8.07 Class Notes Fall 2011

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4 Conservation of energy and momentum in electromagnetism

4.1 Learning objectives

In Section 2 above I talked about the general form of conservation laws for scalar and vector quantities. I now turn to the question of the energy flow and momentum flow in electromagnetism. I introduce the Poynting flux vector and the Maxwell Stress Tensor.

4.2 Maxwell's Equations

Electric and magnetic fields are produced by charges and currents, and Maxwell's equations tell us how the fields are produced by the charge density ρ and current density $\mathbf{J}$. Maxwell's equations relating the fields to their sources are

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_o} \qquad (4.2.1)$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \qquad (4.2.2)$$

$$\nabla\times\mathbf{B}=\mu_o\mathbf{J}+\mu_o\varepsilon_o\frac{\partial\mathbf{E}}{\partial t} \qquad (4.2.3)$$

$$\nabla\cdot\mathbf{B}=0 \qquad (4.2.4)$$

4.3 Conservation of charge

If I take the divergence of (4.2.3) and use (4.2.1), I obtain the differential equation for the conservation of charge (that is, Maxwell's equations contain charge conservation)

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}=0 \qquad (4.3.1)$$

If I consider (4.3.1) in light of our treatment of conservation laws for a scalar quantity in Section 2.2 above, I see that $\mathbf{J}$ is the flux density of charge and that the volume creation rate for charge is zero, that is, electric charge is neither created nor destroyed.

4.4 Conservation of energy

If I use a vector identity in the first step below, and then use Maxwell's equations (4.2.2) and (4.2.3) in the second step, I easily have

$$\begin{aligned}\nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right)&=\frac{1}{\mu_o}\left(\mathbf{B}\cdot\nabla\times\mathbf{E}-\mathbf{E}\cdot\nabla\times\mathbf{B}\right)=\frac{1}{\mu_o}\left(\mathbf{B}\cdot\left(-\frac{\partial\mathbf{B}}{\partial t}\right)-\mathbf{E}\cdot\left(\mu_o\mathbf{J}+\mu_o\varepsilon_o\frac{\partial\mathbf{E}}{\partial t}\right)\right)\\&=\frac{1}{2\mu_o}\frac{\partial\mathbf{B}\cdot\mathbf{B}}{\partial t}-\frac{\varepsilon_o}{2}\frac{\partial\mathbf{E}\cdot\mathbf{E}}{\partial t}-\mathbf{J}\cdot\mathbf{E}\end{aligned} \qquad (4.4.1)$$

This can be re-written as

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon_o E^2 + \frac{B^2}{2\mu_o}\right] + \nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right) = -\mathbf{E}\cdot\mathbf{J} \qquad (4.4.2)$$

From the general form of the conservations laws I considered above, I see that I can interpret $\frac{1}{2}\varepsilon_o E^2 + \frac{B^2}{2\mu_o}$ as the energy density of the electromagnetic field (joules per cubic meter), $\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}$ as the flux density of electromagnetic energy (joules per square meter per second), and $-\mathbf{E}\cdot\mathbf{J}$ as the volume creation rate of electromagnetic energy (joules per cubic meter per second).

4.5 Conservation of momentum and angular momentum

I define the Maxwell Stress Tensor $\overleftrightarrow{\mathbf{T}}$ as

$$\overleftrightarrow{\mathbf{T}} = \varepsilon_o\left[\mathbf{EE} - \frac{1}{2}\overleftrightarrow{\mathbf{I}}\,E^2\right] + \frac{1}{\mu_o}\left[\mathbf{BB} - \frac{1}{2}\overleftrightarrow{\mathbf{I}}\,B^2\right] \qquad (4.5.1)$$

In Problem Set 1, you proved the vector identity

$$\nabla\cdot\left[\mathbf{AA} - \frac{1}{2}\overleftrightarrow{\mathbf{I}}\,A^2\right] = \mathbf{A}(\nabla\cdot\mathbf{A}) + (\nabla\times\mathbf{A})\times\mathbf{A} \qquad (4.5.2)$$

Using this identity and Maxwell's equations, I have

$$\begin{aligned}
\nabla\cdot\overleftrightarrow{\mathbf{T}} &= \varepsilon_o\nabla\cdot\left[\mathbf{EE} - \frac{1}{2}\overleftrightarrow{\mathbf{I}}\,E^2\right] + \frac{1}{\mu_o}\nabla\cdot\left[\mathbf{BB} - \frac{1}{2}\overleftrightarrow{\mathbf{I}}\,B^2\right] \\
&= \varepsilon_o\mathbf{E}\nabla\cdot\mathbf{E} + \varepsilon_o(\nabla\times\mathbf{E})\times\mathbf{E} + \frac{1}{\mu_o}\mathbf{B}\nabla\cdot\mathbf{B} + \frac{1}{\mu_o}(\nabla\times\mathbf{B})\times\mathbf{B} \qquad (4.5.3) \\
&= \mathbf{E}\rho + \varepsilon_o\left(-\frac{\partial\mathbf{B}}{\partial t}\right)\times\mathbf{E} + \mathbf{J}\times\mathbf{B} + \varepsilon_o\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} + \varepsilon_o\frac{\partial(\mathbf{E}\times\mathbf{B})}{\partial t}
\end{aligned}$$

Rearranging terms gives me the following equation

$$\frac{\partial}{\partial t}[\varepsilon_o\mathbf{E}\times\mathbf{B}] + \nabla\cdot\left(-\overleftrightarrow{\mathbf{T}}\right) = -[\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}] \qquad (4.5.4)$$

Given my general form of a conservation law for a vector quantity, equation (2.3.3), I identify $\varepsilon_o \mathbf{E}\times\mathbf{B}$ as electromagnetic momentum density, $-\overleftrightarrow{\mathbf{T}}$ as the flux density of electromagnetic momentum, and $-\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right]$ as the volume creation rate of electromagnetic momentum. Let me show that these three quantities have the appropriate units, e.g. momentum per cubic meter, momentum per square meter per sec, and momentum per cubic meter per sec, respectively.

The relation between the units of E and the units of B is $E = B\,L/T$ (I know this because the force on a charge is $q\left(\mathbf{E}+\mathbf{V}\times\mathbf{B}\right)$). Therefore the units of $\varepsilon_o EB$ are the units of $\varepsilon_o E^2 T/L$, and since $\varepsilon_o E^2$ is an energy density, it has units of $energy/L^3$. But *energy* has units of *force* times *distance*, and *force* has units of *momentum/T*, so *energy density* has units of *momentum /TL*^{2}. So the units of $\varepsilon_o EB$ are *momentum/L*^{3}, as desired. The tensor $\overleftrightarrow{\mathbf{T}}$ has units of energy density, and energy density has units of *momentum /TL*^{2}, or momentum flux density. The units of $-\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right]$ are force per unit volume, and since *force* is *momentum over time*, this has units of *momentum per cubic meter per sec*, as I expect for a creation rate for electromagnetic momentum density.

If I look at (4.5.4) in integral form, I see that for any volume V contained by a closed surface S, I have

$$\frac{d}{dt}\int_V \left[\varepsilon_o \mathbf{E}\times\mathbf{B}\right] d^3x + \int_S \left(-\overleftrightarrow{\mathbf{T}}\right)\cdot\hat{\mathbf{n}}\, da = -\int_V \left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right] d^3x \qquad (4.5.5)$$

The corresponding equations for angular momentum are

$$\frac{\partial}{\partial t}\mathbf{r}\times\left[\varepsilon_o \mathbf{E}\times\mathbf{B}\right] + \nabla\cdot\left(-\mathbf{r}\times\overleftrightarrow{\mathbf{T}}\right) = -\mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right] \qquad (4.5.6)$$

and

$$\frac{d}{dt}\int_V \mathbf{r}\times\left[\varepsilon_o \mathbf{E}\times\mathbf{B}\right] d^3x + \int_S \left(-\mathbf{r}\times\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\right) da = -\int_V \mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right] d^3x \qquad (4.5.7)$$

4.5.1 The Maxwell stress tensor in statics

To get some idea of the properties of the Maxwell stress tensor, I first look at it in cases where this is no time dependence, that is in electrostatics and magnetostatics. In this case, (4.5.5) can be written as

$$\int_V \left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right] d^3x = \int_S \overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\, da \qquad (4.5.8)$$

The term on the left above is the volume force density in electromagnetism, integrated over the volume, so it is the total electromagnetic force on all the charges and currents inside the volume *V*. What (4.5.8) tells me is that I can compute this force in two different ways. First I can do it the obvious way, by sampling the volume, looking at the charge on each little volume element, and adding that up to get the total force. The right hand side says that I can do this calculation a totally different way. I do not ever have to go inside the volume and look at the individual charges and currents and the fields at the location of those charges and currents. Rather I can simply move around on the surface of the sphere containing the charges and currents, and simply look at the fields on the surface of that sphere, calculating $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\,da$ at each little area element, and noting that this depends only on the fields at that area element. Isn't that amazing?

Well actually it is not so amazing, because it is exactly what I should expect for any decent field theory. Remember the fields are the agents which transmit forces between material objects and I should be able to look at the fields themselves and figure out what kind of stresses they are transmitting. As an analogy to illustrate this point, consider the theory of pegboards interacting via connecting strings, as illustrated in Figure 4-1. The right pegboard exerts a net force on the left pegboard because the strings connecting the pegs carry tension. I can calculate the force on the left pegboard in two ways. I can move around in the interior of the pegboard, find each peg, and the strings attached to it, and add up the total force for that peg, and then move on to the other pegs and thus compute the total force on the left pegboard. This process is analogous to doing the volume integral of the electromagnetic force density in (4.5.8).

Or, I surround the left pegboard by an imaginary sphere, as shown in Figure 4-2, and simply walk around on the surface of that sphere, never looking inside the volume it contains. Whenever I see a string piercing the surface of the sphere, I know that that sting is transmitting a force across the imaginary surface and I can measure the direction and magnitude of that force. I explore the entire surface, add up the tension due to all the strings, and then have the total force on the left pegboard. This is analogous to doing the surface integral on the right side of (4.5.8)

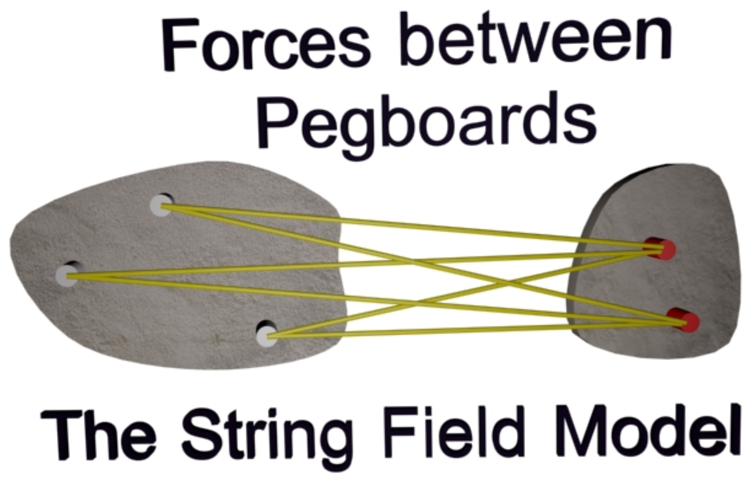


Figure 4-1: Two pegboards interacting through strings attached to the pegs

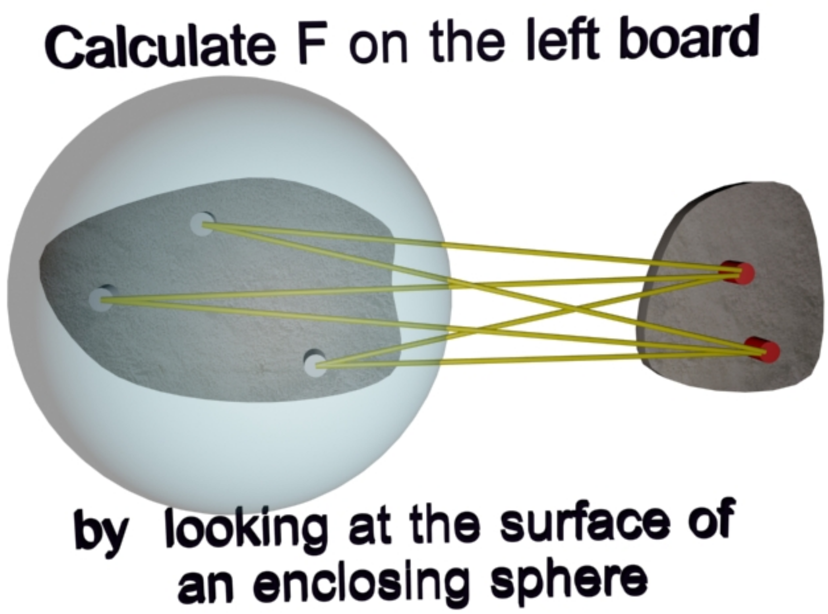


Figure 4-2: Enclosing the left pegboard by a sphere and exploring its surface.

4.5.2 Calculating $\overset{\leftrightarrow}{\mathbf{T}}\cdot\hat{\mathbf{n}}\,da$

So this sounds neat, let me see what is actually involved in calculating $\overset{\leftrightarrow}{\mathbf{T}}\cdot\hat{\mathbf{n}}\,da$ in electrostatics, for example. Figure 4-3 shows a surface element and the local electric field. Unless $\mathbf{E}$ and $\hat{\mathbf{n}}$ are co-linear, they determine a plane. Let the x axis in that plane be along the $\hat{\mathbf{n}}$ direction, and the *y* axis be perpendicular to the $\hat{\mathbf{n}}$ direction and in the **E**-$\hat{\mathbf{n}}$ plane. Let **E** make an angle θ with $\hat{\mathbf{n}}$.

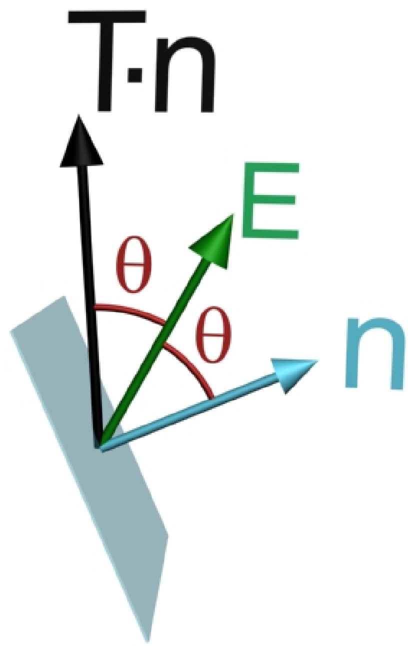


Figure 4-3: The relation between the directions of E, n, and $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$

In this coordinate system, I thus have the components of **E** as

$$\mathbf{E} = E\cos\theta\,\hat{\mathbf{x}} + E\sin\theta\,\hat{\mathbf{y}} \tag{4.5.9}$$

and $\hat{\mathbf{n}} = \hat{\mathbf{x}}$. If I look at the definition of $\overleftrightarrow{\mathbf{T}}$, and the definition of $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$, I see that $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$ is a vector with components

$$\left(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\right)_i = T_{ij}n_j = T_{ix} \tag{4.5.10}$$

and therefore

$$\left(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\right) = T_{xx}\hat{\mathbf{x}} + T_{yx}\hat{\mathbf{y}} + T_{yx}\hat{\mathbf{z}} = \varepsilon_o\left(E_xE_x - \frac{1}{2}E^2\right)\hat{\mathbf{x}} + \varepsilon_o E_xE_y\hat{\mathbf{y}} \tag{4.5.11}$$

If I insert the values of the components of **E** into (4.5.11), and use some trig, I have

$$\begin{aligned}\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}} &= \varepsilon_o E^2\left(\cos^2\theta - \frac{1}{2}\right)\hat{\mathbf{x}} + \varepsilon_o E^2\sin\theta\cos\theta\hat{\mathbf{y}} \\ &= \frac{1}{2}\varepsilon_o E^2\left(\cos 2\theta\,\hat{\mathbf{x}} + \sin 2\theta\,\hat{\mathbf{y}}\right)\end{aligned} \tag{4.5.12}$$

So that I can conclude the following about $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$:

1. $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$ lies in the plane defined by **E** and $\hat{\mathbf{n}}$.
2. If I go an angle θ to get to **E** from $\hat{\mathbf{n}}$, I have to go an angle 2θ to get to $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$, in the same sense.
3. The magnitude of $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$ is always $\frac{1}{2}\varepsilon_o E^2$

Since $\overleftrightarrow{\mathbf{T}}$ is has the same form for **E** and **B**, the only change in these rules for **B** is that the magnitude of $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$ is $B^2/2\mu_o$

Let me take some particular configurations of **E** and $\hat{\mathbf{n}}$ and see what these rules tell me. Figure 4-4 shows various configurations. From studying this figure, I conclude that if **E** and $\hat{\mathbf{n}}$ are parallel or anti-parallel, the **E** field transmits a *pull* across the surface ($\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$ is along $\hat{\mathbf{n}}$ and thus out of the volume of interest). If **E** and $\hat{\mathbf{n}}$ are perpendicular, the **E** field transmits a *push* across the surface ($\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$ is opposite $\hat{\mathbf{n}}$ and thus into the volume of interest). For other orientations $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$ is a combination of a push or a pull and a shear force.

Note that there is no force exerted on *da*. I am just evaluating the force transmitted by the field across *da*, just as I was looking at the force transmitted across *da* in our pegboard string field model above.

I conclude this discussion with an actual calculation for two charges of the same sign. Two charges each with charge +*q* are located a distance *2d* apart, one along the positive *x*-axis a distance *d* from the origin, and the other along the negative *x*-axis also a distance *d* from the origin. The charges are glued in place (that is there is a mechanical force on each that keeps them from moving under the Coulomb repulsion). I enclose the charge on the negative *x*-axis inside a cube of side *H*, with *H* going to infinity. One of the faces of the cube lies in the *z-y* plane at *x = 0*, as shown in Figure 4-5.

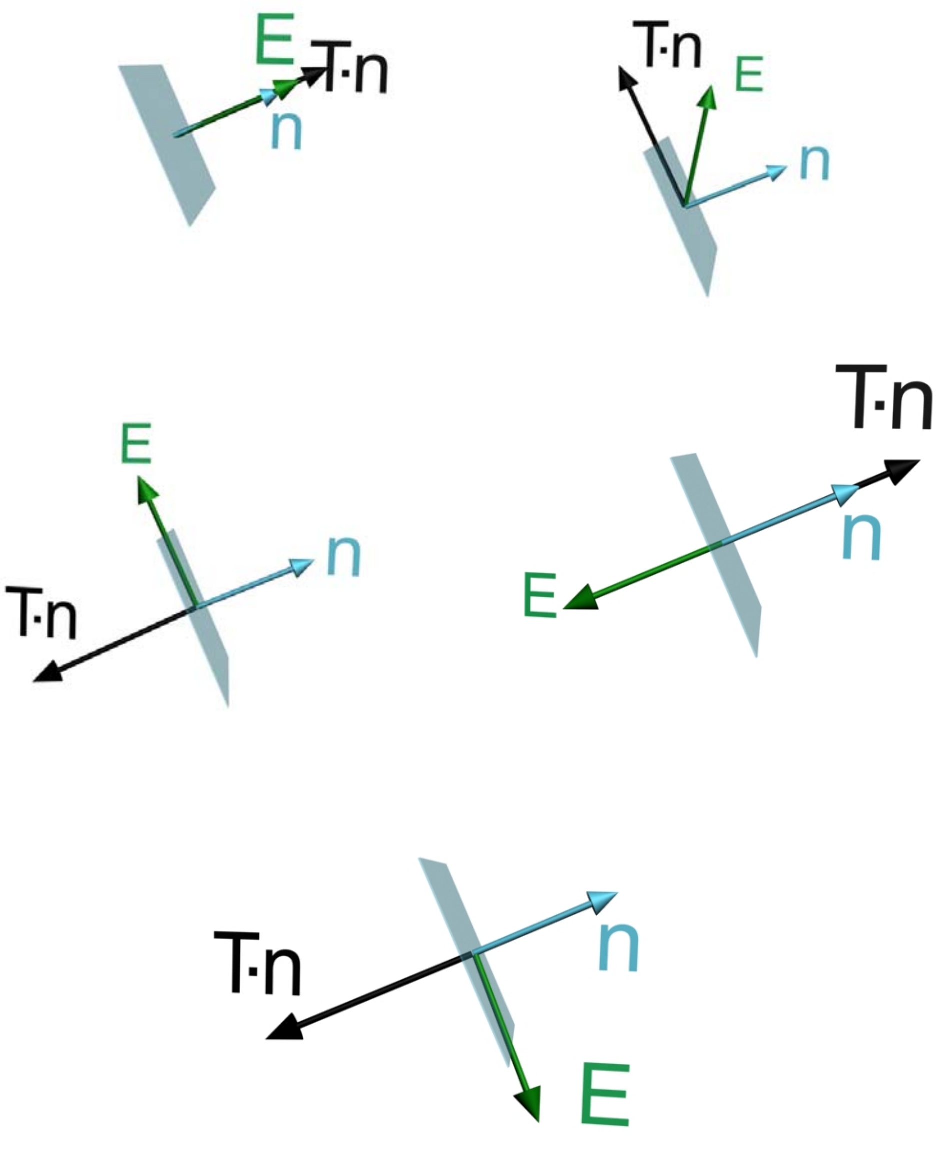


Figure 4-4: Orientation of $\overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{n}}$ for various orientations of $\hat{\mathbf{n}}$ and E.

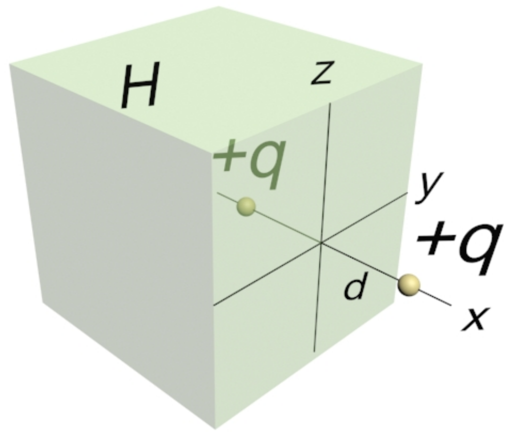
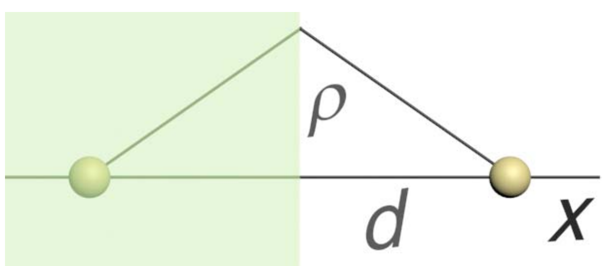


Figure 4-5: An electrostatics problem

If I look at the field configuration on the right *zy* face of the cube, it is perpendicular to the normal to that face, and therefore the electrostatic force is exerting a *push* across the cube's surface (that is, a force in the negative x direction). I can calculate that push by integrating $\frac{1}{2}\varepsilon_o E^2$ over that surface, using the solution to this simple electrostatics problem, and I show that explicit calculation below. I find that the force is a force of repulsion of magnitude $\dfrac{q^2}{4\pi\varepsilon_o\left(2d\right)^2}$, as I expect. The integral of the stress tensor over the other faces goes to zero as *H* goes to infinity.



Here is the explicit calculation of the force. If I look at any face of the cube except the *yz* face at $x = 0$, the stress tensor on that face (which goes as the square of the electric field) will fall off as one over distance to the fourth power and the area will only grow as distance squared, so I will get no contribution from these faces as H goes to infinity. I need only need to evaluate the stress tensor at $x = 0$, so I only need the electric field in the *yz* plane at $x = 0$. This electric field is always perpendicular to the x-axis in this plane, and has magnitude

$$2\frac{q}{4\pi\varepsilon_o}\frac{1}{d^2+\rho^2}\frac{\rho}{\sqrt{d^2+\rho^2}} \tag{4.5.13}$$

where ρ is the perpendicular distance from the *x*-axis to a given point in the *yz* plane. In our stress tensor calculations I need to calculate the following integral:

$$\int_{yz\ plane}\frac{1}{2}\varepsilon_o E^2 da = \frac{\varepsilon_o}{2}\int_0^\infty \rho d\rho \int_0^{2\pi} d\phi E^2 = \pi\varepsilon_o \int_0^\infty \rho d\rho E^2$$

$$= \pi\varepsilon_o \int_0^\infty \rho d\rho \left[\frac{q}{2\pi\varepsilon_o}\frac{\rho}{\left(d^2+\rho^2\right)^{3/2}}\right]^2 = \frac{q^2}{4\pi\varepsilon_o}\int_0^\infty \frac{\rho^3 d\rho}{\left(d^2+\rho^2\right)^3} =$$

$$\int_0^\infty \frac{\rho^3 d\rho}{\left(d^2+\rho^2\right)^3} = \int_0^\infty \frac{\rho d\rho}{\left(d^2+\rho^2\right)^3}\rho^2 = \text{by parts} = \frac{1}{2}\int_0^\infty \frac{\rho d\rho}{\left(d^2+\rho^2\right)^2} = -\frac{1}{4\left(d^2+\rho^2\right)}\Bigg|_0^\infty = \frac{1}{4d^2} \tag{4.5.14}$$

So

$$\int_{yz\ plane}\frac{1}{2}\varepsilon_o E^2 da = \frac{q^2}{4\pi\varepsilon_o}\frac{1}{\left(2d\right)^2}$$

From the properties of the stress tensor, I see that on the *yz* face at $x = 0$, since the electric field is perpendicular to the local normal, $\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$ is a push on the cube, and a push is the negative *x* direction. Therefore

$$\int_{cube\ surface}\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\,da = -\hat{\mathbf{x}}\int_{yz\ plane}\frac{1}{2}\varepsilon_o E^2 da = -\hat{\mathbf{x}}\frac{q^2}{4\pi\varepsilon_o}\frac{1}{\left(2d\right)^2}$$

Therefore the total electromagnetic momentum flux out of the cube surrounding the left charge is

$$\int_{cube\ surface}\left(-\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\right)da = +\hat{\mathbf{x}}\frac{q^2}{4\pi\varepsilon_o}\frac{1}{\left(2d\right)^2}$$

You may well ask how there can be a flux of electromagnetic momentum when the density of electromagnetic momentum ($\varepsilon_o\mathbf{E}\times\mathbf{B}$) is zero. Consider the example of a current carrying wire. The positive ions are at rest and the electrons are moving. So there is no net electric charge, but there ***is*** a flux of electric charge (a current).

If we ask about the total rate at which electromagnetic momentum is being created in the cube, it is

$$\int_{cube\ volume} \left(-\rho\mathbf{E}\right) d^3x = +\hat{\mathbf{x}}\frac{q^2}{4\pi\varepsilon_o}\frac{1}{\left(2d\right)^2} \tag{4.5.15}$$

and thus I see from the last two equations that electromagnetic momentum is being created at exactly the rate at which it is leaving the cube, as I expect, since this is a static situation.

Physically what is happening is the following. Some external agent (the glue holding the left charge down) is applying a force that exactly balances the electrostatic force of repulsion on the positive charge on the negative x-axis, that is there is a mechanical force in the $+\hat{\mathbf{x}}\frac{q^2}{4\pi\varepsilon_o}\frac{1}{\left(2d\right)^2}$. A force is a momentum per time, and thus the external agent is creating momentum at this rate. But the charge is not moving because the electrostatic force just balances this mechanical force. Since I cannot locally put the momentum into the charge, what happens is that the mechanical force is creating electromagnetic momentum, which then flows away from the charge on the left out of the volume containing it, and ultimately to the charge on the right. There it is absorbed by the mechanical force on the charge on the right, which is a sink of momentum in the +x direction.

5 The Helmholtz Theorem

5.1 Learning Objectives

We continue on with the theory of vector fields, particularly as applied to fluids. In this handout, the most important thing that we learn is that under certain assumptions about behavior at infinity, a vector field is specified uniquely by its divergence and curl. We then consider various flow fields derived from specific sources, to get some familiarity with vector fields given their divergence and curl.

5.2 The Helmholtz Theorem

If we specify both the curl $\mathbf{c}(\mathbf{r},t)$ and the divergence $s(\mathbf{r},t)$ of a vector function $\mathbf{F}(\mathbf{r},t)$, and postulate that these functions fall off at least as fast as $1/r^2$ at infinity, and that $\mathbf{F}(\mathbf{r},t)$ itself goes to zero at infinity, then that function $\mathbf{F}(\mathbf{r})$ is uniquely determined. This is known as the Helmholtz Theorem. To prove this theorem, we first construct a function that has the given divergence and curl, and then we show that this function is unique.

5.2.1 Construction of the vector function F

We construct the following function using our given curl and divergence functions.

$$\mathbf{F}(\mathbf{r},t) = -\nabla \frac{1}{4\pi} \int\limits_{all\,space} \frac{s(\mathbf{r}',t)\, d^3x'}{\left|\mathbf{r}-\mathbf{r}'\right|} + \nabla\times \frac{1}{4\pi} \int\limits_{all\,space} \frac{\mathbf{c}(\mathbf{r}',t)\, d^3x'}{\left|\mathbf{r}-\mathbf{r}'\right|} \tag{5.2.1}$$

The divergence of this function is given by

$$\nabla\cdot\mathbf{F}(\mathbf{r},t) = -\nabla^2 \frac{1}{4\pi} \int\limits_{all\,space} \frac{s(\mathbf{r}',t)\, d^3x'}{\left|\mathbf{r}-\mathbf{r}'\right|} + \frac{1}{4\pi}\nabla\cdot\left[\nabla\times \int\limits_{all\,space} \frac{\mathbf{c}(\mathbf{r}',t)\, d^3x'}{\left|\mathbf{r}-\mathbf{r}'\right|}\right] \tag{5.2.2}$$

The divergence of the curl of any function is zero, and we can switch the order of integration and differentiation in (5.2.2) and use (3.2.4), giving

$$\nabla\cdot\mathbf{F}(\mathbf{r},t) = -\frac{1}{4\pi} \int\limits_{all\,space} s(\mathbf{r}',t)\nabla^2 \frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} d^3x' = \int\limits_{all\,space} s(\mathbf{r}',t)\delta^3(\mathbf{r}-\mathbf{r}')d^3x' = s(\mathbf{r},t) \tag{5.2.3}$$

Thus the vector we have constructed with (5.2.1) has the divergence that we desire. In a similar fashion we can show that the curl of this vector field (5.2.1) is $\mathbf{c}(\mathbf{r},t)$. It can be shown that our construction in (5.2.1) is unique as long as the divergence and curl fall off at least as fast as $1/r^2$ at infinity.

5.2.2 The inverse of the Helmholtz Theorem

The theorem above has the two important corollaries. If we have a function $\mathbf{F}(\mathbf{r},t)$ which falls off faster than $1/r$ at infinity, then

$$\nabla\times\mathbf{F} = 0 \quad\Rightarrow\quad \mathbf{F} = -\nabla\phi(\mathbf{r},t) \tag{5.2.4}$$

or

$$\nabla\cdot\mathbf{F} = 0 \quad\Rightarrow\quad \mathbf{F} = \nabla\times\mathbf{A}(\mathbf{r},t) \tag{5.2.5}$$

5.2.3 The Helmholtz Theorem in two dimensions

For future reference, we note that if the vector function **F** is entirely within the *x*-*y* plane and does not depend on *z*, that is

$$\mathbf{F}(\mathbf{r},t) = F_x(x,y,t)\,\hat{\mathbf{x}} + F_y(x,y,t)\,\hat{\mathbf{y}} \tag{5.2.6}$$

then (5.2.1) becomes

$$\mathbf{F}(x,y,t) = -\nabla \frac{1}{2\pi} \int_{xy\,plane} s(x',y',t) \ln\left[\sqrt{(x-x')^2 + (y-y')^2}\right] dx'dy' \\ + \nabla \times \frac{1}{2\pi} \int_{xy\,plane} \mathbf{c}(x',y',t) \ln\left[\sqrt{(x-x')^2 + (y-y')^2}\right] dx'dy' \tag{5.2.7}$$

This can be shown to be the correct construction of F (that is, that it has the proper divergence and curl) by using the two-dimensional version of (3.1.8),

$$\delta(x)\delta(y) = -\frac{1}{2\pi}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] \ln\sqrt{x^2 + y^2} \tag{5.2.8}$$

5.3 Examples of incompressible fluid flows

We consider here some incompressible fluid flow examples where the divergence and curl of the flow is given, and we proceed to construct the flow itself using the Helmholtz construction in (5.2.1). Incompressible means that the density $\rho_{mass}(\mathbf{r},t)$ is constant in time and space at ρ^o_{mass}, so that for incompressible flows the flow velocity **v** satisfies (see equation (2.2.6))

$$\nabla \cdot \mathbf{v} = \frac{s}{\rho^o_{mass}} \tag{5.3.1}$$

5.3.1 Irrotational flows

Let us look at some examples of irrotational flows. An irrotational flow is a flow which satisfies

$$\nabla \times \mathbf{v} = 0 \tag{5.3.2}$$

As a first example, consider an irrotational flow whose source function *s* is given by

$$s(\mathbf{r},t) = s_o \delta^3(\mathbf{r}) = s_o \delta(x)\delta(y)\delta(z) \tag{5.3.3}$$

Since the overall dimensions of the creation rate $s(\mathbf{r},t)$ must be mass per unit time per unit volume, and $\delta^3(\mathbf{r})$ has the dimensions of inverse volume, the dimensions of s_o must be mass per time. Equation (5.2.1) and (5.3.1) tell us that the vector field **v** is given by

$$\mathbf{v}(\mathbf{r}) = -\nabla \frac{1}{4\pi} \int_{all\,space} \frac{s_0 \delta^3(\mathbf{r}')\, d^3x'}{\rho^o_{mass} \left|\mathbf{r}-\mathbf{r}'\right|} = -\nabla \frac{s_o}{4\pi\rho^o_{mass} r} = \frac{s_o \hat{\mathbf{r}}}{4\pi\rho^o_{mass} r^2} \tag{5.3.4}$$

This is just a radial outflow from the source at the origin, with the flow velocity decreasing in magnitude as inverse distance squared. Note that the total rate at which mass is flowing out through a sphere of radius R (the flux of mass through the sphere) is given by

$$\int_S \left[\rho_{mass}\mathbf{v}\right]\cdot\hat{\mathbf{n}}\,da = \int_S \rho^o_{mass}\left(\frac{s_o\hat{\mathbf{r}}}{4\pi\rho^o_{mass}r^2}\right)\cdot\hat{\mathbf{n}}\,da = s_o \tag{5.3.5}$$

This is what we expect to see because the rate at which matter is being created within the sphere is given by

$$\int_V s\,d^3x = \int_V s_o\delta^3(\mathbf{r})\,d^3x = s_o \tag{5.3.6}$$

and so s_o should be the rate at which we see matter flowing out through the surface of the sphere.

As a second example consider the source function for an irrotational flow given by

$$s(\mathbf{r},t) = s_1\delta^3(\mathbf{r}) - s_2\delta^3(\mathbf{r}-L\,\hat{\mathbf{z}}) \tag{5.3.7}$$

This represents a source of fluid at the origin of strength s_1 along with a sink of fluid of strength s_2 at $L\,\hat{\mathbf{z}}$. Using (5.2.1), we can easily find that in this case

$$\mathbf{v}(\mathbf{r}) = \left[-\nabla\frac{s_1}{4\pi\rho^o_{mass}r} + \nabla\frac{s_2}{4\pi\rho^o_{mass}\left|\mathbf{r}-L\,\hat{\mathbf{z}}\right|}\right] = \left[\frac{s_1\hat{\mathbf{r}}}{4\pi\rho^o_{mass}r^2} - \frac{s_2\left(\mathbf{r}-L\,\hat{\mathbf{z}}\right)}{4\pi\rho^o_{mass}\left|\mathbf{r}-L\,\hat{\mathbf{z}}\right|^3}\right] \tag{5.3.8}$$

Note that the total rate at which mass is flowing out through a sphere (the flux of mass through the sphere) of very large radius $R >> d$ is given by

$$\int_S \left[\rho_{mass}\mathbf{v}\right]\cdot\hat{\mathbf{n}}\,da = s_1 - s_2 \tag{5.3.9}$$

This is what we expect to see because the rate at which matter is being created within this sphere is given by

$$\int_V s\,d^3x = s_1 - s_2 \tag{5.3.10}$$

If $s_1 = 5s_2$, then the velocity field topologically has the same form as the field plotted in Figure 1-1 and Figure 1-2 above.

5.3.2 Flows with rotation

Finally let us consider two flows where the curl of the flow is not zero. The first is a "sourceless" flow, that is $\nabla \cdot \mathbf{v} = 0$, but one for which the curl is given by

$$\nabla \times \mathbf{v} = \left[\delta(x)\delta(y-100) - 2\delta(x)\delta(y+140) \right] \hat{\mathbf{z}} \tag{5.3.11}$$

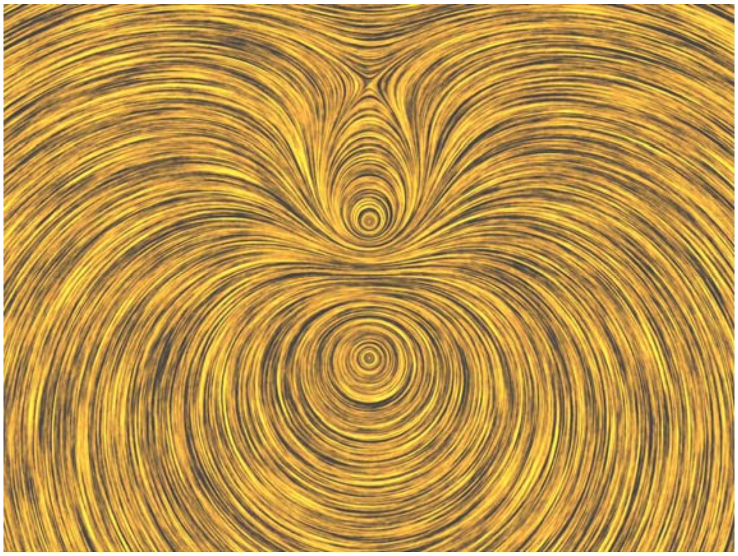


Figure 5-1: A flow with no source but with rotation

To see a movie of this kind of sourceless flow, follow the link below

http://web.mit.edu/viz/EM/visualizations/vectorfields/FluidFlows/FluidFlowCurlCurl02/ffcurlcurl02.htm

The second example of a flow with rotation is a flow with the same curl as (5.3.11) but now instead of a zero divergence, a divergence given by

$$\nabla \cdot \mathbf{v} = \delta(x-250)\delta(y) \tag{5.3.12}$$

Figure 5-2: A flow with a source and rotation

There are a number of movies of different kinds of flows at the link below

http://web.mit.edu/viz/EM/visualizations/vectorfields/FluidFlows/

6 The Solution to the Easy E&M

6.1 Learning Objectives

In this handout, I dive into electromagnetic theory and write out explicitly the solutions to the easy electromagnetism. By easy electromagnetism I refer to the situation where the sources of electromagnetic fields are known for all space and time. The sources of electromagnetic field are charges and currents (*"currents"* are moving charges). *"Known for all space and time"* means that someone gives us those functions, and your task is to deduce the electric and magnetic fields that these sources produce. This task is "easy" in that I can immediately write down a solution for **E** and **B** which can be solved with a straightforward and perfectly well defined algorithmic procedure.

6.2 The Easy Electromagnetism

6.2.1 The Solution to Maxwell's Equations

To solve the equations given in Section 4.2 for **E** and **B** given ρ and $\mathbf{J}$, I first introduce the vector potential **A**. Because the divergence of **B** is zero, we know from the Helmhotz theorem that **B** can be written as the curl of a vector, the vector potential **A**.

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{6.1.1}$$

If we insert (6.1.1) into (4.2.2), we have that

$$\nabla \times \left[\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right] = 0 \tag{6.1.2}$$

Since the curl of the vector $\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}$ is zero, we can write it as the gradient of a scalar function, which we will denote by ϕ, so that

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \tag{6.1.3}$$

If we insert (6.1.1) and (6.1.2) into (4.2.3), we have

$$\nabla \times \nabla \times \mathbf{A} = \mu_o \mathbf{J} + \mu_o \varepsilon_o \frac{\partial}{\partial t} \left[-\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \right] \tag{6.1.4}$$

or

$$\nabla \left(\nabla \cdot \mathbf{A} \right) - \nabla^2 \mathbf{A} = \mu_o \mathbf{J} - \mu_o \varepsilon_o \nabla \frac{\partial \phi}{\partial t} - \mu_o \varepsilon_o \frac{\partial^2}{\partial t^2} \mathbf{A} \tag{6.1.5}$$

If we let

$$c^2 = \frac{1}{\mu_o \varepsilon_o} \tag{6.1.6}$$

then

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \mathbf{A} + \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = -\mu_o \mathbf{J} \tag{6.1.7}$$

We still have the freedom to specify the divergence of **A**, since up to this point we have only specified its curl, and we choose the divergence of **A** so that it satisfies

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0 \tag{6.1.8}$$

This is knownas the Lorentz gauge condition, although it should be more properly called the Lorenz gauge condition (see Jackson (2008)).

With (6.1.8), (6.1.7) becomes

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A} = -\mu_o \mathbf{J} \tag{6.1.9}$$

And we can easily show by inserting (6.1.3) into (4.2.1) and using (6.1.8) that

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi = -\frac{\rho}{\varepsilon_o} \tag{6.1.10}$$

6.2.2 The free space-time dependent Green's function

To solve (6.1.9) and (6.1.10), we first need to solve the equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]G(\mathbf{r},t) = \delta^3(\mathbf{r})\delta(t) \tag{6.2.1}$$

The function $G(\mathbf{r},t)$ is the response of the system to a point disturbance in space and time. Once we know this we can write down a general solution for sources distributed in space and time. The solution to (6.2.1) is

$$G(\mathbf{r},t) = -\frac{1}{4\pi}\frac{\delta(t - r/c)}{r} \tag{6.2.2}$$

which can be verified by direct substitution, as follows. First, we have

$$\nabla\left[\frac{\delta(t-\frac{r}{c})}{r}\right] = \delta(t-\frac{r}{c})\nabla\frac{1}{r} + \frac{1}{r}\nabla\delta(t-\frac{r}{c}) = -\delta(t-\frac{r}{c})\frac{\hat{\mathbf{r}}}{r^2} - \frac{\hat{\mathbf{r}}}{cr}\delta'(t-\frac{r}{c}) \tag{6.2.3}$$

Dotting ∇ into (6.2.3) gives

$$\nabla^2\left[\frac{\delta(t-\frac{r}{c})}{r}\right]=\nabla\cdot\nabla\left[\frac{\delta(t-\frac{r}{c})}{r}\right]$$
$$=-\nabla\cdot\left[\delta(t-\frac{r}{c})\frac{\hat{\mathbf{r}}}{r^2}+\frac{\hat{\mathbf{r}}}{cr}\delta'(t-\frac{r}{c})\right]$$
$$=-\frac{1}{r^2}\frac{\partial}{\partial r}\left[\delta(t-\frac{r}{c})+\frac{r}{c}\delta'(t-\frac{r}{c})\right]$$
$$=-4\pi\delta(t-\frac{r}{c})\delta^3(\mathbf{r})+\frac{1}{cr^2}\delta'(t-\frac{r}{c})-\frac{1}{cr^2}\delta'(t-\frac{r}{c})+\frac{1}{c^2r}\delta''(t-\frac{r}{c})$$
$$=-4\pi\delta(t-\frac{r}{c})\delta^3(\mathbf{r})+\frac{1}{c^2r}\delta''(t-\frac{r}{c}) \tag{6.2.4}$$

But

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\frac{\delta(t-\frac{r}{c})}{r}\right]=\left[\frac{\delta''(t-\frac{r}{c})}{c^2r}\right] \tag{6.2.5}$$

Subtracting the two expressions (6.2.4) and (6.2.5), we have

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\left[\frac{\delta(t-\frac{r}{c})}{r}\right]=-4\pi\delta^3(\mathbf{r})\,\delta(t-\frac{r}{c})=-4\pi\delta^3(\mathbf{r})\,\delta(t) \tag{6.2.6}$$

where the last form on the right hand side is true because the delta function in **r** means that we only have a contribution when $r = 0$, so we can take r to be zero in the argument of the delta function in time.

Then, we see (by shifting the origin of space and time by $\mathbf{r}'$ and t') that

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]G(\mathbf{r},\mathbf{r}',t,t')=\delta^3\left(\mathbf{r}-\mathbf{r}'\right)\delta(t-t') \tag{6.2.7}$$

where

$$G(\mathbf{r},\mathbf{r}',t,t')=-\frac{1}{4\pi}\frac{\delta(t-t'-\left|\mathbf{r}-\mathbf{r}'\right|/c)}{\left|\mathbf{r}-\mathbf{r}'\right|} \tag{6.2.8}$$

6.2.3 The solution for $(\phi, \mathbf{A})$ given $(\rho, \mathbf{J})$ for all space and time

We now assert that the solution to (6.1.9) is

$$\mathbf{A}(\mathbf{r},t) = -\mu_o \int_{all\,time} dt' \int_{all\,space} G(\mathbf{r},\mathbf{r}',t,t')\mathbf{J}(\mathbf{r}',t')\,d^3x' \tag{6.2.9}$$

To see that this is indeed the solution to (6.1.9), we apply the operator

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{6.2.10}$$

to equation (6.2.9), yielding (using (6.2.7))

$$\begin{aligned}\Box^2\mathbf{A}(\mathbf{r},t) &= -\mu_o \int_{all\,time} dt' \int_{all\,space} \Box^2 G(\mathbf{r},\mathbf{r}',t,t')\mathbf{J}(\mathbf{r}',t')\,d^3x' \\ &= -\mu_o \int_{all\,time} \delta(t-t')dt' \int_{all\,space} \delta^3(\mathbf{r}-\mathbf{r}')\,\mathbf{J}(\mathbf{r}',t')\,d^3x' = -\mu_o\mathbf{J}(\mathbf{r},t)\end{aligned} \tag{6.2.11}$$

where we have used the delta functions to carry out the time and space integrations. Our solution (6.2.9) can thus be written as (using (6.2.8))

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi} \int_{all\,time} dt' \int_{all\,space} \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|} dt'\delta(t-t'-|\mathbf{r}-\mathbf{r}'|/c)d^3x' \tag{6.2.12}$$

If we use the delta function in time to do the t' integration in (6.2.12) we have finally

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t'_{ret})}{|\mathbf{r}-\mathbf{r}'|}\,d^3x' \tag{6.2.13}$$

where

$$t'_{ret} = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c} \tag{6.2.14}$$

and similarly we have

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\,\varepsilon_o}\int \frac{\rho(\mathbf{r}',t'_{ret})}{|\mathbf{r}-\mathbf{r}'|}\,d^3x' \tag{6.2.15}$$

6.3 What does the observer see at time *t*?

The prescription as to how to do the spatial integrals in (6.2.13) and (6.2.15) using the definition of retarded time (6.2.14) is unusual. Because of the finite propagation time from source to observer at the observer's time *t*, the spatial integrals are sampling what happened in sources more distant from the observer at an earlier time than the sources closer to the observer. That is, the value of t'_{ret} depends on the distance from the a source to the observer. Although we are integrating over all space in the equations above, we are adding up contributions from different volumes of space at different times in the observer's past.

6.3.1 The collapsing information gathering sphere

One way to understand this sampling (Panofsky and Phillips (1962)) is to consider what information is seen by an observer located at the origin at $t = 0$. The information that arrives at the observer at the origin at $t = 0$ has been collected by a sphere of radius $r' = -ct'$ that has been collapsing toward the observer at the speed of light since time began, as shown in Figure 6-1. The observer at time t will see all the light collected by this information collecting sphere. The center of the sphere is at the location of the observer at time t, and the sphere has been contracting since the beginning of time with a radial velocity c such that it has just converged on the observer at time t. The time t'_{ret} at which this information-collecting sphere passes a source at r' at any point is space is then the time at which that source produced the effect which is seen by the observer at time t.

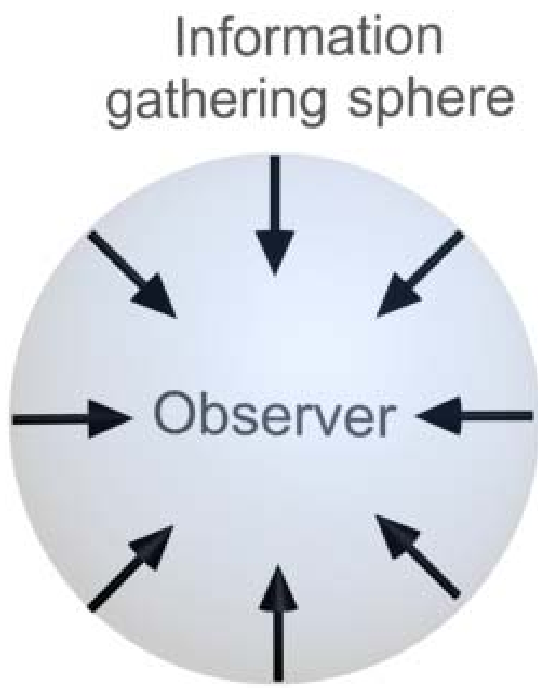


Figure 6-1: The information gathering sphere collapsing toward the origin

6.3.2 The backward light cone

Another way to envisage this process is to look at a space-time diagram. In such a diagram, we plot *ct* along the vertical axes and two spatial coordinates (say *x* and *y*) perpendicular to this axis. An 'event" in space-time, say a firecracker exploding, is located by its time of occurrence and the place at which it occurred. A burst of light emitted by the observer at the origin at $t = 0$ in this diagram propagates outward at the speed of light in all directions, and the locus of space-time points on that outwardly propagating sphere is represented by the forward light cone shown in Figure 6-2. The forward light cone is a cone whose apex is at the origin and whose opening angle is 45 degrees. Only observers at points in space-time lying on the forward light cone will see the burst of light emitted by our observer at $t = 0$. Similarly, the light seen by our observer at the origin at $t = 0$ must have originated at some point in space-time on the observer's backward light cone, since for only those points will the radiation just be arriving at the observer's position at $t = 0$.

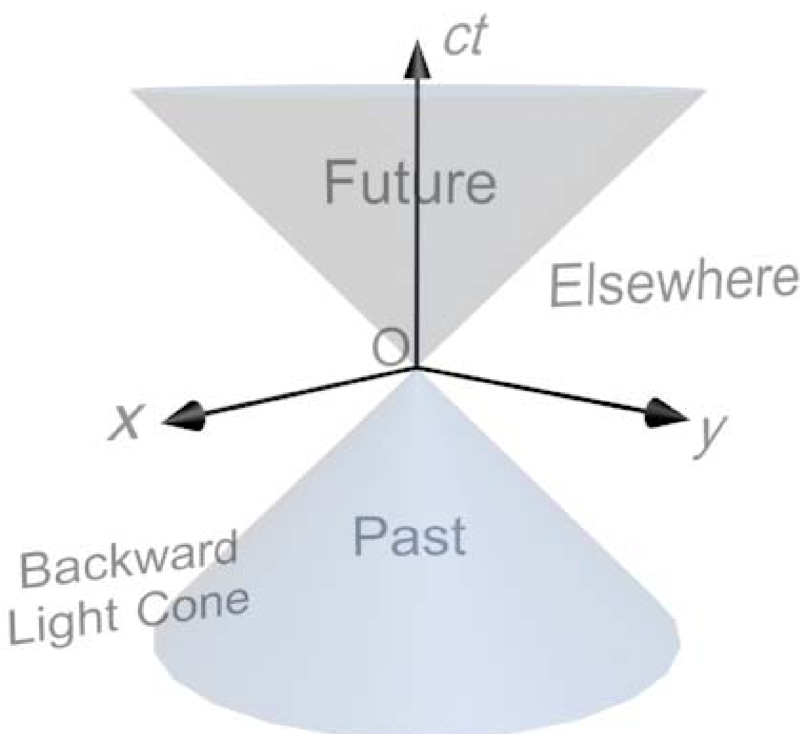


Figure 6-2: The observer's forward and backward light cone