neglect fringing fields.

$$\mathbf{r}\times\left[\varepsilon_o\mathbf{E}\times\mathbf{B}\right] = \mathbf{r}\times\begin{cases}\left[-\frac{R}{r}\sigma\hat{\mathbf{r}} - \frac{r}{2}\varepsilon_o\mu_o\sigma R\frac{d\omega}{dt}\hat{\boldsymbol{\phi}}\right]\times\mu_o\kappa\hat{\mathbf{z}} & r<R \\ 0 & r>R\end{cases}$$

$$\mathbf{r}\times\left[\varepsilon_o\mathbf{E}\times\mathbf{B}\right] = \mu_o\kappa\mathbf{r}\times\begin{cases}\left[\frac{R}{r}\sigma\hat{\boldsymbol{\phi}} - \frac{r}{2c^2}R\sigma\frac{d\omega}{dt}\hat{\mathbf{r}}\right] & r<R \\ 0 & r>R\end{cases}$$

$$\mathbf{r}\times\left[\varepsilon_o\mathbf{E}\times\mathbf{B}\right] = \begin{cases}\left[\frac{R^2}{r}\mu_o\sigma^2\omega\hat{\mathbf{z}}\right] & r<R \\ 0 & r>R\end{cases}$$

$$\mathbf{L}_{EM} = \int \mathbf{r}\times[\varepsilon_o \mathbf{E}\times\mathbf{B}]\,d^3x = \hat{\mathbf{z}}\int_0^L dz \int_0^{2\pi} d\phi \int_0^R r^2 dr\left(\frac{R^2}{r}\mu_o\sigma^2\omega\right) = \hat{\mathbf{z}}L2\pi\frac{1}{2}R^4\mu_o\sigma^2\omega = \hat{\mathbf{z}}L\pi R^4\mu_o\sigma^2\omega$$

(g) Show that the total rate at which electromagnetic angular momentum is being created as the cylinder is being spun up, $\int_{\text{all space}} -\mathbf{r}\times[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}]\,d^3x$, is equal to the rate at which the total electromagnetic angular momentum is increasing.

$$\int_{\text{all space}} -\mathbf{r}\times[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}]\,d^3x = \int_{surface} -\mathbf{r}\times[\sigma\mathbf{E}+\boldsymbol{\kappa}\times\mathbf{B}]\,da = \int_{surface} -\mathbf{r}\times\sigma\mathbf{E}\,da \quad \text{since } \boldsymbol{\kappa}\times\mathbf{B} \text{ is radial.}$$

$$\int_{surface} -\mathbf{r}\times\sigma\mathbf{E}\,da = -\hat{\mathbf{z}}\int_{surface} r\,\sigma E_\phi\,da = \hat{\mathbf{z}}2\pi RL\frac{R^2}{2}\mu_o\sigma^2\frac{d\omega}{dt} = \frac{d}{dt}\hat{\mathbf{z}}\pi R^4 L\mu_o\sigma^2\omega = \frac{d}{dt}\mathbf{L}_{EM}$$

(h) Calculate the flux of electromagnetic angular momentum, $\int_{surface}\left[-\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{r}}\right)\right]da$ through a cylinder of radius *r* for *r* a little greater than *R* and for *r* a little smaller than *R*. Do your results agree with what you expect from (f) and (g)? As in all stress tensor calculations, figure out what components you are going to need before you calculate anything, and then just calculate those.

$$\int_{surface}\left[-\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{r}}\right)\right]da = -\int_{surface}\left[\mathbf{r}\times\left(T_{rr}\hat{\mathbf{r}}+T_{r\phi}\hat{\boldsymbol{\phi}}+T_{rz}\hat{\mathbf{z}}\right)\right]da = -\hat{\mathbf{z}}\int_{surface} rT_{r\phi}\,da = -\hat{\mathbf{z}}\int_{surface} r\varepsilon_o E_r E_\phi\,da$$

For *r* a little greater than *R* the flux is zero because the radial component of the electric field is zero there. For *r* a little smaller than *R*,

$$\int_{surface}\left[-\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{r}}\right)\right]da = -\hat{\mathbf{z}}\int_{surface} r\varepsilon_o\left(-\frac{\sigma}{\varepsilon_o}\right)\left(-\frac{R^2}{2}\mu_o\sigma\frac{d\omega}{dt}\right)da = -\hat{\mathbf{z}}R\varepsilon_o\frac{\sigma}{\varepsilon_o}\frac{R^2}{2}\mu_o\sigma\frac{d\omega}{dt}2\pi RL$$

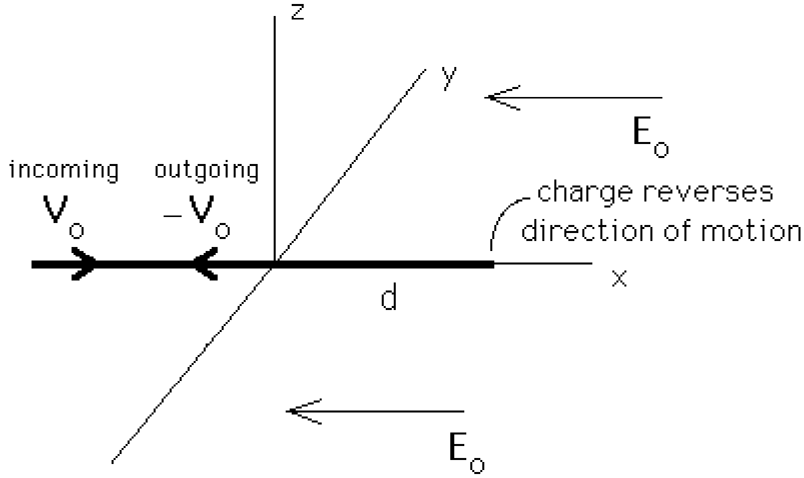
$$= -\frac{d}{dt}\hat{\mathbf{z}}\pi R^4 L\mu_o\sigma^2\omega = -\frac{d}{dt}\mathbf{L}_{EM}$$

The negative sign here is ok because this is the flux in the positive radial direction, the flux inward is just the opposite of that, which is what we want.

Problem 3 (30 points): Radiation by a relativistic charge

We have a static electric field $\mathbf{E}_O$ which is given by

$$\mathbf{E}_o = \begin{cases} -E_o\hat{\mathbf{x}} & x > 0 \\ 0 & x < 0 \end{cases}$$



A particle of mass m and charge q is initially located on the negative x-axis, with velocity $\mathbf{u}_o = \hat{\mathbf{x}}\,u_o$. *We do not assume that the speed* u_o *is small compared to c!*

(a) At $t = 0$, the particle crosses into the region $x > 0$, feels the repulsive force due to the electric field there, and begins to decelerate. It penetrates into a distance d along the positive x axis before its velocity reverses direction, and then it eventually exits the region $x > 0$, returning back down the x-axis at speed $-\mathbf{u}_o = -\hat{\mathbf{x}}\,u_o$. Find an expression for the time T that the particle spends in the region $x > 0$. Assume that the energy radiated is negligible, that is, ignore radiation reaction.

From (13.4.5) of the class notes, we have

$$\frac{d}{dt} m\gamma_u \mathbf{u}(t) = q\left(\mathbf{E} + \mathbf{u}\times\mathbf{B}\right) \Rightarrow \frac{d}{dt} m\gamma_u u(t) = -qE_o \Rightarrow m\gamma_u u(t) = -qE_o t + \frac{mu_o}{\sqrt{1-u_o^2/c^2}}$$

This will be zero at a time $T = \dfrac{m}{qE_o}\dfrac{u_o}{\sqrt{1-u_o^2/c^2}}$ so by symmetry the time spent in the region $x > 0$ will be $\dfrac{2m}{qE_o}\dfrac{u_o}{\sqrt{1-u_o^2/c^2}}$.

(b) What is the total energy radiated in this process? Give your answer in terms of q, m, E_O, u_O, and fundamental constants. The acceleration in the instantaneous rest frame of the charge in this case is particularly simple.

From class notes (14.4.8) we have

$$\frac{dW_{rad}}{dt'} = \frac{1}{4\pi\varepsilon_o}\frac{2q^2}{3c}\gamma^6\left[\left|\dot{\boldsymbol{\beta}}\right|^2 - \left|\boldsymbol{\beta}\ \mathbf{x}\ \dot{\boldsymbol{\beta}}\right|^2\right] = \frac{1}{4\pi\varepsilon_o}\frac{2q^2}{3c^3}a^2_{rest\ frame}$$

where $a^2_{rest\ frame} = c^2\gamma^6\left[\left|\dot{\boldsymbol{\beta}}\right|^2 - \left|\boldsymbol{\beta}\ \mathbf{x}\ \dot{\boldsymbol{\beta}}\right|^2\right] = c^2\gamma^6\left[\left|\dot{\boldsymbol{\beta}}\right|^2\right] = \left[\gamma^3 \frac{du}{dt}\right]^2$. The acceleration in the instantaneous rest frame is $a_{rest\ frame} = -\frac{eE_o}{m}$, since the electric field is the same in that frame as in the laboratory frame (see (11.3.8) of the class handouts). If you want you can show this directly, as below.

$$\frac{d}{dt}\left[\gamma_u u\right] = u\frac{d\gamma_u}{dt} + \gamma_u\frac{du}{dt} = u\frac{d}{dt}\left[1-(u/c)^2\right]^{-1/2} + \gamma_u\frac{du}{dt} = \frac{u^2}{c^2}\left[1-(u/c)^2\right]^{-3/2}\frac{du}{dt} + \gamma_u\frac{du}{dt}$$

$$= \frac{du}{dt}\left[\frac{u^2}{c^2}\gamma_u^3 + \gamma_u\right] = \frac{du}{dt}\gamma_u^3\left[\frac{u^2}{c^2} + \frac{1}{\gamma_u^2}\right] = \frac{du}{dt}\gamma_u^3\left[\frac{u^2}{c^2} + 1 - \frac{u^2}{c^2}\right] = \frac{du}{dt}\gamma_u^3 = -\frac{e}{m}E_o = a_{rest\ frame}$$

So the rate of radiation is constant throughout the time the charge is in the region $x > 0$. Thus the total energy radiated is

$$W_{rad} = \frac{1}{4\pi\varepsilon_o}\frac{2q^2}{3c^3}\frac{q^2E_o^2}{m^2}\frac{2m}{qE_o}\frac{u_o}{\sqrt{1-u_o^2/c^2}} = \frac{1}{4\pi\varepsilon_o}\frac{4q^3}{3c^3}\frac{E_o}{m}\frac{u_o}{\sqrt{1-u_o^2/c^2}}$$

(c) What is the distance d in terms of the given parameters (up to this point you did not have to have d)?

To find d, we need to find $x(t)$, which we can do by noting that

$$\gamma_u u(t) = \alpha \Rightarrow \frac{u(t)}{\sqrt{1-u^2/c^2}} = \alpha \Rightarrow u(t) = \frac{\alpha}{\sqrt{(\alpha/c)^2+1}}$$

$$\gamma_u u(t) = -\frac{eE_o}{m}t + \frac{u_o}{\sqrt{1-u_o^2/c^2}} \Rightarrow u(t) = \frac{-\frac{eE_o}{m}t + \frac{u_o}{\sqrt{1-u_o^2/c^2}}}{\sqrt{1+\frac{1}{c^2}\left[-\frac{eE_o}{m}t + \frac{u_o}{\sqrt{1-u_o^2/c^2}}\right]^2}}$$

We can integrate this to find $x(t)$, with the requirement that $x = 0$ at $t = 0$, to give

$$x(t) = \frac{c^2m}{qE_o}\left\{\frac{1}{\sqrt{1-u_o^2/c^2}} - \sqrt{1+\frac{1}{c^2}\left[-\frac{qE_o}{m}t + \frac{u_o}{\sqrt{1-u_o^2/c^2}}\right]^2}\right\}$$

We can find d by evaluating the above at the time the speed is zero, which gives

$$d = \frac{c^2m}{qE_o}\left\{\frac{1}{\sqrt{1-u_o^2/c^2}} - 1\right\}$$

(c) Let $r_c = \frac{1}{4\pi\varepsilon_o}\frac{q^2}{mc^2}$. What is the ratio of the total radiated energy from part (b) to the initial *kinetic energy* of the particle, $mc^2\left\{\frac{1}{\sqrt{1-u_o^2/c^2}}-1\right\}$? Write this ratio in terms of r_c, d, u_o and c alone. [Hint: your answer should have no dimensions--that is, it should only involve the ratios of lengths, speeds, etc.].

$$W_{rad} = \frac{1}{4\pi\varepsilon_o}\frac{4q^3}{3c^3}\frac{E_o}{m}\frac{u_o}{\sqrt{1-u_o^2/c^2}} = \frac{4c}{3q}4\pi\varepsilon_o\left[\frac{q^2}{4\pi\varepsilon_o mc^2}\right]^2\frac{mE_ou_o}{\sqrt{1-u_o^2/c^2}} = r_c^2\frac{4c}{3q}4\pi\varepsilon_o\frac{mE_ou_o}{\sqrt{1-u_o^2/c^2}}$$

$$\frac{W_{rad}}{mc^2\left\{\frac{1}{\sqrt{1-u_o^2/c^2}}-1\right\}} = r_c^2\frac{4c}{3q}\frac{4\pi\varepsilon_o}{mc^2\left\{\frac{1}{\sqrt{1-u_o^2/c^2}}-1\right\}}\frac{mE_ou_o}{\sqrt{1-u_o^2/c^2}}$$

But from above we have $\left\{\frac{1}{\sqrt{1-u_o^2/c^2}}-1\right\} = \frac{dqE_o}{c^2m}$

$$\frac{W_{rad}}{mc^2\left\{\frac{1}{\sqrt{1-u_o^2/c^2}}-1\right\}} = r_c^2\frac{4c}{3q}\frac{4\pi\varepsilon_o}{mc^2\frac{dqE_o}{c^2m}}\frac{mE_ou_o}{\sqrt{1-u_o^2/c^2}} = r_c^2\frac{4}{3d}\frac{4\pi\varepsilon_o mc^2}{q^2}\frac{u_o/c}{\sqrt{1-u_o^2/c^2}} = \frac{4}{3}\frac{r_c}{d}\frac{u_o/c}{\sqrt{1-u_o^2/c^2}}$$