8.07 Class Notes Fall 2011

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(*LIVE LINKS*)

1 Fields and their Depiction

1.1 Learning objectives

The learning objectives for this section are first to get an overview of electromagnetism and how it changed the way we view the world. I then discuss the way I present the material in this course, and how it differs from the traditional treatments. I will tell you why I choose to teach topics in this order, and why I think this organization will help you get a deeper feel for electromagnetic theory, rather than you becoming lost in the mathematics.

1.2 Maxwell's electromagnetism as a fundamental advance

Classical electromagnetic field theory emerged in more or less complete form in 1873 in James Clerk Maxwell's *A Treatise on Electricity and Magnetism*. Maxwell's treatise had an impact for electromagnetism similar to that of Newton's *Principia* for classical mechanics. It not only provided the mathematical tools for the investigation of electromagnetic theory, but it also altered the basic intellectual framework of physics, in that it finally displaced *action at a distance* and substituted for it the concept of *fields.*

What is action at a distance? It is a world view in which the interaction between two material objects requires no mechanism other than the objects themselves and the empty space between them. That is, two objects exert a force on each other simply because they are present. Any mutual force between them (for example, gravitational attraction or electric repulsion) is instantaneously transmitted from one object to the other through empty space. There is no need to take into account any method or agent of transmission of that force, or any finite speed for the propagation of that agent of transmission. This is known as *action at a distance* because objects exert forces on one another (*action*) with nothing but empty space (*distance*) between them. No other agent or mechanism is needed.

Many natural philosophers, including Newton (1693)[1], criticized the action at a distance model because in our everyday experience, forces are exerted by one object on another only when the objects are in direct contact. In the field theory view, this is always true in some sense. That is, objects that are not in direct contact (objects separated by apparently empty space) exert a force on one another through the presence of an intervening medium or mechanism existing in the space between the objects. The force between the two objects is transmitted by direct *contact* from the first object to an intervening mechanism immediately surrounding that object, and then from one element of space to a neighboring element, in a continuous manner, until it is transmitted to the

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[1] "That Gravity should be innate, inherent and essential to Matter, so that one body may act upon another at a Distance thro' a *Vacuum*, without the Mediation of anything else, by and through which their Action and Force may be conveyed from one to another, is to me so great an Absurdity, that I believe no Man who has in philosophical Matters a competent Faculty of thinking, can ever fall into it."

region of space contiguous to the second object, and thus ultimately to the second object itself.

Thus, although the two objects are not in direct contact with one another, they *are* in direct contact with a medium or mechanism that exists between them. The force between the objects is transmitted (at a finite speed) by stresses induced in the intervening space by the presence of the objects. The field theory view in classical electromagnetism thus avoids the concept of action at a distance and replaces it by the concept of *action by continuous contact*. The contact is provided by a stress, or field, induced in the space between the objects by their presence.

This is the essence of field theory, and is the foundation of all modern approaches to understanding the world around us. Field theory has of course evolved far beyond these beginnings. In the modern view, every aspect of reality is due to *quantized* fields:

> *In its mature form, the idea of quantum field theory is that quantum fields are the basic ingredients of the universe, and the particles are just bundles of energy and momentum of the fields.*
>
> Weinberg, 1999

Our task here is to understand electromagnetism, before quantization, with emphasis on the energy and momentum carried by fields. This is of interest in itself, and will also give us insight, by analogy, into aspects of "matter fields", and how they can carry energy and momentum in their quantized, particle-like realizations.

1.3 Why this course is different

1.3.1 The profound parts of E&M first

The standard way to approach this subject is to present the various topics in electromagnetism in the *historical* order in which they were developed—e.g. electrostatics first, then magnetostatics, then Faraday's Law, and finally the displacement current and radiation, followed by special relativity and the manifestly covariant form of Maxwell's equations. Although there is much to recommend this approach, and perhaps it is the best one to follow in a course that spans two or more semesters, I do not follow it here.

The reasons are as follows. At MIT, Electromagnetism II, 8.07, is a one term course on a semester system, and thus the course is approximately 12 weeks or 37 one hour classes long. I have taught this course many times at MIT, and invariably with the traditional organization I get to the most interesting and profound material near the end of that 12 weeks , when the students (and myself) are exhausted. In contrast, I traditionally have spent a lot of time at the beginning on material which is mathematically difficult but not profound, that is electrostatics and magnetostatics. I count the most profound aspects of classical electromagnetism to be as follows:

The existence of fields which carry energy and momentum

How these fields mediate the interactions of material objects, especially the fact that the shape *of the fields is predictive of the stresses they transmit.*

The nature of light and of the radiation processes by which it is created.

The fact that Maxwell's equations contain the way that space and time transform.

In this course I propose to address the above, more profound, aspects of electromagnetism first. I begin with general solution for the electromagnetic fields given known sources of charge and current density. I will apply these solutions to many different cases, the first of which will be to consider in a relatively brief treatment the static solutions far from sources which do not vary in time. Then I will immediately move on to find the fields far from a spatially localized set of sources which slowly vary in time. Then I discuss the conservation of energy and momentum at length. I then address special relativity. At the end of the course I return to statics and also consider the effects of the presence of material media.

1.3.2 The easy E&M and the hard E&M

Another reason that I depart from the traditional sequence is that there is an "easy" electromagnetism and a "hard" electromagnetism. The first occurs when the behavior of the sources of electromagnetic fields, that is, charges and currents, is given, and that behavior cannot be influenced by the fields that they produce. The second occurs when the behavior of the sources of the fields can be affected by the fields that they produce.

It is in this second situation that electromagnetism becomes difficult, and in many cases, intractable--when the fields that are produced by sources can affect the sources that produce them. When we are dealing with linear dielectric or magnetic material media, the sources *are* in fact affected by the fields they produce, but because of the linearity there are straightforward ways to deal with this situation. But in many other contexts where the sources are affected by the field they produce, there is no good analytic approach to solving problems. For example, in the traditional approach to the subject the really hard E&M appears almost immediately, in boundary value problems in electrostatics, and much effort is expended in investigating the details of solutions to these difficult kinds of problems, which is in parallel with the historical development of the field.

However, it turns out that looking at the easy part of E&M, where the sources are given and are not affected by the fields they produce, is more than enough to show you the nature of fields, the way that they carry energy and momentum, the nature of radiation, and the way in which space and time transform. For that reason, I prefer to separate the treatment of electromagnetism into the easy part and the hard part, doing the easy part first. This allows me to spend more time addressing the profound issues, leaving the less profound (and frequently more mathematically difficult) issues until later. The question of the effects of linear material media can also be grouped with the “easy” electromagnetism, but there are a number of (not so profound) complexities in this, and I therefore leave this subject to the end of my treatment.

1.3.3 Energy and momentum in fields

One of the many amazing things about classical electromagnetic fields is that they carry energy, momentum, and angular momentum just as “ordinary matter” does, and there is a constant interchange of these quantities between their mechanical forms and their electromagnetic field forms. Although all texts in electromagnetism make this point, and derive the appropriate conservation laws, actual examples showing the interchange are rare. In this text I put a lot of emphasis on the processes by which energy and momentum are created in fields and the manner in which that energy and momentum flows around the system, to and from the fields and particles, thereby mediating the interaction of the particles.

For example, consider the electromagnetism of the head-on collision of two charged particles of equal mass and charge. In this process, energy is stored in the field and then retrieved, electromagnetic momentum flux is created at the location of one charge and momentum flows via the electromagnetic field to the other charge, in a dazzling array of interaction between matter and fields. But this problem is almost never discussed in these terms. I will try in most circumstances in this text to describe how the field mediates the interaction of material objects by taking up energy and momentum from them and by transferring this energy and momentum from one particle to the other. This is a different emphasis than the traditional approach, and one which illustrates more clearly the essence and the importance of fields.

1.3.4 Animations and visualizations

In order therefore to appreciate the requirements of the science [of electromagnetism], the student must make himself familiar with a considerable body of most intricate mathematics, the mere retention of which in the memory materially interferes with further progress.

James Clerk Maxwell (1855)

I will spend some time on vector fields and examples of vector fields, and the methods we will use to visualize these fields in the course. The mathematics I delve into is fierce, and as is well known, the level of the mathematics obscures the physical reality that the equations represent. The quote from Maxwell above is from one of his first papers on the subject. I try to offset the tyranny of the mathematics by using many visual depictions of the electromagnetic field, both stills and movies, and also interactive visualizations, as appropriate to the topic at hand.

1.4 Reference texts

There are many excellent electromagnetism texts available, the most popular being the text by Griffiths (1999), which is the course textbook. I will also refer to a number of other classic texts in this area, most notably the various editions of Jackson (1975), but also to texts by Panofsky and Phillips (1962), Hertz (1893), Jefimenko (1989) and others. If you are interested in the history of electromagnetism, the book I would recommend if you read only one is *The Maxwellians* by Hunt (2005).

1.5 Representations of vector fields

1.5.1 The vector field representation of a vector field

A field is a function that has a value at every point in space. A scalar field is a field for which there is a single number associated with every point in space. A vector field is a field in which there is a vector associated with every point in space—that is, three numbers instead of only the single number for the scalar field. An example of a vector field is the velocity of the Earth’s atmosphere, that is, the wind velocity.

Figure 1-1 is an example of a “vector field” representation of a field. We show the charges that would produce this field if it were an electric field due to two point charges, one positive (the orange charge) and one negative (the blue charge). We will always use this color scheme to represent positive and negative charges.

In the vector field representation, we put arrows representing the field direction on a rectangular grid. The direction of the arrow at a given location represents the direction of the vector field at that point. In many cases, we also make the length of the vector proportional to the magnitude of the vector field at that point. But we also may show only the direction with the vectors (that is make all vectors the same length), and color-code the arrows according to the magnitude of the vector. Or we may not give any information about the magnitude of the field at all, but just use the arrows on the grid to indicate the direction of the field at that point.

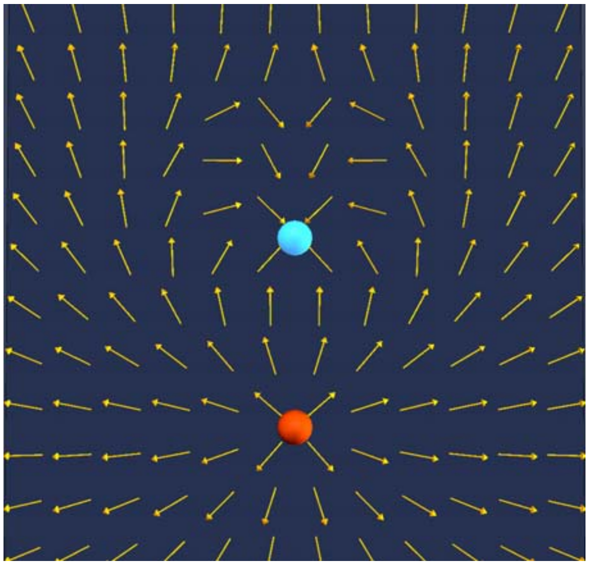

Figure 1-1: A vector representation of the field of two point charges

Figure 1-1 is an example of the latter situation. That is, we use the arrows on the vector field grid to simply indicate the direction of the field, with no indication of the magnitude of the field, either by the length of the arrows or their color. Note that the arrows point away from the positive charge (the positive charge is a "source" for electric field) and towards the negative charge (the negative charge is a "sink" for electric field). In this case the magnitude of the positive charge is five times the magnitude of the negative charge.

1.5.2 The field line representation of a vector field

There are other ways to represent a vector field. One of the most common is to draw field lines. To draw a field line, start out at any point in space and move a very short distance in the direction of the local vector field, drawing a line as you do so. After that short distance, stop, find the new direction of the local vector field at the point where you stopped, and begin moving again in that new direction. Continue this process indefinitely. Thereby you construct a line in space that is everywhere tangent to the local vector field. If you do this for different starting points, you can draw a set of field lines that give a good representation of the properties of the vector field. Figure 1-2 is an example of a field line representation for the same two charges we used inFigure 1-1.

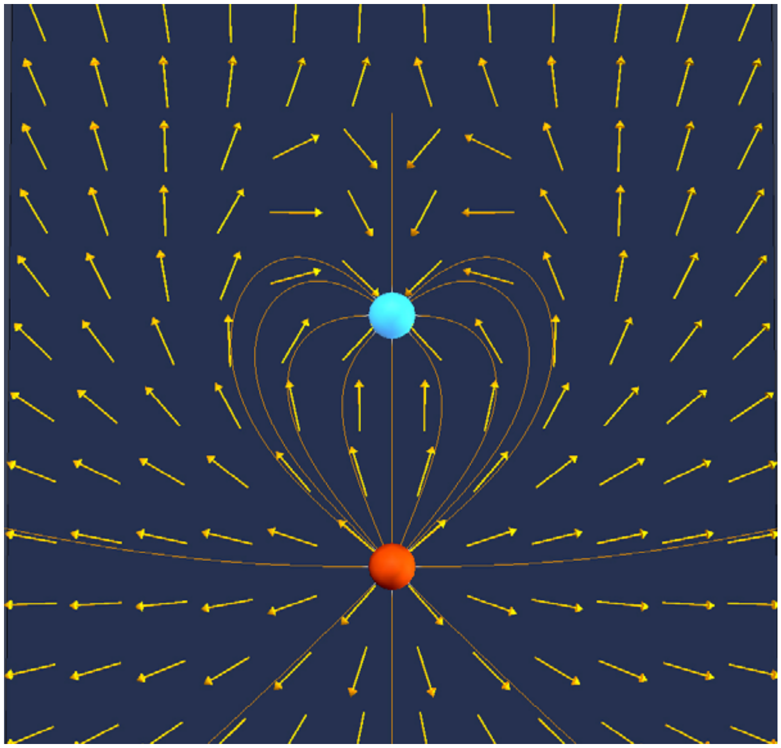

Figure 1-2: A vector field and field line representation of the same field

The mathematics of constructing field lines follows directly from the description above. Let us parameterize a line in space by the arc length along the line, *s*, where the line goes through a point in space $\mathbf{r}_o = (x_o, y_o, z_o)$ and we measure arc length along the field line from the point $\mathbf{r}_0$. A field line going through $\mathbf{r}_0$ of the vector field **F** is a line in space $\mathbf{r}(s) = (x(s), y(s), z(s))$ that satisfies at every point along the line the equations in Cartesian coordinates

$$\frac{dx(s)}{ds} = \frac{F_x}{F} \qquad \frac{dy(s)}{ds} = \frac{F_y}{F} \qquad \frac{dz(s)}{ds} = \frac{F_z}{F} \tag{1.5.1}$$

In some situations we can solve (1.5.1) for the equation describing the field lines analytically, but we can always generate them numerically using this definition.

1.5.3 Line integral convolution representations

The final representation of vector fields is the line integral convolution representation (LIC). The advantage of this representation lies in its spatial resolution. The use of field lines has the disadvantage that small scale structure in the field can be missed depending on the choice for the spatial distribution of the field lines. The vector field grid representation has a similar disadvantage in that the associated icons limit the spatial resolution because of the size of the icons and because of the spacing between icons needed for clarity. These two factors limit the usefulness of the vector field representation in showing small scale structure in the field.

In contrast, the LIC method of Cabral and Leedom (1993) avoids both of these problems by the use of a texture pattern to indicate the spatial structure of the field at a resolution close to that of the display. Figure is a LIC representation of the electric field for the same charges as in the earlier Figures. The local field direction is in the direction in which the texture pattern in this figure is correlated. The variation in color over the figure has no physical meaning; the color is used simply to give visual information about the local field direction. This LIC representation gives by far the most information about the spatial structure of the field. By its nature it cannot indicate the direction of the field: the texture pattern indicates either the field direction or the direction 180 degrees from the field direction.

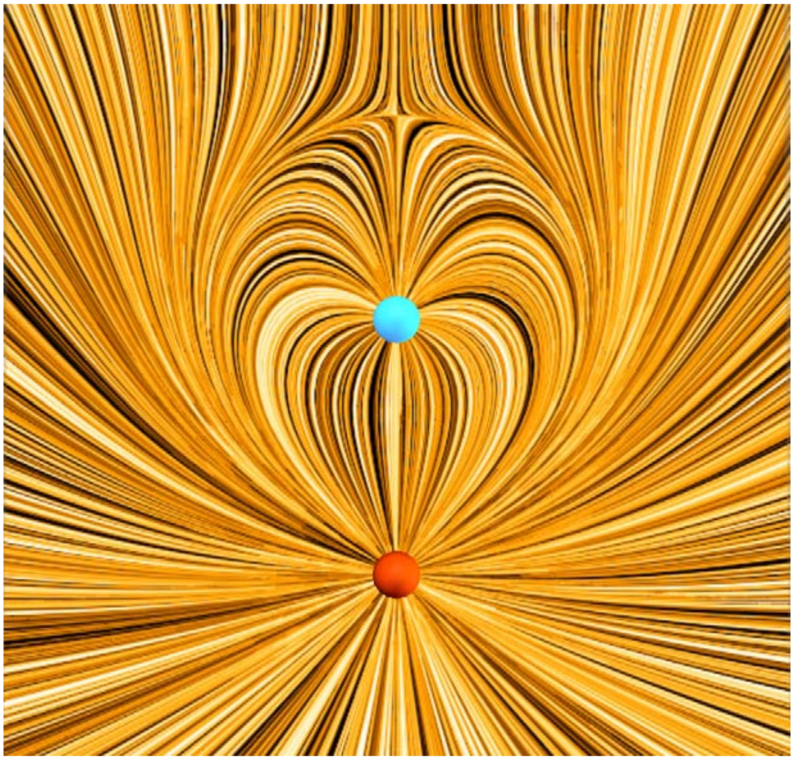

Figure 1-3: A LIC representation of the same field as in earlier figures

The complexity of vector field topologies can be amazing. For example, in Figure 1-4 we show the LIC representation of the vector field

$$\mathbf{F}(\mathbf{r}) = \sin\left(y^2\right)\hat{\mathbf{x}} + \cos(x^2)\hat{\mathbf{y}} \tag{1.5.2}$$

Although this is a simple analytic form for a vector field, the visual representation of the field is complex. This vector field has zero curl, as is apparent from looking at the representation below.

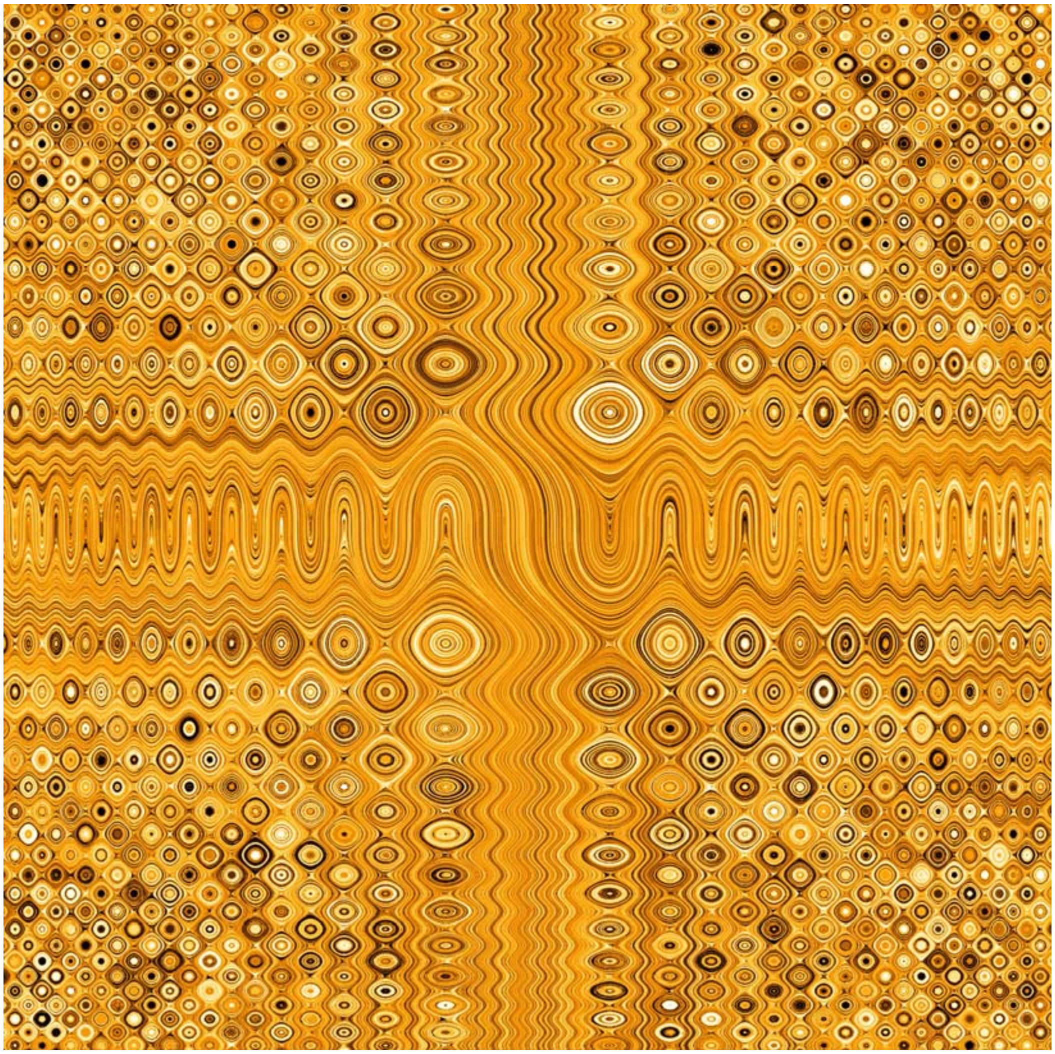

Figure 1-4: A LIC representation of a vector field with no sources

2 Conservations Laws for Scalar and Vector Fields

2.1 Learning objectives

The learning objectives for this handout are to get a feel for the mathematics of vector fields, using fluid flow as an example. We also introduce various nomenclatures that were first introduced in the study of fluids (e.g., flux, sources, sinks) and subsequently taken over to electromagnetism. We learn here what mathematical form we expect to see for conservation of both scalar (e.g. mass) and vector (e.g. momentum) quantities. We then introduce the properties of an important "improper" function, the Heaviside-Dirac delta function. Finally, we discuss briefly the topic of complete sets of functions.

2.2 Conservation laws for scalar quantites in integral and differential form

2.2.1 Density and creation rate of a conserved scalar quantity

Much of the terminology we use in electromagnetism comes from the theory of fluids, and fluid flow represents a tangible example of a vector field which is easily visualized. We thus briefly review the properties of fluid flow. We particularly focus on conservations laws, since we will see many such laws in electromagnetism, and understanding the meaning of these laws is a central part of understanding the physics.

Consider conservations laws for scalar quantities first. Suppose the mass per unit volume of the fluid at $(\mathbf{r},t)$ is given by $\rho_{mass}(\mathbf{r},t)$ and the velocity of a fluid element at $(\mathbf{r},t)$ is given by $\mathbf{v}(\mathbf{r},t)$. Suppose also that mass is being created at $(\mathbf{r},t)$ at a rate given by $s_{mass}(\mathbf{r},t)$ (units of *s* are mass per unit volume per unit time). Consider an arbitrary closed surface *S* containing a volume *V*, as shown in Figure 2-1. We assume that the surface and the enclosed volume are fixed in space and time (e.g. the surface of the volume is not moving). At time *t*, the amount of mass inside of the closed surface is given by the volume integral of the mass density, and the rate at which matter is being created inside the volume is given by the volume integral of the mass creation rate per unit volume. On physical grounds, it is obvious that the time rate of change of the amount of matter inside of the volume is given by the rate at which it is being created inside the volume minus the rate at which matter is flowing out through the stationary surface of the volume. That is,

$$\frac{d}{dt}\int_V \rho_{mass}(\mathbf{r},t)\,d^3x = \int_V s_{mass}(\mathbf{r},t)\,d^3x - \text{rate of mass flow out of volume} \quad (2.2.1)$$

2.2.2 The flux and flux density of a conserved scalar quantity

What is the rate at which mass is flowing out of the volume? Consider a infinitesimal surface element on the surface $\hat{\mathbf{n}}\,da$, where $\hat{\mathbf{n}}\,da$ is the local normal to the surface, defined so that it points away from the volume of interest (in this case outward). The rate at which mass is flowing through this surface element at time t is $\rho_{mass}\mathbf{v}\cdot\hat{\mathbf{n}}\,da$. To see this, imagine that you are an observer sitting on da and you measure the total amount of matter which flows across da in a time dt. This amount is the volume of the matter that will cross da in time dt, $\left[\mathbf{v}\cdot\hat{\mathbf{n}}\,da\,dt\right]$, times the mass density, that is, $\rho_{mass}\left[\mathbf{v}\cdot\hat{\mathbf{n}}\,da\,dt\right]$. Note that the dot product of $\mathbf{v}$ and $\hat{\mathbf{n}}$ in this expression is very important. If there is no component of the flow velocity along $\hat{\mathbf{n}}$, there is no matter flow across da.

To find the *rate* of matter flow we simply divide this amount of matter in time dt by the infinitesimal time dt, giving us $\rho_{mass}\mathbf{v}\cdot\hat{\mathbf{n}}\,da$. If we want the total rate at which mass if flowing through the entire surface S we integrate this quantity over the entire surface. Then (2.2.1) becomes

$$\frac{d}{dt}\int_V \rho_{mass}\left(\mathbf{r},t\right)d^3x = \int_V s_{mass}\left(\mathbf{r},t\right)d^3x - \int_S \left[\rho_{mass}\mathbf{v}\right]\cdot\hat{\mathbf{n}}\,da \tag{2.2.2}$$

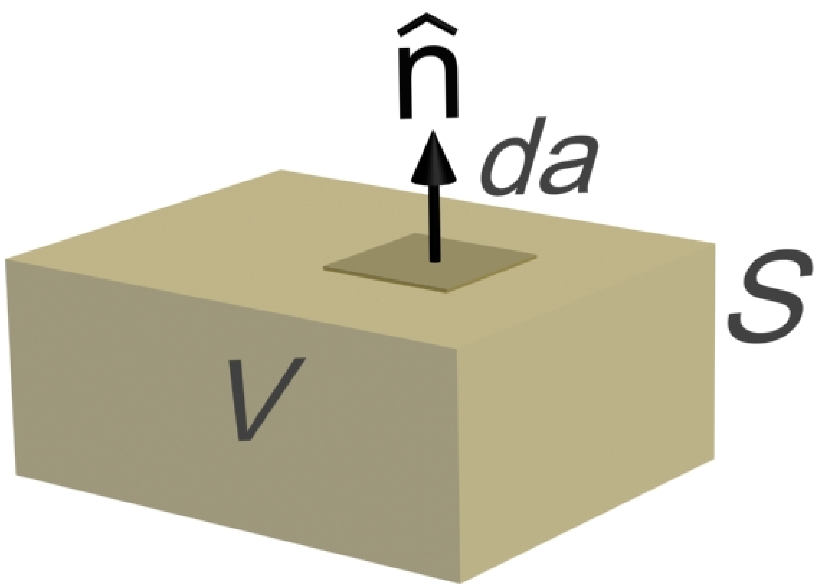

Figure 2-1: A closed surface *S* with enclosed volume *V*

We call $\int_S \left[\rho_{mass}\mathbf{v}\right]\cdot\hat{\mathbf{n}}\,da$ the flux (flow) of matter through the surface S. This quantity is a scalar. A point of frequent confusion is that the vector quantity $\rho_{mass}\mathbf{v}$ is sometimes also called the flux. In these notes we will always call quantities like $\rho_{mass}\mathbf{v}$ the flux density (a vector), and reserve the use of flux to mean the scalar we get from integrating the flux density over a specific surface.

The more standard way of writing (2.2.2) is to bring the flux to the left and side, that is,

$$\frac{d}{dt}\int_V \rho_{mass}\left(\mathbf{r},t\right)d^3x+\int_S\left[\rho_{mass}\mathbf{v}\right]\cdot\hat{\mathbf{n}}\,da=\int_V s_{mass}\left(\mathbf{r},t\right)d^3x \tag{2.2.3}$$

2.2.3 Gauss's Theorem and the differential form

We now want to write (2.2.3) in differential form. To do this we invoke one of the fundamental theorems of vector calculus, the divergence theorem. This theorem states that if **F** is a reasonably behaved vector function, then the surface integral of **F** over a closed surface is related to the volume integral over the volume that surface encloses of the divergence of **F**

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,da=\int_V \nabla\cdot\mathbf{F}\,d^3x \tag{2.2.4}$$

where in Cartesian coordinates

$$\nabla\cdot\mathbf{F}=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z} \tag{2.2.5}$$

If we use (2.2.5) in (2.2.3), and use the fact that our surface *S* is arbitrary, we see that we must have for the conservation of mass in differential form the equation

$$\frac{\partial\rho_{mass}}{\partial t}+\nabla\cdot\left[\rho_{mass}\mathbf{v}\right]=s \tag{2.2.6}$$

2.3 Conservation laws for vector quantities in integral and differential form

Above we were talking about the conservation of a scalar quantity, e.g. the scalar field $\rho_{mass}(\mathbf{r},t)$. However many of our conserved quantities in electromagnetism are vector quantities, for example electromagnetic momentum. To see how we handle conservation laws involving vector quantities, let us first consider the fluid context, specifically the conservation of the momentum density of the fluid. The momentum density of the fluid is a vector, and is given by the mass density of the fluid times the vector velocity of the fluid element, that is $\rho_{mass}\mathbf{v}$. If the momentum of fluid is a conserved quantity, we should have a statement similar to (2.2.1), that is

$$\frac{d}{dt}\int_V \rho_{mass}\mathbf{v}\,d^3x=\int_V\left[\text{creation rate of momentum}\right]d^3x-\text{rate of momentum flow out of volume} \tag{2.3.1}$$

Note now that our creation rate of momentum is a vector quantity, as is the rate at which momentum flows out of the volume through *da*. In analogy with the mass flowing out of the volume through $\hat{\mathbf{n}}\, da$, the momentum flowing out of the volume as carried by the flow is $\left[\rho_{mass}\mathbf{v}\right]\mathbf{v}\cdot\hat{\mathbf{n}}\, da$. That is, we take the density of momentum, $\left[\rho_{mass}\mathbf{v}\right]$, multiply it by the volume of the fluid which flows across *da* in time *dt*, $\mathbf{v}\cdot\hat{\mathbf{n}}\, da\, dt$, and divide by *dt* to get the rate. Thus (2.3.1) becomes (bringing the flux of momentum to the left hand side),

$$\frac{d}{dt}\int_V \rho_{mass}\mathbf{v}\, d^3x + \int_S \left[\rho_{mass}\mathbf{v}\right]\mathbf{v}\cdot\hat{\mathbf{n}}\, da = \int_V \left[\text{creation rate of momentum}\right] d^3x \qquad (2.3.2)$$

So the quantity $\int_S \left[\rho_{mass}\mathbf{v}\right]\mathbf{v}\cdot\hat{\mathbf{n}}\, da$ is the flux of momentum through the surface S, and the flux density of momentum is a second rank tensor, $\rho_{mass}\mathbf{v}\,\mathbf{v}$. This makes sense because the flux of a scalar quantity like mass is a vector, e.g. $\rho_{mass}\mathbf{v}$, and therefore the flux of a vector quantity like momentum must be more complicated than a vector, just as a vector is a more complicated object than a scalar. We only mention second rank tensors in passing here. Later we will consider them in detail.

In differential form, the conservation for momentum is written as

$$\frac{\partial}{\partial t}\left[\rho_{mass}\mathbf{v}\right] + \nabla\cdot\left[\rho_{mass}\mathbf{v}\,\mathbf{v}\right] = \left[\text{volume creation rate of momentum}\right] \qquad (2.3.3)$$

3 The Dirac delta function and complete othorgonal sets of fucntions

3.1 Basic definition

One of the functions we will find indispensable in proving various vector theorems in electromagnetism is the Dirac[2] delta function. In one dimension, the delta function is defined by

$$\delta(x) = 0 \qquad x \neq 0 \qquad (3.1.1)$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \qquad (3.1.2)$$

From (3.1.2) we see that the one-dimensional delta function must have the dimensions of one over the dimensions of its argument. To get an intuitive feel for this "improper"

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[2] Although this function is usually attributed to Dirac, it was first introduced by Heaviside, and is a good example of the zeroth theorem of the history of science: a discovery named after someone often did not originate with that person. See Jackson (2008).

function, it is probably best to think of it as the limit of a sequence of proper functions. There are many sequences which in a limit become delta functions, for example the function

$$\mathrm{H}_a(x) = \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2} \tag{3.1.3}$$

It is easy to show that $\mathrm{H}_a(x)$ satisfies (3.1.1) and (3.1.2) in the limit that *a* goes to zero. Our intuitive picture of a delta function is thus a function which is highly peaked around *x = 0*, with unit area under the function. It is then plausible that if we take a well-behaved continuous function $f(x)$, we have

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0) \tag{3.1.4}$$

In higher dimensions we define the delta function as a product of one-dimensional delta functions, e.g.

$$\delta^3(\mathbf{r}) = \delta(x)\,\delta(y)\,\delta(z) \tag{3.1.5}$$

with

$$\delta^3(\mathbf{r}) = 0 \qquad |\mathbf{r}| \neq 0 \tag{3.1.6}$$

and

$$\int_{all\,space} \delta^3(\mathbf{r})\, d^3x = 1 \tag{3.1.7}$$

One of the relations that we will find most useful in this course is the following:

$$\delta^3(\mathbf{r}) = -\frac{1}{4\pi} \nabla^2 \frac{1}{r} \tag{3.1.8}$$

To prove this relation, we first note that it is easy to show that $\nabla^2 \frac{1}{r} = 0$ for $r \neq 0$. To see that (3.1.7) holds, we use the divergence theorem (2.2.4)

$$\begin{aligned} \int_{all\,space} \nabla^2 \frac{1}{r} d^3x &= \int_{all\,space} \nabla \cdot \left[\nabla \frac{1}{r}\right] d^3x = \int_{surface\,sphere} \left[\nabla \frac{1}{r}\right] \cdot \hat{\mathbf{n}}\, da = \\ &= \int_{surface\,sphere} \left[-\frac{\hat{\mathbf{n}}}{r^2}\right] \cdot \hat{\mathbf{n}}\, da = \int_{surface\,sphere} \left[-\frac{r^2 d\Omega}{r^2}\right] = -4\pi \end{aligned} \tag{3.1.9}$$

3.2 Useful relations

From the definitions above, there are a number of useful relations for the delta function which we list here

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a) \tag{3.2.1}$$

$$\int_{-\infty}^{\infty} \delta'(x-a) f(x) dx = -f'(a) \tag{3.2.2}$$

$$\int_{-\infty}^{\infty} f(x) \delta[g(x)] dx = \frac{f(a)}{|g'(a)|} \quad where \quad g(a) = 0 \tag{3.2.3}$$

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\,\delta^3(\mathbf{r}-\mathbf{r}') \tag{3.2.4}$$

3.3 Complete sets of orthorgonal functions on a finite interval

A denumerably infinite set of functions $\{f_n(x)\}\big|_{n=0}^{\infty}$ of the interval $[-1,1]$ is *orthorgonal* if for any *n* and *m*,

$$\int_{-1}^{1} f_n(x) f_m(x) dx = C_n \delta_{nm} \tag{3.3.1}$$

Here δ_{jn} is the Kronecker delta (δ_{jn} is 1 if $j = n$ and 0 otherwise). The set is complete is for any "nice" function $g(x)$ defined on $[-1,1]$, we can expand $g(x)$ as

$$g(x) = \sum_{n=0}^{\infty} a_n f_n(x) \tag{3.3.2}$$

where $\{a_n\}\big|_{n=0}^{\infty}$ is a set of constants. We can use the orthogonality property of our complete set of functions by first multiplying (3.3.2) by $f_m(x)$

$$g(x) f_m(x) = \sum_{n=0}^{\infty} a_n f_n(x) f_m(x) \tag{3.3.3}$$

and then integrating (3.3.3) from -1 to 1, yielding

$$\int_{-1}^{1} g(x)\, f_m(x)\, dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} f_n(x)\, f_m(x)\, dx = \sum_{n=0}^{\infty} a_n C_n \delta_{nm} \tag{3.3.4}$$

I thus find that my constants are given by

$$a_m = \frac{1}{C_m} \int_{-1}^{1} g(x)\, f_m(x)\, dx \tag{3.3.5}$$

If for every *m*, $C_m = 1$, then my set of functions is said to be *orthonormal.*

3.4 Representation of a delta function in terms of a complete set of functions

Suppose my function $g(x)$ is a delta function; that is, suppose

$$g(x) = \delta(x - x_o) \tag{3.4.1}$$

Then from (3.3.5) I have

$$a_m = \frac{1}{C_m} \int_{-1}^{1} \delta(x - x_o)\, f_m(x)\, dx = \frac{1}{C_m} f_m(x_o) \tag{3.4.2}$$

and if I insert this into (3.3.2) I find that

$$\delta(x - x_o) = \sum_{n=0}^{\infty} \frac{f_n(x)\, f_n(x_o)}{C_n} \tag{3.4.3}$$

Thus I have a representation of a one dimensional delta function for every complete set of functions, and there are literally an infinite number of complete sets of function on any finite interval.

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