8.07 Class Notes Fall 2011

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If we compare equation (17.8.2) and (17.8.3), it is natural to conclude that

$$\mathbf{F}_{\text{radiation reaction}} = \frac{1}{4\pi\varepsilon_o}\frac{2}{3}\frac{q^2}{c^3}\ddot{\mathbf{v}} = \frac{2}{3}m_e\frac{r_e}{c}\ddot{\mathbf{v}} = \frac{2}{3}m_e\tau_e\ \ddot{\mathbf{v}} \tag{17.8.4}$$

where

$$\tau_e = \frac{r_e}{c} \tag{17.8.5}$$

is the speed of light transit time across the classical electron raduis. This is the form we have in (17.8.1).

It is important to emphasize that the radiation reaction force represents an irreversible loss of energy to infinity. We never get this energy back, it disappears forever from the system.

18 Basic Electrostatics

18.1 Learning Objectives

We first motivate what we are going to do in the next four or five sections. We then go back to the origins of electromagnetism—electrostatics, and spend some time going through the classic aspects of this subject, including the energy we put into assembling a configuration of charges.

18.2 Where are we going?

In Section 1.3.1, I enumerated what I consider to be the profound part of classical electromagnetism, which I repeat here.

1) The existence of fields which carry energy and momentum, and the ways in which they mediate the interactions of material objects.

2) The nature of light and the radiation process.

3) The explicit prescription for the way that space and time transform which is contained in Maxwell's equations.

We have finished with (2) and (3) above, and we have touched on various aspects of (1). We now focus our attention on (1). In particular we will be looking at the electromagnetic interactions of particles and fields in the "near zone", where we neglect the effects of radiative losses, and look at the reversible exchange of energy, momentum, and angular momentum between charged particles and fields, and how that proceeds.

As a preview of the sorts of things we want to explore, consider the application showing the interaction of charged particles, as shown in Figure 18-1 and Figure 18-2.

Charges in the application interact via the Coulomb force, with a Pauli repulsive force at close distances, plus a damping force proportional to velocity. The Pauli repulsive force goes at inverse radius to the sixth power, so it is very "stiff". That is it either dominates the interaction or it is more or less negligible compared to the Coulomb force. So as you watch the charges in the application interact, they "bounce" at close distances, when the Pauli repulsion very quickly overpowers any Coulomb attraction between charges. The damping proportional to the velocity allows our particles to settle down to a meta-stable configuration. If we did not have the damping there would be a continuous interchange of energy between the kinetic energy of the charges and the energy stored in the field. The damping allows us to drain away that kinetic energy so that the particles end up in a meta-stable state with zero kinetic energy and a local minimum in the electrostatic energy.

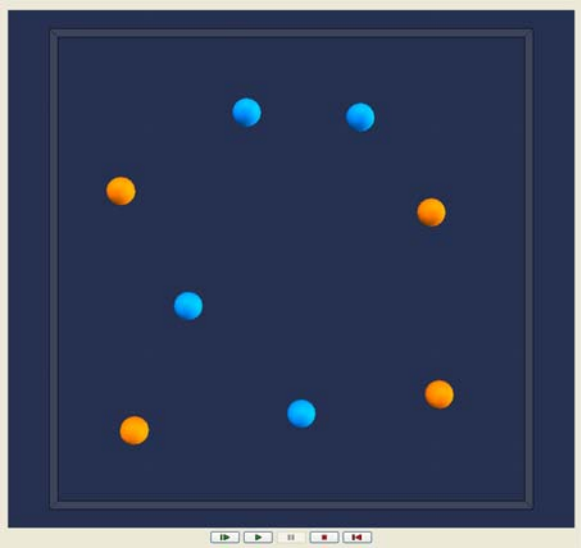

Figure 18-1: An applicaton showing charges interacting via Coulomb's Law
http://web.mit.edu/viz/EM/visualizations/electrostatics/InteractingCharges/

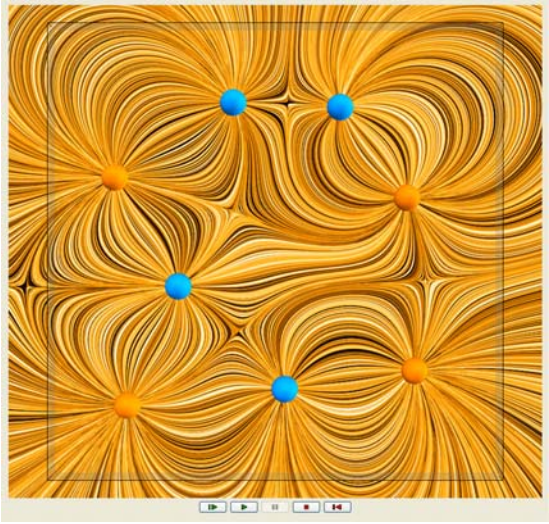

Figure 18-2: The same application as above except with a "grass seeds" representation of the electric fields

So the non-relativistic equations we are soving in this application, assuming we have *N* interacting charges $\{q_i\}_{i=1}^N$ with masses $\{m_i\}_{i=1}^N$ and frictional damping rates $\{\gamma_i\}_{i=1}^N$ located at positions $\{\mathbf{r}_i(t)\}_{i=1}^N$, with velocities $\{\mathbf{u}_i(t)\}_{i=1}^N$, are

$$\mathbf{u}_i(t) = d\,\mathbf{r}_i(t)/dt \tag{18.2.1}$$

$$m_i \frac{d\,\mathbf{u}_i(t)}{dt} = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{q_i\,q_j}{4\pi\varepsilon_o} \frac{\left(\mathbf{r}_i - \mathbf{r}_j\right)}{\left|\,\mathbf{r}_i - \mathbf{r}_j\,\right|^3} + \sum_{\substack{j=1 \\ j\neq i}}^{N} P \frac{\left(\mathbf{r}_i - \mathbf{r}_j\right)}{\left|\,\mathbf{r}_i - \mathbf{r}_j\,\right|^7} - m_i\gamma_i\mathbf{u}_i \tag{18.2.2}$$

Coulomb force Pauli repulsion drag

This is a system of first order differential equations which in two (three) dimensions has 4 (6) dependent variables for each particle (a position vector and a velocity vector) and thus a total of 4*N* (6*N*) dependent variables. For the system shown in Figure 18-1, where we have eight particles, this is 32 dependent variables. Solving this set of coupled dependent variable equations as a function of the independent variable *t* can be carried out by standard numerical techniques, for example fifth order Runge-Kutta. The first term on the right in (18.2.2) is either repulsive or attractive depending on whether the signs of the ith particle and that of the jth particle are the same or opposite, and goes as inverse distance squared between the charges. The second term is always repulsive, and goes as inverse distance to the sixth between the charges. The third term is dissipative, that is it always drains kinetic energy from the charges unless they are stationary.

As can be easily explored by playing with this application, there is an enormous wealth of complex intereactons between charged particles when their interaction is governed by the Coulomb interaction. The particles initially rapidly try to form electric dipoles, that is a positive charge in close proximity to a negative charge. Once dipoles form, the forces between dipoles is greatly reduced, and therefore the time scale for changes in position is greatly reduced, because the dipole fields fall off as inverse distance cubed rather than inversed distance squared. Eventually the charges will aggregate into a clump, forming stable “crystals”.

These behaviours mimic what we see in the real world, where the everyday interactions between matter are dominated by electrostatic forces. Even though we can solve for the dynamics of our charges in this application simpy using the Coulomb repulsion or attraction (along with the Pauli repulsion at small distances and the frictional damping), that calculation hides an enormous amount of the physics. What is not apparent when we these dynamical calculations is the complex exchange of energy between field and charges as these interactions proceed. We will really not be able to explain the complexity of this exchange until we also consider magnetostatics, because we need **B** as well as **E** to understand the flow of energy in these interactions via the Poynting flux $\mathbf{E}\times\mathbf{B}/\mu_o$. But that is where we are going.

First though we need to explore the electric fields of stationary particles, as we do in the present section. Then we will look at the magnetic fields generated when we allow electric charges to move. Then we consider the electric fields that are present whenever we see time changing magnetic fields—that is, Faraday's Law. Then will come back and try to understand the complexity of the interactions shown in the application. There is far more than meets the eye here, which we will discuss eventually, but first we look at electrostatics.

18.3 The electric field E and potential ϕ of a set of point charges

The force $\mathbf{F}$ on a test charge Q located at $\mathbf{r}$ due to a charge q located at $\mathbf{r}'$ is given by

$$\mathbf{F} = \frac{qQ}{4\pi\varepsilon_o}\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3} = \frac{qQ}{4\pi\varepsilon_o}\frac{\hat{\mathbf{n}}}{\left|\mathbf{r}-\mathbf{r}'\right|^2} \tag{18.3.1}$$

where as always, the unit vector $\hat{\mathbf{n}}$ points from the source to the observation point, that is

$$\hat{\mathbf{n}} = \frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|} \tag{18.3.2}$$

This force is attractive if the two charges have the opposite sign. It is repulsive if the two charges have opposite signs.

We define the electric field $\mathbf{E}$ at $\mathbf{r}$ to be the ratio of the force on the test charge to the test charge Q. That is, the electric field is given by

$$\mathbf{E}(\mathbf{r}) = \frac{\mathbf{F}}{Q} = \frac{q}{4\pi\varepsilon_o}\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3} \tag{18.3.3}$$

Since we can write $\dfrac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3} = -\nabla\dfrac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}$, we can also write (18.3.3) as

$$\mathbf{E}(\mathbf{r}) = -\nabla\left[\frac{q}{4\pi\varepsilon_o}\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\right] = -\nabla\phi(\mathbf{r}) \quad \text{where} \quad \phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_o}\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} + \text{Constant} \tag{18.3.4}$$

where $\phi(\mathbf{r})$ is the electrostatic potential of a point charge. We will discuss the physical meaning of the electrostatic potential below. The unit of the electric field are thus Newtons/Coulomb.

If we have a set of charges $\{q_i\}_{i=1}^N$ located at positions $\{\mathbf{r}_i'\}_{i=1}^N$, then the electric field of this collection of charges is simply the sum of the electric field of each individual charge, that is

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_o} \frac{\mathbf{r}-\mathbf{r}_i'}{\left|\mathbf{r}-\mathbf{r}_i'\right|^2} = -\nabla\phi(\mathbf{r}) \qquad \text{where} \qquad \phi(\mathbf{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_o} \frac{1}{\left|\mathbf{r}-\mathbf{r}_i'\right|} + \text{Constant} \tag{18.3.5}$$

We note that since the curl of any gradient of a scalar is zero, we have for electrostatic fields that

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0 \tag{18.3.6}$$

18.4 The electric field E and potential ϕ of a continuous charge distribution

Suppose we have a continuous distribution of electric charge defined by a volume change density $\rho(\mathbf{r}')$. Then the amount of charge in an infinitesimal volume element d^3x' is $dq' = \rho(\mathbf{r}')\,d^3x'$, and our sums in (18.3.5) go into integrals, as follows.

$$\mathbf{E}(\mathbf{r}) = \int_{all\ space} \frac{\rho(\mathbf{r}')\,d^3x'}{4\pi\varepsilon_o} \frac{(\mathbf{r}-\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|^3} = -\nabla\phi(\mathbf{r}) \tag{18.4.1}$$

$$\phi(\mathbf{r}) = \int_{all\ space} \frac{\rho(\mathbf{r}')\,d^3x'}{4\pi\varepsilon_o} \frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} \tag{18.4.2}$$

We can get the second equation for electrostatics from (18.4.1) by taking the divergence of (18.4.1), which yields

$$\nabla\cdot\mathbf{E}(\mathbf{r}) = -\nabla^2\phi(\mathbf{r}) = -\nabla^2 \int_{all\ space} \frac{\rho(\mathbf{r}')\,d^3x'}{4\pi\varepsilon_o} \frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} = -\int_{all\ space} \frac{\rho(\mathbf{r}')\,d^3x'}{4\pi\varepsilon_o} \nabla^2 \frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} \tag{18.4.3}$$

But of course we have $\nabla^2 \dfrac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$, and using this to do the integral in (18.4.3) yields

$$\nabla\cdot\mathbf{E}(\mathbf{r}) = \rho(\mathbf{r})/\varepsilon_o \tag{18.4.4}$$

18.5 The physical meaning of the electric potential ϕ

In an early problem in our problem sets, we showed that if $\mathbf{E}(\mathbf{r})$ is a vector function which has zero curl, then the scalar function $-\int_{\mathbf{r}_o}^{\mathbf{r}} \mathbf{E}(\mathbf{r}')\cdot d\,\mathbf{l}'$, which is the line integral of $\mathbf{E}$ along any path connecting a fixed reference location $\mathbf{r}_o$ and the observer's position $\mathbf{r}$, is independent of the path taken. Therefore this scalar function is a single valued and unique function given E and the reference location $\mathbf{r}_o$. We also saw how to get $\mathbf{E}$ given $-\int_{\mathbf{r}_o}^{\mathbf{r}} \mathbf{E}(\mathbf{r}')\cdot d\,\mathbf{l}'$ by looking at, for example

$$-\int_{\mathbf{r}}^{\mathbf{r}+\Delta x\hat{\mathbf{x}}} \mathbf{E}(\mathbf{r}')\cdot d\,\mathbf{l}' \approx -\Delta x\, E_x \tag{18.5.1}$$

If we generalize (18.5.1) to all three components we have

$$\mathbf{E} = -\nabla\int_{\mathbf{r}_o}^{\mathbf{r}} \mathbf{E}(\mathbf{r}')\cdot d\,\mathbf{l} \tag{18.5.2}$$

Comparing (18.5.2) to (18.4.2) or (18.3.5), we see that we must have

$$\phi(\mathbf{r}) = -\int_{\mathbf{r}_o}^{\mathbf{r}} \mathbf{E}(\mathbf{r}')\cdot d\,\mathbf{l}' \tag{18.5.3}$$

where we are free to choose the "reference" point $\mathbf{r}_o$. If possible we will choose that point to be at infinity, but that is not possible in some situations. So we now know that $\mathbf{E}(\mathbf{r})$ and $\phi(\mathbf{r})$ are related not only by $\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$ but also by (18.5.3).

Equation (18.5.3) allows us to interpret the meaning of $\phi(\mathbf{r})$ in physical terms. To do so we must talk about moving a test electric charge *Q* around in the presence of fixed charges whose electric field is **E**. How much work do I do in moving that charge from one point to another, say from **a** to **b**? Well first of all I must assume that I do this very slowly in some sense, because I do not want to have a substantial amount of energy radiated away to infinity, but if I do this slowly enough I can make the radiated energy as small as I desire. I can thus ignore any radiation reaction term, and the force I must exert to move the charge around is simply the force I need to counter balance the force associated with the electric field that the test charge feels, plus a little bit more. That is I need to exert a force

$$\mathbf{F}_{me} = -Q\mathbf{E}(\mathbf{r}) \tag{18.5.4}$$

plus a little tiny bit more, to get the test charge to actually move. Again, I can make this additional "little bit more" force and its associated work arbitrarily small.

Thus the work I must do in moving our test charge from **a** to **b** is given by

$$W_{me}^{\mathbf{a}\to\mathbf{b}} = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F}_{me}\cdot(\mathbf{r}')\cdot d\,\mathbf{l}' = -\int_{\mathbf{a}}^{\mathbf{b}} Q\mathbf{E}(\mathbf{r}')\cdot d\,\mathbf{l}' = -Q\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}(\mathbf{r}')\cdot d\,\mathbf{l}' \tag{18.5.5}$$

But using (18.5.3), we have

$$W_{me}^{\mathbf{a}\to\mathbf{b}} = -Q\left[\int_{\mathbf{r}_o}^{\mathbf{b}} \mathbf{E}(\mathbf{r}')\cdot d\,\mathbf{l}' - \int_{\mathbf{r}_o}^{\mathbf{a}} \mathbf{E}(\mathbf{r}')\cdot d\,\mathbf{l}'\right] = Q\left[\phi(\mathbf{b})-\phi(\mathbf{a})\right] \tag{18.5.6}$$

or

$$\left[\phi(\mathbf{b})-\phi(\mathbf{a})\right] = \frac{W_{me}^{\mathbf{a}\to\mathbf{b}}}{Q} \tag{18.5.7}$$

Thus the difference in electric potential between two points **a** and **b** is the amount of work I must do to move a unit test charge from **a** to **b** against the electric field. The units of ϕ are thus joules per coulomb, or *volts*. The units of electric field are Newtons per coulomb, which are also joules per coulomb per meter, so the units of electric field are also volts/meter, and this is how we most often quote the units of electric field.

18.6 The energy required to assemble charges

18.6.1 A set of point charges

Now we can calculate the amount of energy we need to do to assemble a set of point charges, bringing them in from infinitiy. We do this one by one. It takes no enerty to bring in the first charge q_1 from infinity to its final position $\mathbf{r}_1$, because there is no electric field to work against. Once the first particle is there we have a potential given by (see (18.3.4)) $\phi_1(\mathbf{r}) = \dfrac{q_1}{4\pi\varepsilon_o}\dfrac{1}{|\mathbf{r}-\mathbf{r}_1|}$, where have taken our reference point for zero potential at infinity.

Now if we bring in the second charge q_2 from infinity to its final position $\mathbf{r}_2$, we can calculate the work we need to do this by using the meaning of the potential we discussed above, to find that the work we need to do to bring in this charge is $q_2\phi_1(\mathbf{r}_2) = \dfrac{q_1 q_2}{4\pi\varepsilon_o}\dfrac{1}{|\mathbf{r}_2-\mathbf{r}_1|}$. Now our potential with these two charges present is

$$\phi_{1+2}(\mathbf{r}) = \frac{q_1}{4\pi\varepsilon_o}\frac{1}{|\mathbf{r}-\mathbf{r}_1|} + \frac{q_2}{4\pi\varepsilon_o}\frac{1}{|\mathbf{r}-\mathbf{r}_2|} \tag{18.6.1}$$

To bring in a third charge q_3 from infinity to its final position $\mathbf{r}_3$ requires energy $q_3\phi_{1+2}(\mathbf{r}_3)$, or

$$q_3\phi_{1+2}(\mathbf{r}_3) = \frac{q_3 q_1}{4\pi\varepsilon_o}\frac{1}{|\mathbf{r}_3 - \mathbf{r}_1|} + \frac{q_3 q_2}{4\pi\varepsilon_o}\frac{1}{|\mathbf{r}_3 - \mathbf{r}_2|} \tag{18.6.2}$$

So the total work we have done thus far is the sum of (18.6.2) and $\frac{q_1 q_2}{4\pi\varepsilon_o}\frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|}$, or

$$W_3 = \frac{q_1 q_2}{4\pi\varepsilon_o}\frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} + \frac{q_3 q_1}{4\pi\varepsilon_o}\frac{1}{|\mathbf{r}_3 - \mathbf{r}_1|} + \frac{q_3 q_2}{4\pi\varepsilon_o}\frac{1}{|\mathbf{r}_3 - \mathbf{r}_2|} \tag{18.6.3}$$

It is fairly easy to see that if we do this *N* times, we do total work

$$W_N = \sum_{j<i}\sum_{i=1}^{N}\frac{q_i q_j}{4\pi\varepsilon_o}\frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{18.6.4}$$

If we put in a factor of ½ to account for double counting, we can write this as

$$W_N = \frac{1}{2}\sum_{j\neq i}^{N}\sum_{i=1}^{N}\frac{q_i q_j}{4\pi\varepsilon_o}\frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{18.6.5}$$

18.6.2 A continuous distribution of charges

If we go over to a continuous distribution of charges, our double sum in (18.6.5) becomes a double integral, as follows.

$$W = \frac{1}{2}\int\int\frac{\left(\rho(\mathbf{r})\,d^3x\right)\left(\rho(\mathbf{r}')\,d^3x'\right)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}'|} = \frac{1}{2}\int\rho(\mathbf{r})\,d^3x\int\frac{\rho(\mathbf{r}')\,d^3x'}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}'|}$$
$$W = \int\frac{1}{2}\rho(\mathbf{r})\phi(\mathbf{r})\,d^3x \tag{18.6.6}$$

where we have used (18.4.2) to get the last form in (18.6.6). To put (18.6.6) in a more familiar form, we use $\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$ and $\nabla\cdot\mathbf{E}(\mathbf{r}) = \rho(\mathbf{r})$ to obtain

$$W = \int\frac{\varepsilon_o}{2}(\nabla\cdot\mathbf{E})\phi(\mathbf{r})\,d^3x = \int\frac{\varepsilon_o}{2}\nabla\cdot\left(\phi(\mathbf{r})\mathbf{E}\right)d^3x - \int\frac{\varepsilon_o}{2}\mathbf{E}\cdot\left(\nabla\phi(\mathbf{r})\right)d^3x$$
$$W = \int\frac{1}{2}\varepsilon_o E^2\,d^3x \tag{18.6.7}$$

where in the second term in (18.6.7) we have converted the volume integral of a divergence to a surface integral at infinity and set it to zero because the integrand falls off faster than inverse *r* cubed. Thus we recover in this expression the term we have seen before for the energy density in the electric field, $\frac{\varepsilon_o E^2}{2}$.

Note that the expression in (18.6.7) is always positive, whereas the similar expression for point charges (18.6.5) can be positive or negative. The difference is that in our sum in (18.6.5) we explicity exclude the $i = j$ terms, which corresponds to the energy required to assemble the point charges themselves, which is infinite. In a more realistic model we would have the point charges having some small but finite radius *R*, and the energy to assemble them would be $W = \frac{q^2}{4\pi\varepsilon_o R}$ (see the next paragraph for a more quantitative justification of this). This of course blows up as *R* approaches zero, which is why we do not include it in the sum in (18.6.5). As long as we are not disassembling point charges, however, there is no harm in ignoring these infinite terms, because it is only the changes in energy we are interested in as the configuration changes, not the total energy.

18.7 Where is the energy really located in space?

If we look at (18.6.6) and (18.6.7) you might think that we have two equally good expressions for the energy density of the electromagnetic field, either $\frac{1}{2}\rho\phi$ or $\frac{1}{2}\varepsilon_o E^2$. These give very different spatial distributions of energy density however. To illustrate this, let us consider the energy necessary to assemble a spherical shell of charge of radius *R*, carrying total charge *Q*, distributed uniformly over the sphere in a surface change density $\sigma = Q/4\pi R^2$. We can easily use Gauss's Law to calculate the electric field everywhere (see next Section) and then use the electric field and (18.5.3) to calculate the electric potential from the electric field, to obtain the following results.

$$\mathbf{E}(\mathbf{r}) = \begin{cases} 0 & \text{r < R} \\ \hat{\mathbf{r}}\dfrac{Q}{4\pi\varepsilon_o r^2} & \text{r > R} \end{cases} \tag{18.7.1}$$

$$\phi(\mathbf{r}) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_o R} & \text{r < R} \\ \dfrac{Q}{4\pi\varepsilon_o r} & \text{r > R} \end{cases} \tag{18.7.2}$$

Calculating the energy required to put this distribution of charge together using (18.6.7) is straightforward, giving

$$W = \int \frac{1}{2}\varepsilon_o E^2 d^3x = \int d\Omega \int_R^\infty dr \frac{1}{2}\varepsilon_o \left[\frac{Q}{4\pi\varepsilon_o r^2}\right]^2 r^2 = \frac{Q^2}{8\pi\varepsilon_o R} \tag{18.7.3}$$

This calculation is even simpler using (18.6.6), since $\rho(\mathbf{r})$ is only non-zero at $r = R$.

$$W = \int \frac{1}{2}\rho(\mathbf{r})\phi(\mathbf{r}) d^3x = \frac{1}{2}\phi(R)\int \rho(\mathbf{r}) d^3x = \frac{Q^2}{8\pi\varepsilon_o R} \tag{18.8.1}$$

These two methods of calculating W give the same result, as they must, even though the integrands are very different. So what is the correct expression for the energy density of the electric field? Electrostatics actually gives us little guidance in how to answer this question, but when we introduce time dependence, the answer is quite clear. If we go back to (4.4.2) for the differential form of the conservation of energy, we had

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon_o E^2 + \frac{B^2}{2\mu_o}\right] + \nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right) = -\mathbf{E}\cdot\mathbf{J} \tag{18.8.2}$$

and this form clearly chooses $\frac{1}{2}\varepsilon_o E^2$ for the local energy density of the electrostatic field. Moreover if we look at the time dependent process by which the electrostatic energy is located at a given point in space, we clearly see the flow of energy from where we are doing work to create it (where the creation rate for electromagnetic energy, $-\mathbf{E}\cdot\mathbf{J}$, is non-zero) to where it resides in space, as indicated by the local value of $\frac{1}{2}\varepsilon_o E^2$, through the agency of the Poynting flux, $\mathbf{E}\times\mathbf{B}/\mu_o$.

To take a concrete example of what I mean by this, consider the following scenario. An electric field is created by an external agent who separates charges. We start out with five negative electric charges and five positive charges, all at the same point in space. Since there is no net charge, there is no electric field. Now the agent moves one of the positive charges at constant velocity from its initial position to a distance L away along the horizontal axis. After doing that, the agent moves the second positive charge in the same manner to the position where the first positive charge sits. The agent continues on with the rest of the positive charges in the same manner, until all of the positive charges are sitting a distance L from their initial position along the horizontal axis. We have color coded the "grass seeds" representation in the still below to represent the strength of the electric field. Very strong fields are white, very weak fields are black, and fields of intermediate strength are yellow.

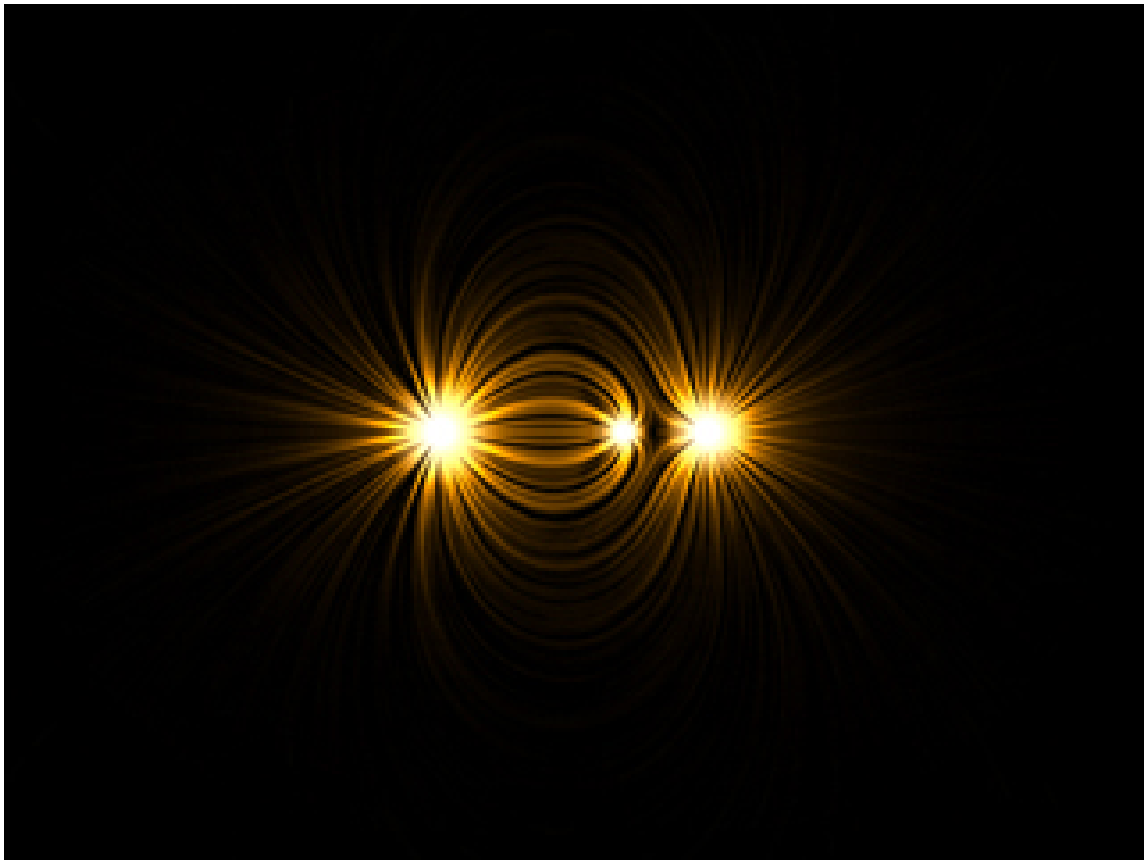

Figure 18-3: An external agent creating an electric field by separating + and – charges.

http://web.mit.edu/viz/EM/visualizations/electrostatics/CreatingDestroyingEFields/

The field lines move in the direction of the energy flow of the electromagnetic field. Over the course of the animation, the strength of the electric field grows as each positive charge is moved into place. That energy flows out from the path along which the charges move, because that is where $-\mathbf{E}\cdot\mathbf{J} > 0$ is non-zero and positive, and nowhere else. That energy is being provided by the agent moving the charge against the electric field of the other charges. The work that this agent does to separate the charges against their electric attraction appears as energy in the electric field, and we can see it flow out from where it is created and take up its position in space.

When we do the reverse process, that is the external agent now moves the charges back to where they initially were, the energy stored in the electrostatic field moves back from where it is stored in space to the path of the particle, because that is where is non-zero and $-\mathbf{E}\cdot\mathbf{J} < 0$. It is then returned reversibly to the agent moving the charges back into their original positions. We are neglecting any energy radiated away in this process, which is fine as long as the charge speeds are non-relativisitic, so that the external agent recovers exactly the amount of energy he expended increating the electric fields in the first place.

The amazing thing about electromagnetism is that the fields contain energy, and we can see exactly how the electric energy is (how it is distributed in space) and how it got to where it is.

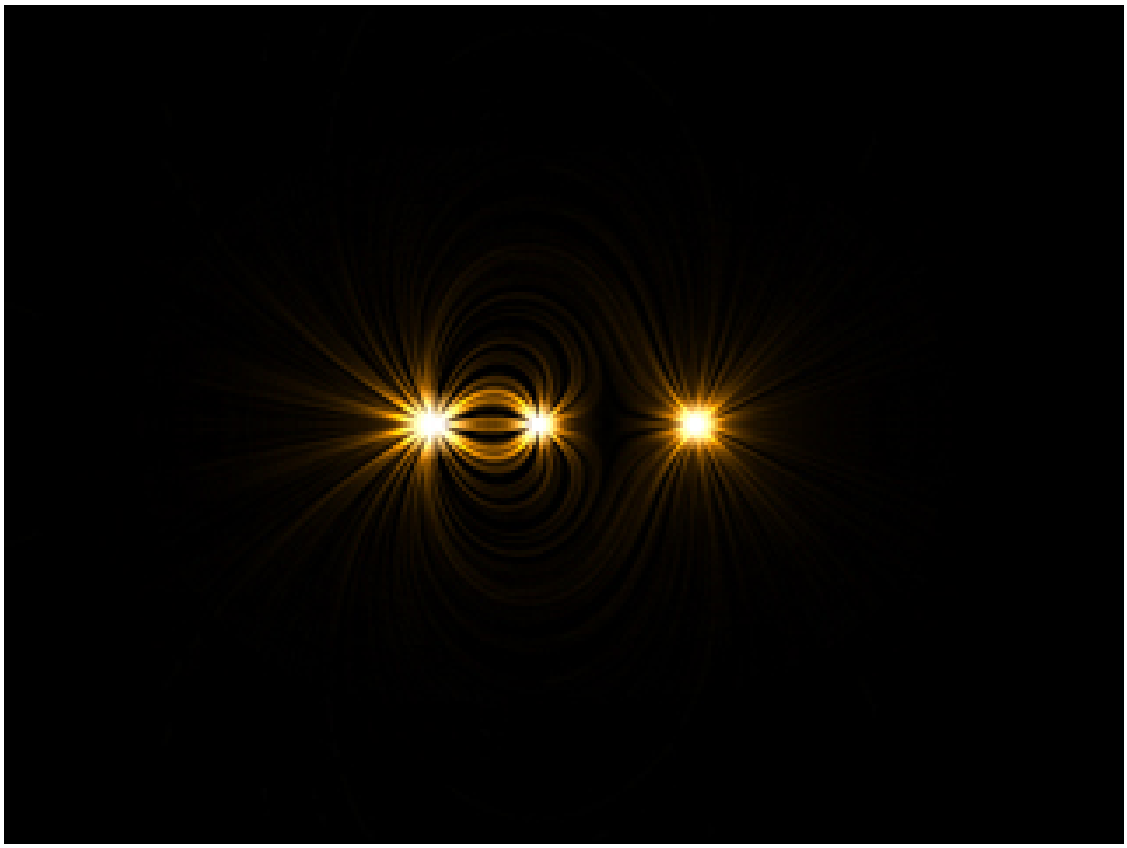

Figure 18-4: An external agent destroying an electric field by bringing together + and – charges.

18.8 Gauss's Law

18.8.1 The general statement

Gauss's Law is extremely useful in solving electrostatic problems with a high degree of symmetry, as we shall see. If we consider any closed surface surrounding a volume, we have from Gauss's Theorem that

$$\int_{volume} \nabla \cdot \mathbf{E}(\mathbf{r})\, d^3x = \int_{surface} \mathbf{E}(\mathbf{r}) \cdot \hat{\mathbf{n}}\; da \tag{18.8.3}$$

If we use $\nabla \cdot \mathbf{E}(\mathbf{r}) = \rho(\mathbf{r}) / \varepsilon_o$ ((18.4.4), we have for any closed surface that

$$\int_{surface} \mathbf{E}(\mathbf{r}) \cdot \hat{\mathbf{n}}\; da = \frac{1}{\varepsilon_o} \int_{volume} \rho(\mathbf{r})\, d^3x = \frac{Q_{inside}}{\varepsilon_o} \tag{18.8.4}$$

Equation (18.8.4) is always true, but it is not always useful in solving problems. As an example of this, in Figure 18-5, we show a "Gaussian cylinder" in the presence of two point charges. The surface integral of the electric field dotted into the normal to the surface for this cylinder times *da* is shown by the electric field on the surface of the cylinder evalualted at a number of ponts, where we also indicate the surface normal. In the scenario shown, there is zero flux through the cylinder because there is zero change in the cylinder, even though at every point on the surface there is an electric field. This is an example where Gauss's Law is true but useless in solving a problem.

18.8.2 The field of a point charge from Gauss's Law

To have Gauss's Law actually be useful for problem solving, we need a situation like that shown in Figure 18-6. Here we have a sphere centered on the charge, so that everywhere on the sphere of radius *r* the electric field is radially outward and thus parallel to the surface normal. It is also plausible to assume that the electric field magnitude is only a function of the radius *r*. Thus (18.8.4) becomes

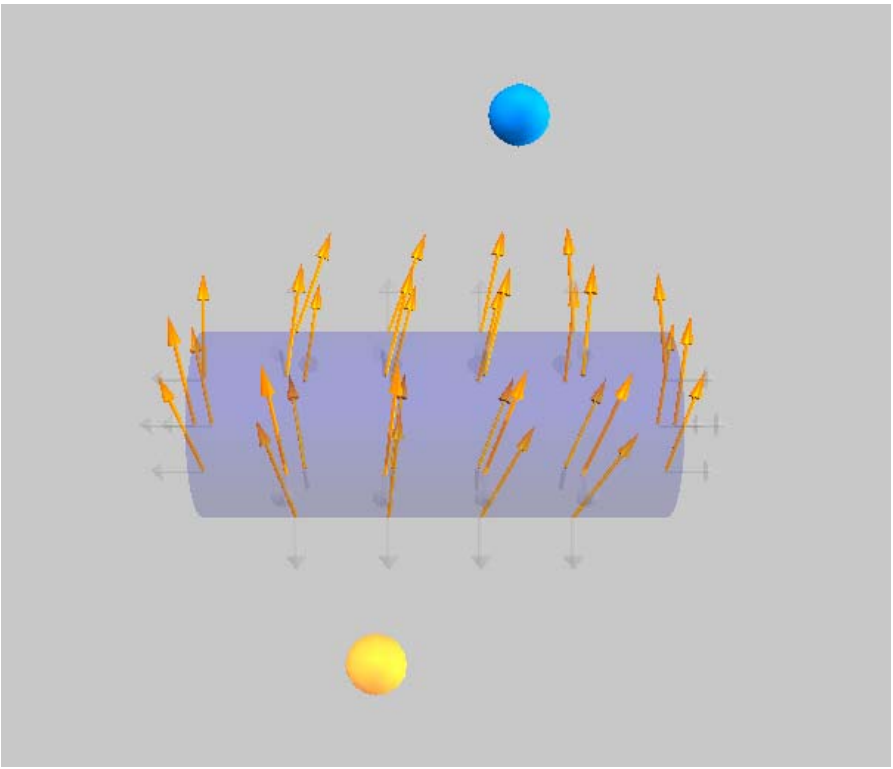

Figure 18-5: Gauss's Law application for point charges

http://web.mit.edu/viz/EM/visualizations/electrostatics/GaussLawProblems/

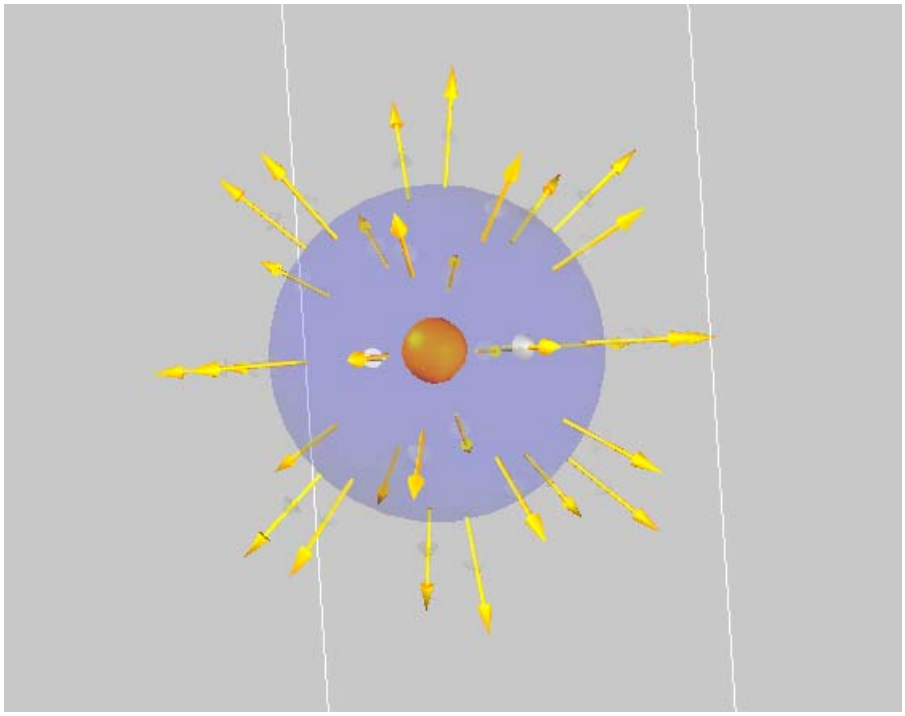

Figure 18-6: Gauss's Law application for a spherical Gaussian surface

$$\int_{surface} \mathbf{E}(\mathbf{r})\cdot\hat{\mathbf{n}}\; da = \frac{q}{\varepsilon_o} = \int_{surface} E(r)\; da = E(r)\int_{surface} da = E(r)4\pi r^2 \qquad (18.8.5)$$

and in this form Gauss's Law has actually allowed us to get the field of a point charge, that is it is radial and varies in magnitude as (18.8.5) prescribes, e.g. $E(r) = q/4\pi\varepsilon_o r^2$.

18.8.3 The field of a line charge from Gauss's Law

Lets do something more difficult with Gauss's Law—the field of an infinite line of charge. We can of course calculate this electric field using the procedure embodied in (18.4.1), but this turns out to be comparatively tedius, whereas the Gauss's Law procedure yields the answer in just a few steps. We must first assume that our electric field is along the cylindrically outward radial direction. Then for our imaginary Gaussian surface we choose a cylinder whose axis is the line of charge, with length *L* and cylindrical radius *r* (see Figure 18-7).

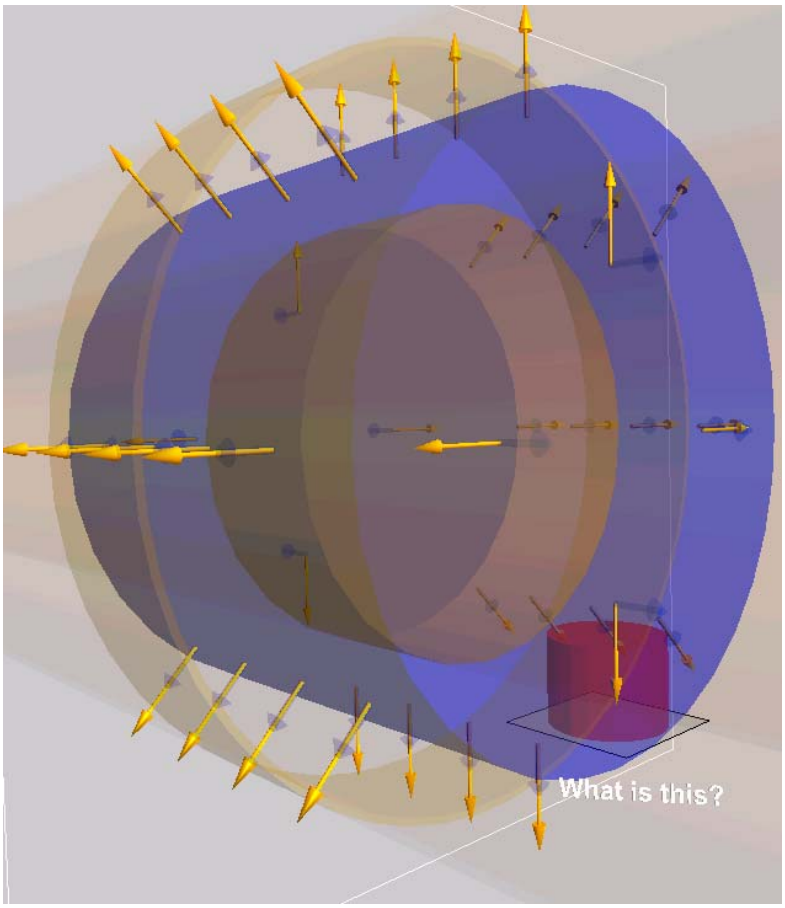

Figure 18-7: An imaginary Gaussian surface (blue cylinder) in a problem with cylindrical symmetry.

Since the cylinder is centered on the line charge, everywhere on the cylinder the electric field is radially outward in the cylindrical sense and thus parallel to the surface normal. It is also plausible to assume that the electric field magnitude is only a function of the cylindrical radius *r*. Thus (18.8.4) becomes (the integrals over the ends of the cylinder vanish because the electric field and the normal are perpendicular there)

$$\int_{sides} \mathbf{E}(\mathbf{r})\cdot\hat{\mathbf{n}}\; da = \frac{\lambda L}{\varepsilon_o} = \int_{sides} E(r)\; da = E(r)\int_{sides} da = E(r)2\pi rL \qquad (18.8.6)$$

And we easily recover from (18.8.6) that the field of a line charge is given in terms of the cylindrical

$$\mathbf{E}(\mathbf{r}) = \frac{\lambda}{2\pi\varepsilon_o r}\hat{\mathbf{r}} \qquad (18.8.7)$$

18.8.4 The field of a plane of charge from Gauss's Law

Finally, lets obtain the electric field of a plane of charge with charge per unit area σ. We choose a Gaussian surface as shown in Figure 18-8.

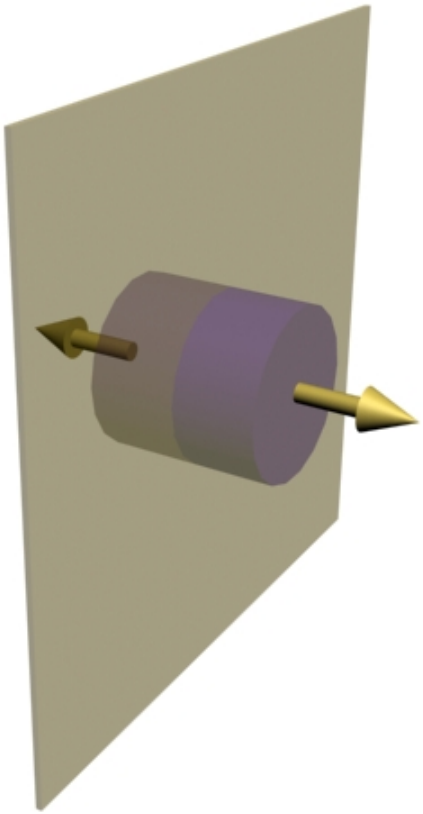

Figure 18-8: The Gaussian surface for a plane of charge.

With this choice and using (18.8.4), we find that

$$\mathbf{E}(\mathbf{r}) = \frac{\sigma}{2\varepsilon_o}\hat{\mathbf{x}}\, sign(x) \tag{18.8.8}$$

where the normal to the plane is $\hat{\mathbf{x}}$.

19 Boundary Value Problems in Electrostatics

19.1 Learning Objectives

We look at boundary value problems in electrostatics. First we define a typical boundary problem, and then we explain why these problems fall into the "hard" category of electromagnetism problems. We then discuss a classic method of solving these problems, the image charge method. The method only works for problems with a high degree of symmetry, so we move on to a method that has more general application, which

is separation of variables in various coordinate systems. We only discuss this method for the case of spherical coordinates,

19.2 The Typical Boundary Value Problem

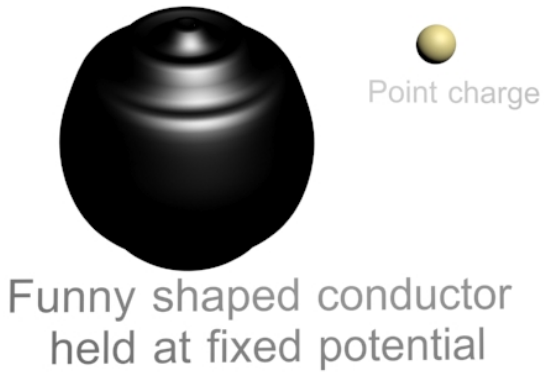

Figure 19-1: The typical boundary value problem

Figure Figure 19-1 shows a typical problem that we want to solve. We have a conductor of some shape, which is held at a fixed potential. We have a point charge with charge *q* located at some point in space outside the conductor. We want to find the electric field everywhere in space.

19.2.1 Poisson's and Laplace's Equations

Here is the way we go about solving such a problem. We know from (18.4.4) and (18.4.1) that we have

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) \qquad \text{and} \qquad \nabla\cdot\mathbf{E}(\mathbf{r}) = \rho(\mathbf{r})/\varepsilon_o \tag{19.2.1}$$

These two equations immediately lead to

$$\nabla^2\phi(\mathbf{r}) = -\rho(\mathbf{r})/\varepsilon_o \tag{19.2.2}$$

This equation is known as Poisson's Equation (if $\rho(\mathbf{r}) = 0$ in the region outside of the conductor, it is also known as Laplace's Equation). So we want to find a function $\phi(\mathbf{r})$ which satisfies (19.2.2) for some specified "free" charge distribution $\rho(\mathbf{r})$ outside of the

conductor ("free" means we control the location of those charges), with the additional requirement that the potential goes to a specified value on the surface of the conductor. This is a "hard" problem because we don't know where all the charges are. If we knew where all the charges were, we could simply use

$$\phi(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_o}\int\frac{\rho(\mathbf{r}',t'_{ret})}{|\mathbf{r}-\mathbf{r}'|}\,d^3x' \tag{19.2.3}$$

in the limit of no time dependence, which is

$$\phi(\mathbf{r})=\frac{1}{4\pi\varepsilon_o}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3x' \tag{19.2.4}$$

and we would be done. But we cannot use (19.2.4) to calculate the potential, because we ***do not know where all the charges are located***. All we know is where the "free" charges are, and that the charges in the conductor will have arranged themselves to make the conductor an equipotential, but we ***do not know*** *a priori* ***how they have done that***. This is an example of the "hard" electromagnetism: the charges move in response to the fields they create. In this kind of situation, even though (19.2.4) ***is always true***, it is useless in solving the problem, because we have unknowns on both sides of the equation.

19.2.2 Electric fields in and near isolated conductors

Below we will discuss various techniques for dealing with these kinds of problems, but first let's discuss the properties of electric fields in and near isolated conductors. First of all, the electric field inside an isolated conductor will vanish, because if there where any electric field, the charges in the conductor will move so as to cancel out that electric field. Since the electric field is zero inside the conductor, we can conclude using Gauss's Law for any volume inside the conductor, that the charge inside that volume must also be zero. Therefore any charge on a conductor must lie entirely on its surface.

Moreover, the surface of any conductor must be an equipotenial. If it were not an equipotential, that would mean that there would be an electric field on the surface of the conductor which is tangential to the surface of the conductor. But the charges in the conductor will move to cancel out any such tangential electric fields on the surface just as they move to cancel out any electric fields inside the conductor. Therefore an isolated conductor is an equipotential, and the electric field must everywhere be normal to the surface of the conductor. Using Gauss's Law for a small pillbox on the surface of any conductor, we can easily deduce that the component of the electric field normal to the surface, E_n, is related to the surface charge on the conductor by

$$E_n=-\frac{\partial\phi}{\partial n}=-\hat{\mathbf{n}}\cdot\nabla\phi=\frac{\sigma}{\varepsilon_o} \tag{19.2.5}$$

In (19.2.5) we have assumed that the normal $\hat{\mathbf{n}}$ points *out of* the conductor.

19.3 The Uniqueness Theorem

Before we go any further, we pause and prove the Uniqueness Theorem. The Uniqueness Theorem says that if we find by hook or crook any solution to the above problem that has the correct value of $\nabla^2\phi(\mathbf{r})$ in the space outside of the conductor and has the specified value on the surface of the conductor, then that is the solution, because there is one and only one such solution. The usefulness of this theorem is that it allows us to find a solution using any kind of trick we can think of (the method of images discussed below is such a trick), and we know that if we have one, we are done, it is the only possible solution.

Proof of the Uniqueness Theorem: Suppose we have two solutions to our boundary value problem, $\phi_1(\mathbf{r})$ and $\phi_2(\mathbf{r})$ that satisfy our requirements. That is,

$$\nabla^2\phi_1(\mathbf{r}) = -\rho(\mathbf{r})/\varepsilon_o \quad \text{and} \quad \nabla^2\phi_2(\mathbf{r}) = -\rho(\mathbf{r})/\varepsilon_o \tag{19.3.1}$$

$$\phi_1(\mathbf{r})\big|_{\mathbf{r}\text{ on surface of condutor}} = \phi_2(\mathbf{r})\big|_{\mathbf{r}\text{ on surface of condutor}} = \text{given function} \tag{19.3.2}$$

Define the function $W(\mathbf{r})$ to be the difference between these two solutions.

$$W(\mathbf{r}) = \phi_2(\mathbf{r}) - \phi_1(\mathbf{r}) \tag{19.3.3}$$

Then we must have that

$$\nabla^2 W = 0 \text{ outside of conductor} \quad \text{and} \quad W(\mathbf{r})\big|_{\text{r on surface of conductor}} = 0 \tag{19.3.4}$$

We have the general vector identity for any scalar potential that

$$\nabla\cdot(W\nabla W) = (\nabla W)^2 + W\nabla^2 W \tag{19.3.5}$$

We apply Gauss's Theorem to $\nabla\cdot(W\nabla W)$ for the volume outside of the conductor and the surface of the conductor, as follows

$$\int_{\text{vol where }\rho\text{ defined}} \nabla\cdot(W\nabla W)d^3x = \int_{\text{bounding surface}} (W\nabla W)\cdot\hat{\mathbf{n}}\,da \tag{19.3.6}$$

But we know that W is zero on the surface of the conductor, so we have

$$\int_{\text{vol where }\rho\text{ defined}} \nabla\cdot(W\nabla W)d^3x = \int_{\text{vol where }\rho\text{ defined}} \left[(\nabla W)^2 + W\nabla^2 W\right] d^3x = 0 \tag{19.3.7}$$

where we have used (19.3.5) to get the final form in (19.3.7). But we have that $\nabla^2 W = 0$ in the volume, so we must have

$$\int_{\text{vol where } \rho \text{ defined}} \left[\left(\nabla W \right)^2 \right] d^3x = 0 \tag{19.3.8}$$

Since $\left(\nabla W \right)^2$ is positive definite, we must have that everywhere in the volume ∇W =0. Thus W can only be a constant, and since it is zero on the surface of the conductor, that constant must be zero. Therefore $\phi_1(\mathbf{r})$ and $\phi_2(\mathbf{r})$ are the same, and the solution is unique.

19.4 The Method of Images

The method of images is basically a trick that takes advantage of uniqueness. It can only be used in situations with high degrees of symmetry. Basically you take the problem as stated, which we call Problem I, which has a conductor with some unknown surface charge induced by the presence of the known free charge. You find a ***different free space*** problem, which we call Problem II, ***which has no conductor***, where you arrange charges so that you get the proper value of the potential on the surface of the conductor from Problem I. We then take that part of Problem II relevant to Problem I and transfer it back to Problem I. Since it satisfies all the conditions for Problem I, we know it must be the solution to Problem I. The best way to illustrate how this works is to take examples, as follows.

19.4.1 Point Charge and a Conducting Plane

Suppose a point charge *q* is on the z-axis a distance *d* above the origin. The *xy* plane at *z* = 0 is an infinite grounded conducting plane. This is Problem I, as illustrated in Figure 19-2.

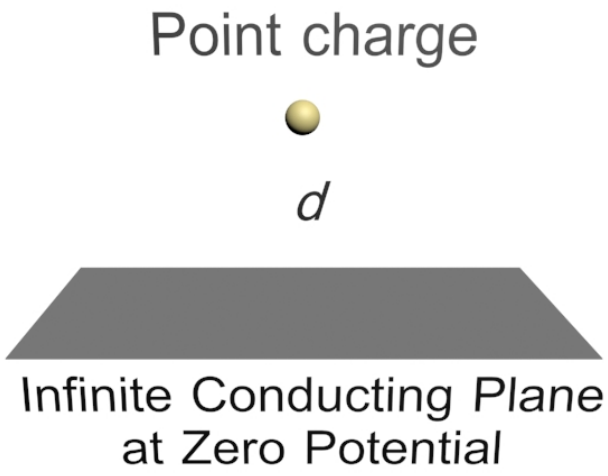

Figure 19-2: Problem I: A point charge above in infinite grounded couducting plane.

Here is Problem II, as illustrated in Figure 19-3. What we are doing with Problem II is to duplicate the charge distribution for $z > 0$ that we have in Problem I, and arrange charges in the region where the conductor is in Problem I ($z \le 0$) so that we satisfy the boundary conditions at $z = 0$ from Problem I, that is zero potential at $z = 0$. We simply guess that if we put a charge of $-q$ in Problem II a distance d down the negative z-axis, we will get zero potential in Problem II at $z = 0$.

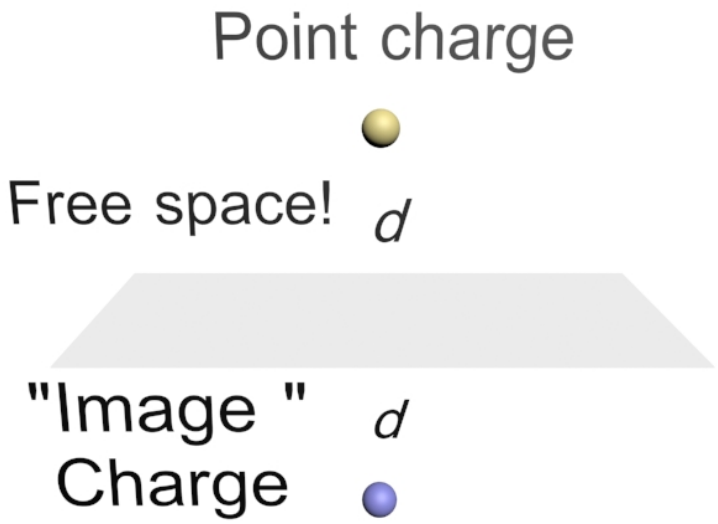

Figure 19-3: Problem II: A free space problem with two charges

And indeed this is the case. If we write down the solution for the potential for Problem II, it is

$$\phi_{II}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right] \tag{19.4.1}$$

This potential has the right behavior at $z = 0$ (it is zero there), and

$$\nabla^2\phi_{II}(\mathbf{r}) = q\left[\delta^3(\mathbf{r} - d\,\hat{\mathbf{z}}) - \delta^3(\mathbf{r} + d\,\hat{\mathbf{z}})\right]/\varepsilon_o \tag{19.4.2}$$

so in the upper half plane it also has the right value for $\nabla^2\phi(\mathbf{r})$, a delta function at the position of the point charge q. It also has a delta function at the position of the negative point charge, but that is not in the upper half plane and we don't care about that one. That is why we are free to put any charges for z < 0 that we want to try to satisfy the boundary condition at $z = 0$—they don't count.

So the Uniqueness Theorem guarantees that if we simply take over to Problem I the solution to Problem II in the upper half plane, and put the potential for Problem I to zero for $z < 0$, then we have the unique solution to Problem I, which is

$$\phi_I(\mathbf{r}) = \begin{cases} \dfrac{1}{4\pi\varepsilon_o}\left[\dfrac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \dfrac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right] & z > 0 \\ 0 \quad z \le 0 \end{cases} \tag{19.4.3}$$

Once we have the solution to Problem I, we can go back and see what the mysterious surface charge induced on the conducting plane by the presence of the free charge q at $d\,\hat{\mathbf{z}}$ is. Using (19.2.5) we see that

$$\sigma = -\varepsilon_o\frac{\partial\phi_I}{\partial n} \qquad \text{or} \qquad \sigma = -\varepsilon_o\left.\frac{\partial\phi_I}{\partial z}\right|_{z=0+} \tag{19.4.4}$$

But

$$\left.\frac{\partial\phi_I}{\partial z}\right|_{z\ge 0} = \frac{1}{4\pi\varepsilon_o}\left[-\frac{q(z-d)}{\left(x^2+y^2+(z-d)^2\right)^{3/2}} + \frac{q(z+d)}{\left(x^2+y^2+(z+d)^2\right)^{3/2}}\right] \tag{19.4.5}$$

so

$$\sigma(x,y) = -\frac{1}{2\pi}\frac{qd}{\left(x^2+y^2+d^2\right)^{3/2}} \tag{19.4.6}$$

If we want to know the total induced charge on the plane, it is

$$\int_0^\infty dr \int_0^{2\pi} rd\phi\sigma = -\int_0^\infty \frac{dr\left(rqd\right)}{\left(r^2+d^2\right)^{3/2}} = +\frac{qd}{\left(r^2+d^2\right)^{1/2}}\Bigg|_0^\infty = -q \tag{19.4.7}$$

19.4.2 Point Charge outside a Conducting Grounded Spherical Shell

We give one more example of using image charges. A point charge of charge q is located at position $\mathbf{r}'$ outside of a grounded conducting sphere of radius R. The point charge is a distance $r' > R$ from the center of the shell. Using the method of images, we find that the potential outside the sphere is the potential due to the point charge plus the potential due to an image charge with charge $-qR/r'$ located inside the shell at a distance R^2/r'. Therefore our solution is

$$\phi(\mathbf{r}) = \begin{cases} \dfrac{1}{4\pi\varepsilon_o}\left[\dfrac{q}{|\mathbf{r}-\mathbf{r}'|} - \dfrac{qR}{r'}\dfrac{1}{\left|\mathbf{r}-\dfrac{\mathbf{r}'R^2}{(r')^2}\right|}\right] & r > R \\ 0 & r \le R \end{cases} \tag{19.4.8}$$

19.5 Separation of Variables in Spherical Coordinates

Now we turn to another standard technique, which has much wider usage than the image charge method. What we do is look at solutions to Laplaces's equation, $\nabla^2\phi(\mathbf{r}) = 0$, and investiage the form of solutions to this equation in various coordinate systems. Although we could do this in cartesion and cylindrical coordinates, we will only look at spherical coordinates in these notes, as it is illustrative of the general technique. You might ask what good finding solutions to $\nabla^2\phi(\mathbf{r}) = 0$ is. But in many problems, $\nabla^2\phi(\mathbf{r})$ is zero almost everywhere, and we try to piece together a solution to a general problem, which charge located in limited regions, out of solutions to $\nabla^2\phi(\mathbf{r}) = 0$, appropriately chosen.

19.5.1 A Typical Problem Involving Seperation of Variables

Here is a typical boundary value problem where separation of variables is useful. We have a sphere of radius R, and on the surface of the sphere we know the values of the potential. That is, some one has given us the function $f(\theta,\phi)$, such that on the surface of the sphere

$$\phi(r,\theta,\phi)\big|_{r=R} = f(\theta,\phi) \tag{19.5.1}$$

We also know that the potential vanishes at infinity, and that is no free charge for $r > R$. Given all this, we want to find the potential for all $r \geq R$. Since our boundary condition is given in spherical coordinates, we see if we can find a solution by adding up many solutions to $\nabla^2\phi(\mathbf{r}) = 0$ in spherical coordinates. If we can satisfy all of the boundary conditions, then uniqueness tells us that out solution is the only solution, no matter how we come by it.

Laplace's equation in spherical polar coordinates is

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial^2\phi} = 0 \qquad (19.5.2)$$

We are going to assume that our solution $\phi(r,\theta,\phi)$ is separable, that is, that it can be written as

$$\phi(r,\theta,\phi) = R(r)P(\theta)W(\phi) \qquad (19.5.3)$$

The reason we might think it is separable in these coordinates is that our boundary condition is in these coordinates. But the ultimate rationale is that we can find solutions when we make this assumption that satisfy our boundary conditions, and $\nabla^2\phi(\mathbf{r}) = 0$, and uniqueness tells us that this is the solution. If we insert (19.5.3) into (19.5.2) and divide by $\frac{r^2\sin^2\theta}{R(r)P(\theta)W(\phi)}$, we have

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{1}{W}\frac{d^2W}{d^2\phi} = 0 \qquad (19.5.4)$$

We have isolated all of the ϕ dependence in (19.5.4) in the last term, and the only way this can be true is if there is a constant m^2 such that

$$\frac{1}{W}\frac{d^2W}{d^2\phi} = -m^2 \qquad (19.5.5)$$

The solutions to (19.5.5) are $\sin m\phi$ or $\cos m\phi$. In order for W to be single valued, we must have that m be an integer. Our remaining equation is

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = 0 \qquad (19.5.6)$$

where we have divided by $\sin^2\theta$ so as to again isolate all of the *r* dependence in the first term of (19.5.6). Again, this means that there must be a constant which we call $l(l+1)$ such that

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1) \tag{19.5.7}$$

The solutions $R(r)$ are

$$R(r)=Ar^l+Br^{-l-1} \tag{19.5.8}$$

This leaves the remaining equation for *P* as

$$\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+l(l+1)-\frac{m^2}{\sin^2\theta}=0 \tag{19.5.9}$$

If we make the substitution $x=\cos\theta$, the equation for P becomes

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P=0 \tag{19.5.10}$$

This is the generalized Legendre equation and its solutions are called the associated Legendre polynomials.

The Legendre functions are the solutions to the above equation with *m* = 0, that is

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right]+l(l+1)P=0 \tag{19.5.11}$$

If we impose the requirement that solutions to the above equation converge for $x^2\le 1$, then this requires that *l* be zero or a positive integer[5]. By convention our functions are normalized to have the value of unity at $x=+1$. The first few solutions, the Legendre polynomials, are given by

$$\begin{aligned} &P_o(x)=1 \qquad P_1(x)=x \qquad P_2(x)=\frac{1}{2}\left(3x^2-1\right)\\ &P_3(x)=\frac{1}{2}\left(5x^3-3x\right) \qquad P_4(x)=\frac{1}{8}\left(35x^4-30x^2+3\right)\end{aligned} \tag{19.5.12}$$

In general one can show that

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[5] See Jackson 2nd Edition, page 86.

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2 - 1\right)^l \tag{19.5.13}$$

This formula is known as *Rodrigues' formula*. From Rodriques' formula it can be shown that

$$\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} - (2l+1)P_l = 0 \tag{19.5.14}$$

This equation together with the differential equation can be use to show that

$$P_{l+1} = \frac{(2l+1)x\,P_l - lP_{l-1}}{(l+1)} \tag{19.5.15}$$

This recursion relation for Legendre polynomials is very useful for any numerical work with the Legendre polynomials because you need only know that $P_0(x) = 1$ and $P_1(x) = x$ and you can find the value of $P_l(x)$ at any *x* for $l > 1$ by applying this recursion relation $l - 1$ times.

If the solutions to the associated Legendre equation, where *m* is not equal to zero, are to converge for $x^2 \le 1$, we find similarly that *l* must be zero or a positive integer and that the integer m can only take on the values –*l*, -(*l*-1),….,0….,(*l*-1), *l*. The associated Legendre function $P_l^m(x)$ is given by

$$P_l^m(x) = (-1)^m \left(1 - x^2\right)^{m/2} \frac{d^m}{dx^m} P_l(x) \tag{19.5.16}$$

Since the differential equation depends only on m^2, P_l^m and P_l^{-m} must be proportional, and it can be shown that

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \tag{19.5.17}$$

$$(\ell - m + 1)P_{\ell+1}^m(x) - (2\ell+1)xP_\ell^m(x) + (\ell + m)P_{\ell-1}^m(x) = 0 \tag{19.5.18}$$

$$P_{\ell+1}^m(x) = \frac{(2\ell+1)xP_\ell^m(x) + (\ell+m)P_{\ell-1}^m(x)}{(\ell - m + 1)} \tag{19.5.19}$$

$$(\ell - m + 1)P_{\ell+1}^m(x) + (1-x^2)^{1/2} P_\ell^{m+1}(x) - (\ell + m + 1)xP_\ell^m(x) = 0 \tag{19.5.20}$$

For a given *m*, the set of functions $\left\{P_l^m(x)\right\}$ is a complete set of functions, with normalization given by

$$\int_{-1}^{1} P_l^m(x)P_{l'}^m(x)dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'} \tag{19.5.21}$$

19.5.2 Spherical Harmonics

It is usual to combine the θ and ϕ to define the spherical harmonics $Y_{lm}(\theta,\phi)$ defined by

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{+im\phi} \tag{19.5.22}$$

The Y_{lm}'s form a complete set of functions in θ and ϕ. The normalization and orthorgonality conditions are

$$\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi} d\phi Y_{lm}^*(\theta,\phi)Y_{l'm'}(\theta,\phi) = \delta_{l'l}\delta_{m'm} \tag{19.5.23}$$

We can expand any function $g(\theta,\phi)$ as

$$g(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}Y_{lm}(\theta,\phi) \tag{19.5.24}$$

where

$$A_{lm} = \int_0^{\pi}\sin\theta d\theta\int_0^{2\pi} d\phi\, Y_{lm}^*(\theta,\phi)\,g(\theta,\phi) \tag{19.5.25}$$

With the definition of the Y_{lm}'s and (19.5.8), we thus see that the most general form of the solutions to $\nabla^2\phi(\mathbf{r}) = 0$ that is separable in spherical coordinates is given by

$$\phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^l + B_{lm}r^{-l-1}\right]Y_{lm}(\theta,\phi) \tag{19.5.26}$$

19.5.3 The Solution to the Typical Problem

We now return to our "typical" problem above, and see if we can find a solution to that problem that is of the form given in (19.5.26). If we want to prevent the potential from blowing up at infinity, we must take all of our A_{lm} to be zero. To satisfy our boundary condition at the surface of the sphere (19.5.1) we must have

$$f(\theta,\phi)=\phi(r,\theta,\phi)\big|_{r=R}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[B_{lm}R^{-l-1}\right]Y_{lm}(\theta,\phi) \tag{19.5.27}$$

But since we know that the Y_{lm}'s are complete, we know this can be done, and in fact we can write down the coefficients as follows

$$B_{lm}=R^{l+1}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi\,Y_{lm}^{*}(\theta,\phi)\,f(\theta,\phi) \tag{19.5.28}$$

And we are done. We have solved the "typical" problem involving separation of variables in spherical coordinates.

19.5.4 Azimuthal Symmetry

Let us suppose that we have a problem with azimuthal symmetry. Then our integer *m* above must be zero, and our complete solution given by (19.5.26) reduces to

$$\phi(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{19.5.29}$$

Our normalization condition for the P_l's in this case (cf. (19.5.21) for $m = 0$) is

$$\int_{-1}^{1}P_l(x)P_{l'}(x)dx=\frac{2}{2l+1}\delta_{ll'} \tag{19.5.30}$$

It is clear that we can do any potential problem where the potential at $r = R$ is given as a function of θ using the same technique as above. But there are other interesting things to do as well.

Consider the following problem. We put a surface charge $\sigma(\theta)=\sigma_o P_n(\cos\theta)$ on the surface of a spherical shell. There are no charges inside or outside the shell other than these charges. What is the potential and the electric field everywhere? The lack of charges other than at the shell means that the potential must vanish at infinity and not blow up at the origin. Since we also know that the potential must be continous across $r = R$, we therefore we must have

$$\phi(r,\theta)=\begin{cases}\displaystyle\sum_{l=0}^{\infty}A_l\left(\frac{r}{R}\right)^l P_l(\cos\theta)\\ \displaystyle\sum_{l=0}^{\infty}A_l\left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta)\end{cases} \tag{19.5.31}$$

Gauss's Law applied to a pill box on the surface of the sphere yields

$$E_{r=R+}-E_{r=R-}=\sigma(\theta)/\varepsilon_o=\left.\frac{\partial\phi}{\partial r}\right|_{r=R-}-\left.\frac{\partial\phi}{\partial r}\right|_{r=R+}=\sum_{l=0}^{\infty}\left(\frac{A_l l}{R}+\frac{A_l(l+1)}{R}\right)P_l(\cos\theta) \quad (19.5.32)$$

$$\sigma_0 P_n(\theta)/\varepsilon_o=\sum_{l=0}^{\infty}\frac{A_l(2l+1)}{R}P_l(\cos\theta) \quad (19.5.33)$$

Since the P_l's are orthogonal, we must there have all the A_l's are zero except A_n, with $A_n=\dfrac{\sigma_0 R}{(2n+1)\varepsilon_o}$. Therefore our solution (19.5.31) is

$$\phi(r,\theta)=\frac{\sigma_0 R}{\varepsilon_o(2n+1)}\begin{cases}\left(\dfrac{r}{R}\right)^n P_n(\cos\theta) & r<R\\ \left(\dfrac{R}{r}\right)^{n+1} P_n(\cos\theta) & r>R\end{cases} \quad (19.5.34)$$

As an example of (19.5.34), suppose we take $n = 0$, that is the surface charge density on the surface of the sphere is constant. Then we have for this case

$$\phi(r,\theta)=\frac{\sigma_0 R}{\varepsilon_o}\begin{cases}1 & r<R\\ \left(\dfrac{R}{r}\right) & r>R\end{cases}=\frac{\sigma_0 4\pi R^2}{4\pi\varepsilon_o}\begin{cases}\dfrac{1}{R} & r<R\\ \dfrac{1}{r} & r>R\end{cases}=\frac{q_o}{4\pi\varepsilon_o}\begin{cases}\dfrac{1}{R} & r<R\\ \dfrac{1}{r} & r>R\end{cases} \quad (19.5.35)$$

where $q_o=4\pi R^2$. This is what we expect for a uniformly charged spherical shell.

As a second example of (19.5.34), suppose we take $n = 1$, that is the surface charge density on the surface of the sphere goes as $\cos\theta$. Then we have for this case

$$\phi(r,\theta)=\begin{cases}\dfrac{\sigma_0}{3\varepsilon_o}\,r\cos\theta & r<R\\ \dfrac{\sigma_0 R^3}{3\varepsilon_o}\dfrac{\cos\theta}{r^2} & r>R\end{cases} \quad (19.5.36)$$

This potential gives the following electric field

$$\mathbf{E}(r,\theta)=\phi(r,\theta)=\begin{cases} -E_0\hat{\mathbf{z}} & r<R \\ \dfrac{p_0}{4\pi\varepsilon_o}\left(\dfrac{2\cos\theta}{r^3}\hat{\mathbf{r}}+\dfrac{\sin\theta}{r^3}\hat{\boldsymbol{\theta}}\right) & r>R \end{cases} \tag{19.5.37}$$

where $E_o=\dfrac{\sigma_0}{3\varepsilon_o}$ and $p_o=\dfrac{4\pi\sigma_0 R^3}{3}$. Thus we have a constant and downward field inside the sphere and a perfect dipole outside the sphere, if our surface charge goes as $\cos\theta$.

19.6 Boundary Conditions on the Electric Field

Before leaving electrostatics, we discuss the boundary conditions that must be true across any thin interface in electrostatics. From Gauss'a Law, $\nabla\cdot\mathbf{E}(\mathbf{r})=\rho(\mathbf{r})/\varepsilon_o$, we see that any change in the normal component of the electric field across a thin interface must be given by

$$E_{2n}-E_{1n}=\hat{\mathbf{n}}\cdot\left(\mathbf{E}_2-\mathbf{E}_1\right)=\sigma/\varepsilon_o \tag{19.5.38}$$

where the normal $\hat{\mathbf{n}}$ is assumed to point *from* 1 *to* 2.

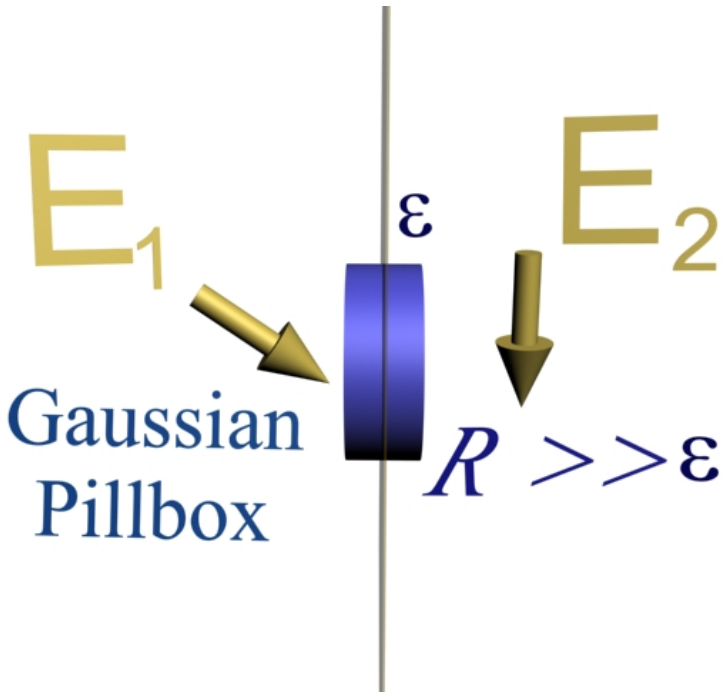

Figure 19-4: Boundary condition on the normal component of E across an interface

To make absolutely sure we understand where (19.5.38) comes from, consider the figure above. We apply Gauss's Law to the pillbox shown, making sure that the radius R of the pillbox is arbitrarily large compared to its height ε. The area of the sides of the

pillbox is therefore given by $2\pi R\varepsilon$ and the area of the ends is given by πR^2. By making ε small enough compared to *R*, we can insure that any flux through the sides of the pillbox due to tangential components of **E** do not contribute to the integral over the surface area of the pillbox. Only the normal components of **E** contribute to the integral, and we easily obtain (19.5.38).

What about the tangential components of **E**? These must be continuous, even in a time varying situation. Take an amperean loop that spans the interface, with width ε perpendicular to the interface and length *l* tangential to the interface, as shown in the figure below. If we use Faraday's Law, and integrate around this loop, we have

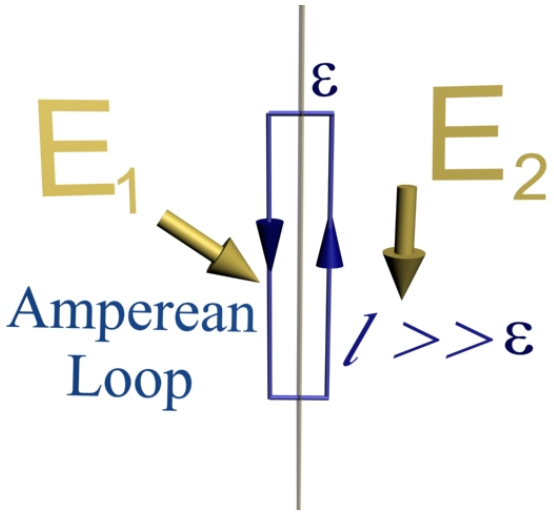

Figure 19-5: Boundary condition on the tangential component of E across an interface.

$$\oint \mathbf{E}\cdot d\,\mathbf{l} = \int_{\text{open surface}} \frac{\partial \mathbf{B}}{\partial t}\cdot \hat{\mathbf{n}}\, da \qquad (19.5.39)$$

If we make the width ε very small compared to the length *l*, the only component that will enter into the left hand side of (19.5.39) will be the tangential **E**, since any normal component will be multiplied by $\varepsilon << l$. And the magnitude of that component will be $E_t\, l$. If we look at the right hand side of (19.5.39), it involves an area $l\varepsilon$, and again if we make $\varepsilon << l$ we can make the area integral insignificant, even in a time varying situation. Therefore we will always have for the electric field that

$$\mathbf{E}_{2t} = \mathbf{E}_{1t} \qquad (19.5.40)$$