DETACH THESE SHEETS & USE AT YOUR CONVENIENCE

Maxwell in differential form:

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_o}$$

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla\times\mathbf{B} = \mu_o\mathbf{J} + \mu_o\varepsilon_o\frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla\cdot\mathbf{B} = 0$$

Maxwell in integral form:

$$\int_{\substack{\text{closed}\\\text{surface}}} \mathbf{E}\cdot\hat{\mathbf{n}}\,da = \frac{q_{\text{enclosed}}}{\varepsilon_o}$$

$$\int_{\substack{\text{bounding}\\\text{contour}}} \mathbf{E}\cdot d\,\mathbf{l} = -\frac{d}{dt}\int_{\substack{\text{open}\\\text{surface}}} \mathbf{B}\cdot\hat{\mathbf{n}}\,da$$

$$\int_{\substack{\text{bounding}\\\text{contour}}} \mathbf{B}\cdot d\,\mathbf{l} = \mu_o\left[I_{\text{through}} + \varepsilon_o\frac{d}{dt}\int_{\substack{\text{open}\\\text{surface}}} \mathbf{E}\cdot\hat{\mathbf{n}}\,da\right]$$

$$\int_{\substack{\text{closed}\\\text{surface}}} \mathbf{B}\cdot\hat{\mathbf{n}}\,da = 0$$

$$C = \frac{Q}{|\Delta V|} \qquad \Delta V = -\int_a^b \mathbf{E}\cdot d\vec{\mathbf{l}}$$

$$Energy = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$$

$$\Delta V = IR$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A} = -\mu_o\mathbf{J}$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi = -\frac{\rho}{\varepsilon_o}$$

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$$

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\,\varepsilon_o}\int\frac{\rho(\mathbf{r}',t'_{ret})}{|\,\mathbf{r}-\mathbf{r}'\,|}d^3x'$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t'_{ret})}{|\,\mathbf{r}-\mathbf{r}'\,|}d^3x'$$

$$t'_{ret} = t - |\,\mathbf{r}-\mathbf{r}'\,|/c$$

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{r^3}\frac{\left[3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}})-\mathbf{p}\right]}{4\pi\,\varepsilon_o} + \frac{1}{c\,r^2}\frac{\left[3\hat{\mathbf{n}}(\dot{\mathbf{p}}\cdot\hat{\mathbf{n}})-\dot{\mathbf{p}}\right]}{4\pi\,\varepsilon_o} + \frac{1}{rc^2}\frac{(\ddot{\mathbf{p}}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{4\pi\,\varepsilon_o}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\left[\frac{\dot{\mathbf{p}}}{r^2} + \frac{\ddot{\mathbf{p}}}{c\,r}\right]\times\hat{\mathbf{n}}$$

all evaluated at $t' = t - r/c$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon_o E^2 + \frac{B^2}{2\mu_o}\right] + \nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right) = -\mathbf{E}\cdot\mathbf{J}$$

Energy density of field $= \frac{1}{2}\varepsilon_o E^2 + \frac{B^2}{2\mu_o}$

Energy flux density = $\dfrac{\mathbf{E}\times\mathbf{B}}{\mu_o}$

If $\overleftrightarrow{\mathbf{T}}$ is a second rank tensor, then $\mathbf{C}\cdot\overleftrightarrow{\mathbf{T}}$ and $\overleftrightarrow{\mathbf{T}}\cdot\mathbf{C}$ are both vectors, where

$$\left(\mathbf{C}\cdot\overleftrightarrow{\mathbf{T}}\right)_i = C_j T_{ji}$$

$$\left(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{C}\right)_i = C_j T_{ij}$$

$$\frac{\partial}{\partial t}\left[\varepsilon_o\mathbf{E}\times\mathbf{B}\right] + \nabla\cdot\left(-\overleftrightarrow{\mathbf{T}}\right) = -\left[\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\right]$$

Momentum density = $\varepsilon_o\mathbf{E}\times\mathbf{B}$

Momentum flux density = $-\overleftrightarrow{\mathbf{T}}$

$$\frac{\partial}{\partial t}\left[\varepsilon_o \mathbf{r}\times\left(\mathbf{E}\times\mathbf{B}\right)\right]+\nabla\cdot\left(-\mathbf{r}\times\overleftrightarrow{\mathbf{T}}\right)=-\mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right]$$

Angular momentum density = $\varepsilon_o \mathbf{r}\times\left(\mathbf{E}\times\mathbf{B}\right)$

Angular momentum flux density = $-\mathbf{r}\times\overleftrightarrow{\mathbf{T}}$

$$\overleftrightarrow{\mathbf{T}}=\varepsilon_o\left[\mathbf{EE}-\frac{1}{2}\overleftrightarrow{\mathbf{I}}\,E^2\right]+\frac{1}{\mu_o}\left[\mathbf{BB}-\frac{1}{2}\overleftrightarrow{\mathbf{I}}\,B^2\right]$$

$\overleftrightarrow{\mathbf{T}}_{\mathbf{E}}\cdot\hat{\mathbf{n}}$ lies in the plane defined by $\mathbf{E}$ and $\hat{\mathbf{n}}$.

The magnitude of $\overleftrightarrow{\mathbf{T}}_{\mathbf{E}}\cdot\hat{\mathbf{n}}$ is always $\frac{1}{2}\varepsilon_o E^2$

If you go an angle θ to get to $\mathbf{E}$ from $\hat{\mathbf{n}}$, go an angle 2θ in the same sense to get to $\overleftrightarrow{\mathbf{T}}_{\mathbf{E}}\cdot\hat{\mathbf{n}}$

All of the above apply to $\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\cdot\hat{\mathbf{n}}$ except for the fact that $\left|\overleftrightarrow{\mathbf{T}}_{\mathbf{B}}\cdot\hat{\mathbf{n}}\right|$ is always $B^2/2\mu_o$

$$\mathbf{p}(t')=\int\mathbf{r}'\rho(\mathbf{r}',t')d^3x'$$

$$\frac{dW}{d\Omega\ dt}=\frac{c}{\mu_o}(\mathbf{E}\times\mathbf{B})\cdot(r^2\,\hat{\mathbf{r}})$$

$$\frac{dW_{el\ dip}}{d\Omega\,dt}=\frac{c\,r^2}{\mu_o}\left[\frac{\mu_o}{4\pi}\frac{\ddot{\mathbf{p}}\,\mathbf{x}\,\hat{\mathbf{n}}}{c\ r}\right]^2=\frac{\mu_o\,\ddot{p}^{\,2}}{(4\pi)^2 c}\sin^2\theta$$

$$\frac{dW_{el\ dip}}{dt}=\frac{1}{4\pi\,\varepsilon_o}\frac{2}{3}\frac{\left|\ddot{\mathbf{p}}\right|^2}{c^3}$$

$$\frac{dW_{el\ dip}}{dt}=\frac{1}{4\pi\,\varepsilon_o}\frac{2}{3}\frac{q^2a^2}{c^3}$$

$$\mathbf{m}(t')=\frac{1}{2}\int\mathbf{r}'\times\mathbf{J}(\mathbf{r}',t')d^3x'$$

$$\frac{dW_{mag\ dip}}{dt}=\frac{\mu_o}{4\pi}\frac{2\left|\ddot{\mathbf{m}}\right|^2}{3c^3}$$

$$\rho\left(\mathbf{r},t\right)=q\delta^3\left(\mathbf{r}-\mathbf{X}(t)\right)$$

$$\mathbf{J}\left(\mathbf{r},t\right)=q\mathbf{u}\left(t\right)\delta^3\left(\mathbf{r}-\mathbf{X}(t)\right)$$

$$\mathbf{u}\left(t\right)=\frac{d}{dt}\mathbf{X}(t)$$

$$\mathbf{p}(t')=\int\mathbf{r}'\rho(\mathbf{r}',t')d^3x'=q\mathbf{X}(t')$$

$$\mathbf{m}(t')=\frac{1}{2}\int\mathbf{r}'\times\mathbf{J}(\mathbf{r}',t')d^3x'=\frac{q}{2}\mathbf{X}\left(t'\right)\times\mathbf{u}\left(t'\right)$$

For a localized distribution of current,

$$\int_{\textbf{all space}}\mathbf{J}(\mathbf{r}',t')d^3x'=-\int_{\textbf{all space}}\left(\mathbf{r}'\left[\nabla'\cdot\mathbf{J}(\mathbf{r}',t')\right]\right)d^3x'$$

$$\frac{\partial\rho(\mathbf{r}',t')}{\partial t'}+\nabla'\cdot\mathbf{J}(\mathbf{r}',t')=0$$

$$\left|\mathbf{r}-\mathbf{r}'\right|=r-\hat{\mathbf{n}}\cdot\mathbf{r}'+\ldots.\qquad\hat{\mathbf{n}}=\mathbf{r}/\left|\mathbf{r}\right|$$

$$\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}=\frac{1}{r}+\frac{\hat{\mathbf{n}}\cdot\mathbf{r}'}{r^2}+\ldots.$$

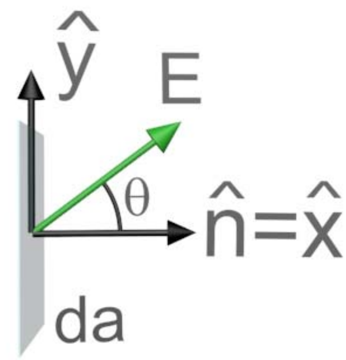
$$E_{radiation}=cB_{radiation}$$

$$\sin 2\theta=2\sin\theta\cos\theta$$

$$\cos 2\theta=2\cos^2\theta-1$$

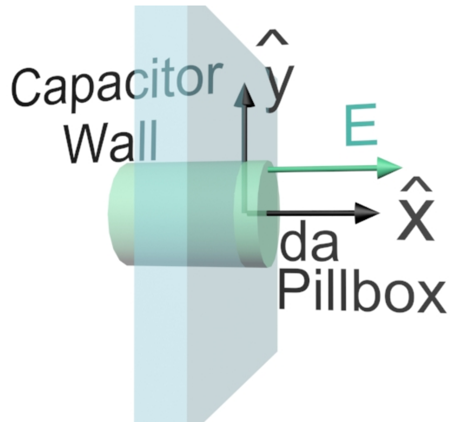
Problem 1

We have an imaginary vector area $d\vec{\mathbf{a}} = da\,\hat{\mathbf{n}} = da\,\hat{\mathbf{x}}$, as shown in the sketch. The electric field $\mathbf{E}$ makes an angle θ with respect to the $\hat{\mathbf{n}} = \hat{\mathbf{x}}$ direction, and the $\hat{\mathbf{y}}$ axis is in the plane formed by $\mathbf{E}$ and $\hat{\mathbf{n}}$, as indicated.

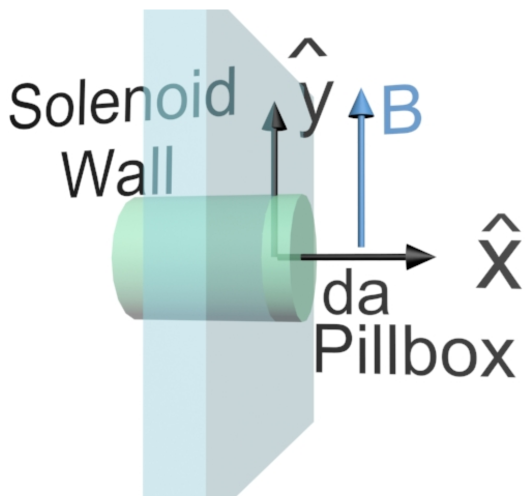


a. If the second rank tensor $\overleftrightarrow{\mathbf{T}} = \varepsilon_o \left[\mathbf{EE} - \frac{1}{2} \overleftrightarrow{\mathbf{I}} E^2 \right]$, what are the x, y, and z components of vector $\overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{n}} = \overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{x}}$ in terms of the quantities given? You must ***derive*** your answers from the definition of $\overleftrightarrow{\mathbf{T}}$ and $\overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{n}}$ and ***show*** the derivation, to get credit.

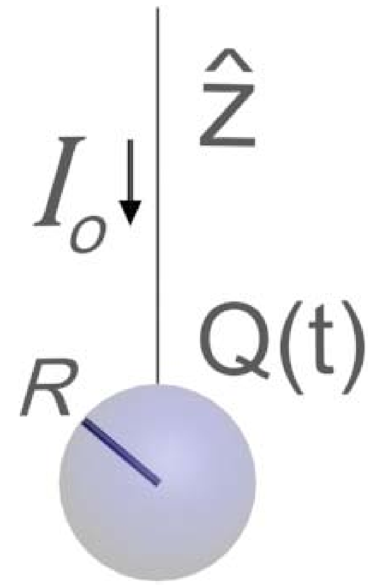
b. One side of a capacitor plate is shown in the figure, along with an imaginary pill-box enclosing the plate, with the right end of the pill box extending into the space between the capacitor plates. The electric field strength between the plates is E with the orientation shown. What is the value of $\oint \overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{n}}\, da$ integrated over the pillbox. The area of the top and bottom of the pillbox is da. The electric field is zero in the region to the left of the capacitor wall shown. Justify your answer.



c. One side of a solenoid is shown in the figure, along with an imaginary pill-box enclosing the solenoid wall, with the right end of the pill box extending into the space interior to the solenoid. The magnetic field strength in the interior of the solenoid is B, with the orientation shown. What is the value of $\oint \overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{n}}\, da$ integrated over the pillbox? The area of the top and bottom of the pillbox is da. The magnetic field is zero in the region to the left of the solenoid wall shown. Justify your answer.



Problem 2:



A metal sphere of radius R whose center is at the origin is being charged by a constant current I_o carried by a wire extending from the surface of the sphere vertically upwards to infinity along the z-axis (see sketch). The current in the wire is flowing downward in the $-\hat{\mathbf{z}}$ direction, and the current density $\mathbf{J}_{wire}$ is given by

$$\mathbf{J}_{wire} = -I_o \delta(x)\delta(y)\,\hat{\mathbf{z}} \quad \text{for } z \geq R$$

We have $dQ/dt = I_o \to Q(t) = I_o t$.

Once the charge carried by the wire arrives at the surface of the sphere, we assume that it spreads out uniformly over the surface of the sphere, carried by currents on the sphere which are ***always perpendicular*** to the radial direction.

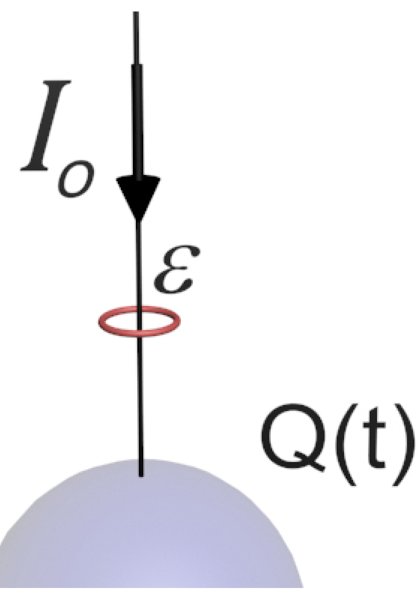
We also assume that the electric field for this situation is just the quasi-static field, that is, we assume that

$$\mathbf{E}(\mathbf{r},t)) = \begin{cases} \dfrac{Q(t)}{4\pi\varepsilon_o r^2}\hat{\mathbf{r}} & r \geq R \\ 0 & r < R \end{cases}$$

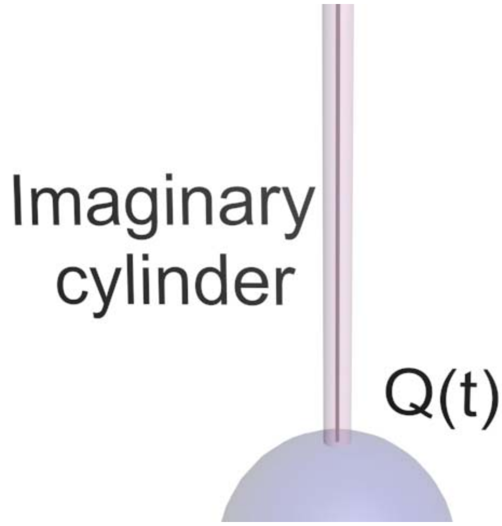
a. What is the total electric energy stored at time t, e.g. what is $\int_{\text{all space}} \frac{1}{2}\varepsilon_o E^2 \, d^3x$, in terms of the quantities given?

b. What is the rate at which electromagnetic energy is being created over all space, that is what is $\int_{\textbf{all space}} -\mathbf{J}\cdot\mathbf{E}\, d^3x$?

c. We want to find the magnetic field just outside the wire on a small circle of radius ε at a distance $z > R$ up the z-axis. Find this magnetic field by using the integral form of the Ampere-Maxwell equation applied to the Amperean loop with radius ε shown in the figure. Why can we neglect the displacement term $\propto dE/dt$ when we find the expression for the magnetic field, if we assume that ε is arbitrarily small, but not zero? Show your work.

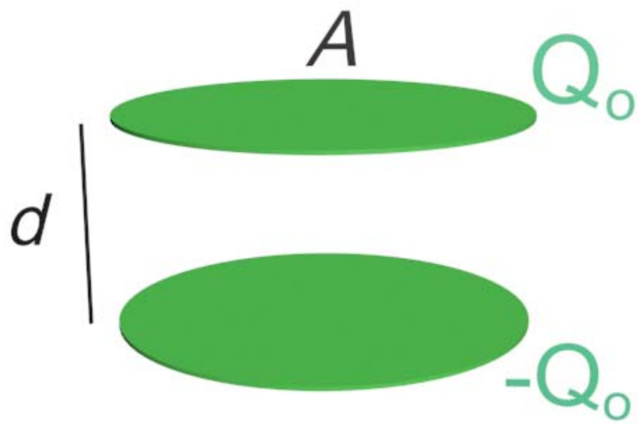


d. Find the total energy flow though the surface of an imaginary cylinder of radius ε extending from the surface of the sphere up to infinity on the z-axis (see sketch), where ε is arbitrarily small but not zero.



d. How do you interpret your answers from a), b) and d) in terms of the creation, flow, and storage of electric energy? You can ignore any energy stored in the magnetic field in this interpretation. Why are you justified in ignoring that magnetic energy? Explain.

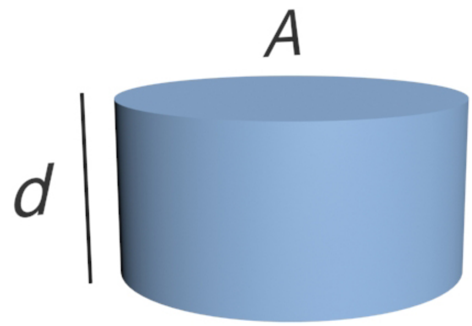
Problem 3:



A charged capacitor consists of identical circular plates of area *A* separated by a distance *d* (see sketch). The plates initially carry charges $\pm Q_o$.

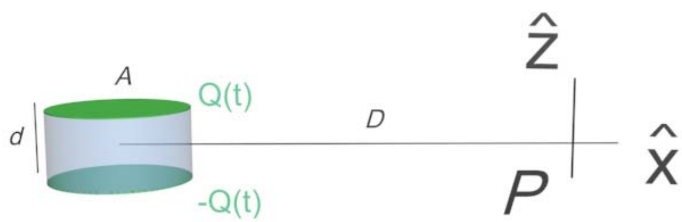
a. Assuming that the electric field is uniform between the plates of the capacitor and zero outside of the plates (that is, ignore fringing fields), show that the capacitance of this capacitor is given by $C = \varepsilon_o A / d$.

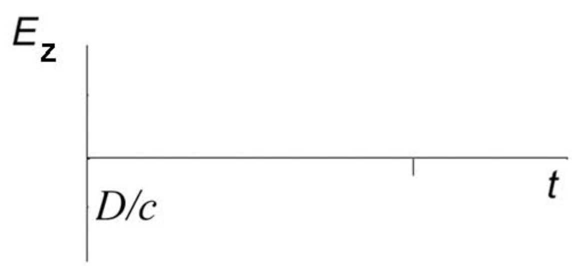
b. We have a cylinder of resistive material, resistivity $\rho_{resistivity}$. The cylinder has area A and height d. The resistance R of this cylinder when inserted between the plates of the capacitor above is $R = d\rho_{resistivity} / A$. We insert this resistive material between the plates of the capacitor and it begins to discharge. Show that the charge on the plates will subsequently decay in time according to



$$Q = Q_o e^{-t/RC} = Q_o e^{-t/\left(\varepsilon_o \rho_{resistivity}\right)}$$

c. Consider the electrical dipole radiation from this capacitor, assuming that the electric dipole moment vector is given by $\mathbf{p} = Qd\,\hat{\mathbf{z}}$. An observer P sits a distance $D >> d \approx \sqrt{A}$ from the capacitor on the x-axis, which goes through the middle of the capacitor (see sketch). Sketch on the two graphs below the time behavior of the x and the z components of the radiation electric field (and *only* the radiation electric field). You do not need to label the vertical axis, just get the shape right. You *do* need to label a time along the horizontal axis. Be careful of signs.





d. Compute the total energy radiated away into all solid angles in electric dipole radiation as the capacitor discharges. Find the ratio of this radiated energy to the energy originally stored in the capacitor before it began discharging (your ratio should be dimensionless). What condition do you have to put on d and A for the energy radiated away to be very small compared to the original electrostatic energy stored?