8.07 Class Notes Fall 2011

Comments and questions to jbelcher@mit.edu

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(*LIVE LINKS*)

20 Basic Magnetostatics

20.1 Learning Objectives

We now consider magnetostatics, and in particular concentrate on how to calculate magnetic fields using the Biot-Savart Law and Ampere's Law.

20.2 The relation between $\mathbf{J}\,d^3x$ and $I\ d\mathbf{l}$

In magnetostatics, we do a lot of switching back and forth between volume integrals involving $\mathbf{J}(\mathbf{r}')\,d^3x'$ and line integrals involving $I\,d\mathbf{l}'$, since many currents are carried by wires. The correspondence is as follows. If we have a wire carrying current *I*, then the current density **J** is always parallel to the local tangent to the wire, $I\,d\mathbf{l}$. Consider a small segment of the wire centered at point *Q* (see Figure 20-1). With

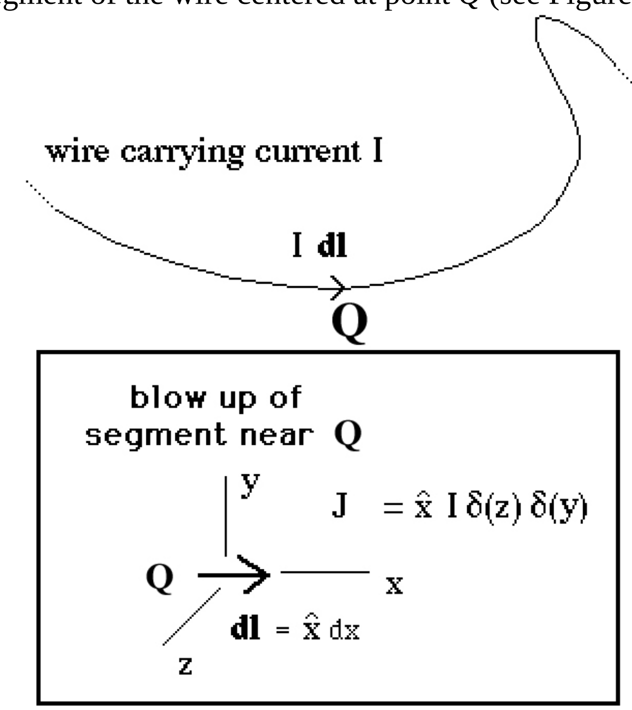

Figure 20-1: The relation between J and $I\,d\mathbf{l}$

no loss of generality, we can set up a local coordinate system where $d\mathbf{l}$ is along the x-axis. In that coordinate system, near the point *Q*, the current density **J** is given by $\mathbf{J} = \hat{\mathbf{x}}\,I\,\delta(z)\,\delta(y)$. Note that since the delta functions have dimensions of inverse argument, this expression for **J** has the correct units, that is, coulombs/m²s. If we look at the integral of **J** over a small volume enclosing point *Q*, we have

$$\int_{volume} \mathbf{J}\,d^3x = \int_{volume} \hat{\mathbf{x}}\,I\,\delta(x)\,\delta(y)\,dx\,dy\,dz = \int_{line} I\,\hat{\mathbf{x}}\,dz = \int_{line} I\,d\,\mathbf{l} \qquad (20.2.1)$$

where we have used the delta functions to do two of the spatial integrations, and in the last step we have used the fact that $\hat{\mathbf{x}}\,dx = d\,\mathbf{l}$. Thus the correspondence we want is that the volume integral of $\mathbf{J}\ d^3x$ goes over to a line integral of $I\,d\mathbf{l}$ when wires carry the current. Conversely, if we have a formula that involves a line integral of $I\,d\mathbf{l}$, we can generalize it to a volume integral by replacing $I\,d\mathbf{l}$ by $\mathbf{J}\;d^3x$ For example, in equation (8.2.14) we defined the magnetic dipole moment of a distribution of currents to be $\mathbf{m} = \frac{1}{2}\int \mathbf{r}' \times \mathbf{J}(\mathbf{r}')\,d^3x'$. We can use the prescription above to also write this as a line integral, $\mathbf{m} = \frac{1}{2} I \oint \left(\mathbf{r}' \times d\mathbf{l}' \right)$.

20.3 The Biot Savart Law

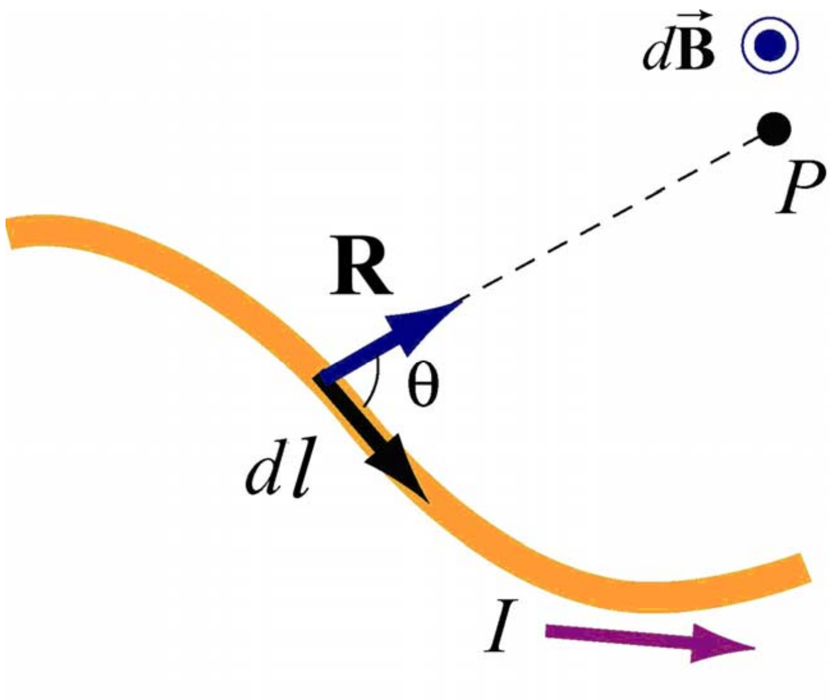

Figure 20-2: The Biot-Savart Law

The Biot-Savart Law tells us how magnetic fields are generated by steady currents. In reference to Figure 20-2, if **R** is the vector from the current carrying element $d\mathbf{l}$ to the point *P*, then the magnetic field d**B** generated at the point *P* by the current element $d\mathbf{l}$ is given by

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I\,d\mathbf{l} \times \mathbf{R}}{R^3} \tag{20.3.1}$$

If the current element $d\mathbf{l}$ is located at $\mathbf{r}'$ and the observer at point P is located at $\mathbf{r}$, then this equation is

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_o I}{4\pi} d\mathbf{l}' \times \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{20.3.2}$$

where we have introduced a prime on $d\mathbf{l}$ to indicate that it is located at $\mathbf{r}'$. To compute the total magnetic field at point P due to the complete wire, we evaluate

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o I}{4\pi} \oint d\mathbf{l} \times \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{20.3.3}$$

Taking the cross-product in (20.3.2) is a particularly hard process to imagine in one's head, and I urge you to go to the Shockwave visualization show below to get a firm grasp of how this works.

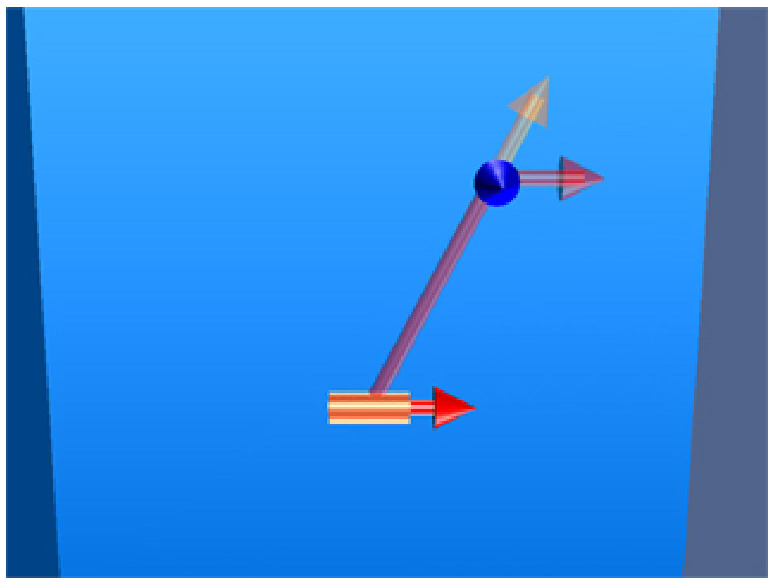

Figure 20-3: Shockwave visualization to illustrate the nature of Biot-Savart

http://web.mit.edu/viz/EM/visualizations/magnetostatics/MagneticFieldConfigurations/CurrentElement3d/CurrentElement.htm

If we go over to a continuous distribution of current density, then our expression in (20.3.3) becomes

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi} \int_{volume} d^3x' \, \mathbf{J}(\mathbf{r}') \times \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{20.3.4}$$

If we go back to our proof of the Helmholtz Theorem, we see that we can write

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) \tag{20.3.5}$$

where

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi} \int_{volume} d^3x' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{20.3.6}$$

We can also deduce from (20.3.5) that

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \tag{20.3.7}$$

and from (20.3.6) that

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_o \mathbf{J} \tag{20.3.8}$$

20.4 The magnetic field on the axis of a circular current loop

A circular loop of radius R in the xy plane carries a steady current I, as shown in Figure 20-4. We want to use the Biot-Savart Law to calculate the magnetic field on the axis of the loop.

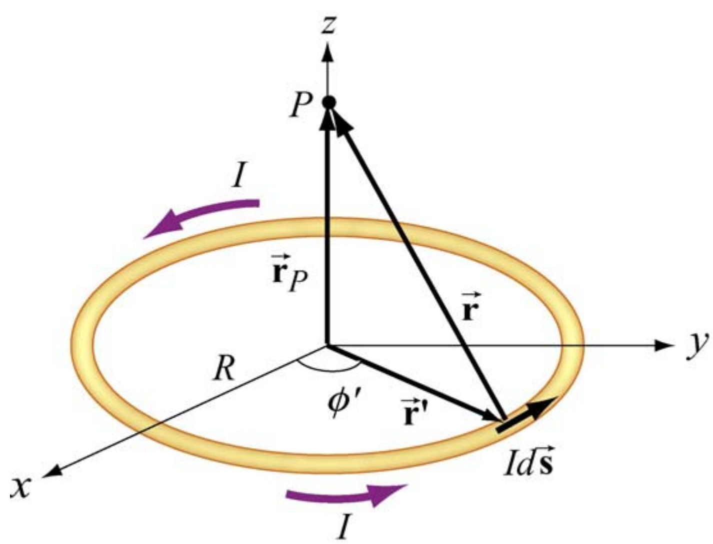

Figure 20-4: Calculating the magnetic field on the axis of a circular current loop

In Cartesian coordinates, the differential current element located at

$$\mathbf{r}' = R(\cos\phi'\,\hat{\mathbf{i}} + \sin\phi'\,\hat{\mathbf{j}}) \tag{20.4.1}$$

And $Id\mathbf{l}$ can be written as

$$Id\mathbf{l} = IRd\phi'(-\sin\phi'\,\hat{\mathbf{i}} + \cos\phi'\,\hat{\mathbf{j}}) \tag{20.4.2}$$

Since the field point P is on the axis of the loop at a distance z from the center, its position vector is given by $\vec{\mathbf{r}}_P = z\hat{\mathbf{k}}$. Thus we have

$$\mathbf{r}_P - \mathbf{r}' = -R\cos\phi'\,\hat{\mathbf{i}} - R\sin\phi'\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} \tag{20.4.3}$$

and

$$|\mathbf{r}_P - \mathbf{r}'| = \sqrt{(-R\cos\phi')^2 + \left(-R\sin\phi'\right)^2 + z^2} = \sqrt{R^2 + z^2} \tag{20.4.4}$$

In (20.3.3), we need to compute the cross product $d\,\mathbf{l}' \times (\mathbf{r}_P - \mathbf{r}')$ which can be simplied as

$$\begin{aligned} d\,\mathbf{l}' \times (\mathbf{r}_P - \mathbf{r}') &= R\,d\phi'\left(-\sin\phi'\,\hat{\mathbf{i}} + \cos\phi'\,\hat{\mathbf{j}}\right) \times [-R\cos\phi'\,\hat{\mathbf{i}} - R\sin\phi'\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}] \\ &= R\,d\phi'[z\cos\phi'\,\hat{\mathbf{i}} + z\sin\phi'\,\hat{\mathbf{j}} + R\,\hat{\mathbf{k}}] \end{aligned} \tag{20.4.5}$$

Using the Biot-Savart law, the contribution of the current element to the magnetic field at P is

$$\begin{aligned} d\vec{\mathbf{B}} &= \frac{\mu_0 I}{4\pi}\frac{d\,\mathbf{l} \times (\mathbf{r}_P - \mathbf{r}')}{|\mathbf{r}_P - \mathbf{r}'|^3} \\ &= \frac{\mu_0 IR}{4\pi}\left[\frac{z\cos\phi'\,\hat{\mathbf{i}} + z\sin\phi'\,\hat{\mathbf{j}} + R\,\hat{\mathbf{k}}}{(R^2 + z^2)^{3/2}}\right]d\phi' \end{aligned} \tag{20.4.6}$$

Carrying out the integration gives the magnetic field at P as

$$\mathbf{B} = \frac{\mu_0 IR}{4\pi}\int_0^{2\pi}\frac{\left[z\cos\phi'\,\hat{\mathbf{i}} + z\sin\phi'\,\hat{\mathbf{j}} + R\,\hat{\mathbf{k}}\right]}{(R^2 + z^2)^{3/2}}d\phi' \tag{20.4.7}$$

The x and the y components of $\vec{\mathbf{B}}$ can be readily shown to be zero, with the final result that

$$\mathbf{B} = \hat{\mathbf{z}}\frac{\mu_0 IR^2}{2(R^2 + z^2)^{3/2}} \tag{20.4.8}$$

Thus, we see that along the symmetric axis, B_z is the only non-vanishing component of the magnetic field. The conclusion can also be reached by using the symmetry arguments.

To calculate the field off axis is beyond our present mathematical capabilities, but we can see in principle what the field looks like. Figure 20-5 is a Shockwave visualization that shows how this is done.

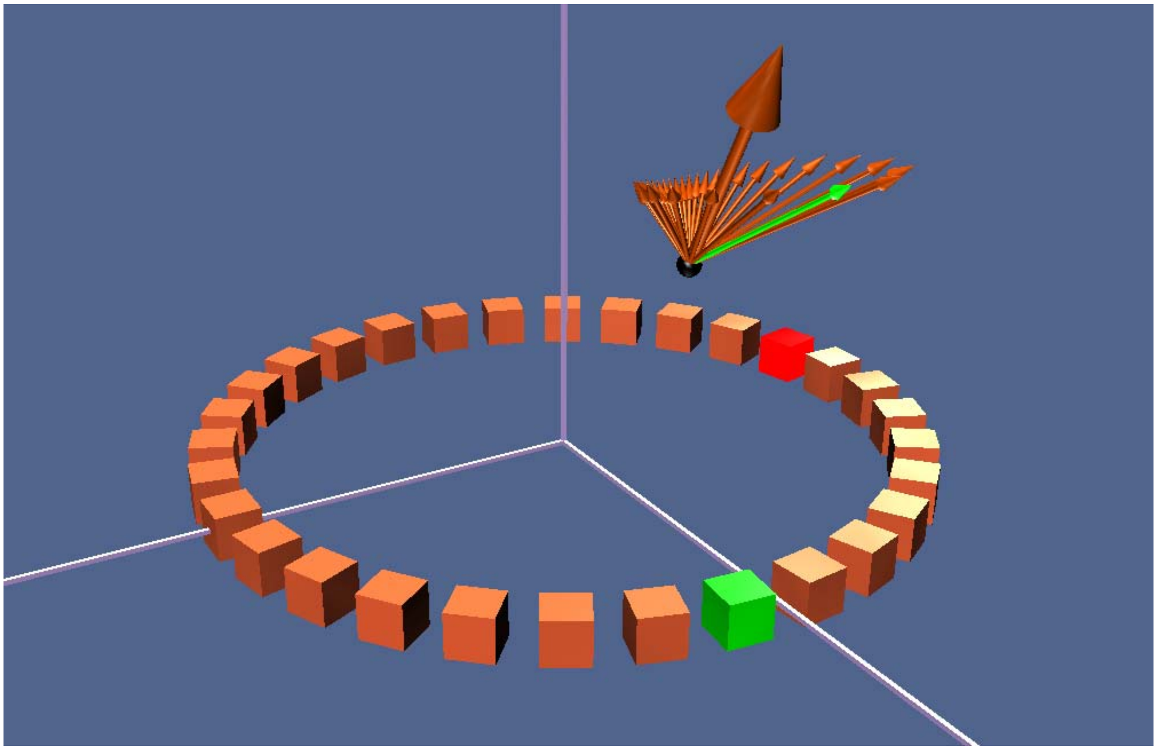

Figure 20-5: A shockwave visualization for constructing the field of a current ring

http://web.mit.edu/viz/EM/visualizations/magnetostatics/calculatingMagneticFields/RingMagField/RingMagFieldFullScreen.htm

20.5 Ampere's Law

Ampere's Law is the integral form of (20.3.8). If we take any open surface *S* with contour *C*, then from (20.3.8) we have

$$\int_S (\nabla\times\mathbf{B})\cdot\hat{\mathbf{n}}\,da = \mu_o\int_S \mathbf{J}\cdot\hat{\mathbf{n}}\,da \qquad (20.5.1)$$

We can use Stoke's theorem to convert the left hand side of (20.5.1) to a line integral, and the right hand side of (20.5.1) is simply the current through the surface S, with the positive direction of current defined right-handedly with respect to the direction of the contour integration. This gives us what is known as Ampere's Law.

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_o I_{through} \qquad (20.5.2)$$

Ampere's Law can be used to do many straightforward problems which have a lot of symmetry. Examples are given on the problem set. As with Gauss's Law, Ampere's Law is always true, but it is usually useless for solving problems, unless there is a lot of symmetry. You can find a java applet on the web at the *url* below which illustrates what is going on with Ampere's Law in the general case without a lot of symmetry.

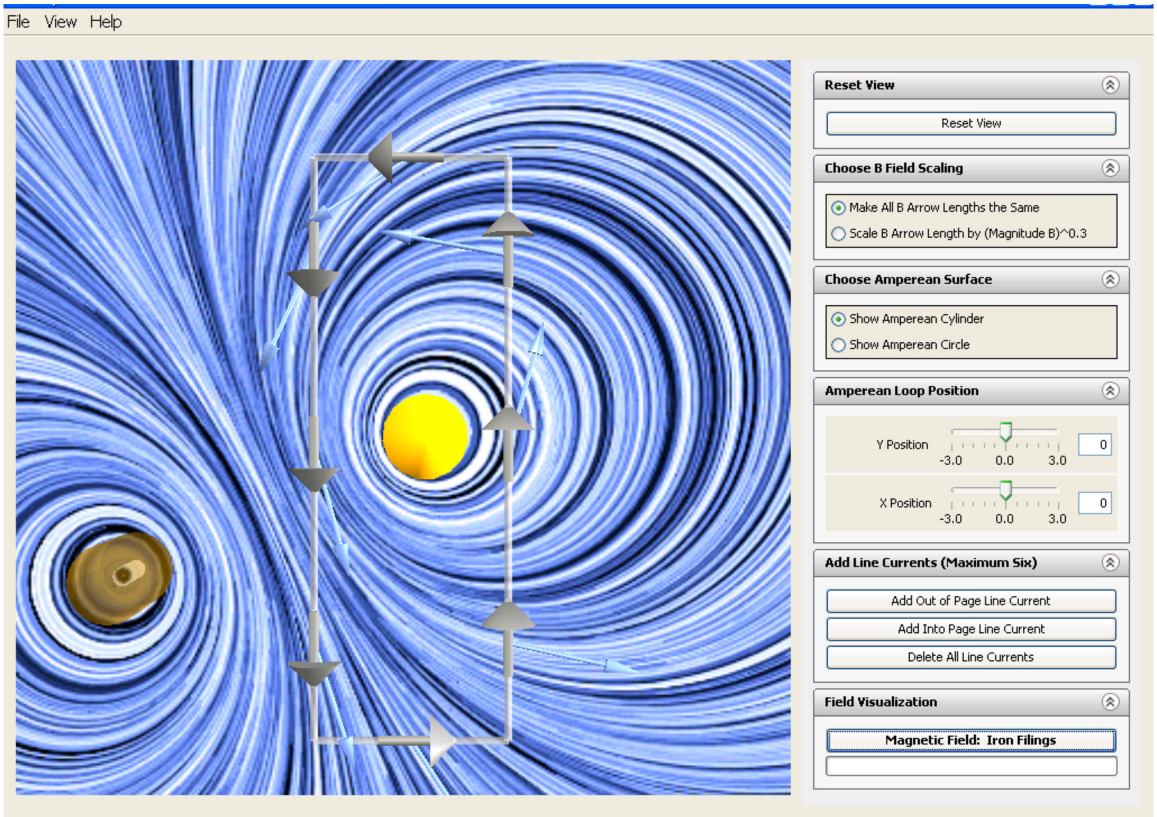

Figure 20-6: A Java applet illustrating Ampere's Law

http://web.mit.edu/viz/EM/simulations/ampereslaw.jnlp

20.6 The Magnetic Potential

In general in magnetostatics the magnetic field cannot be derived from the gradient of a scalar function, since its curl is not zero ($\nabla\times\mathbf{B}(\mathbf{r})=\mu_o\mathbf{J}$). However if there are extensive regions where the current density **J** is zero, in those regions the curl of **B** is zero, and it can be derived from the gradient of the "magnetic" potential.

As an example of this, consider a spherical shell which carried a surface current $\kappa(\theta)$ in the azimuthal direction, that is

$$\mathbf{J}=\delta(r-R)\kappa(\theta)\hat{\phi} \tag{20.6.1}$$

In this situation we have that the curl of the magnetic field is zero everywhere in space except at $r = R$. Therefore the magnetic field can be written as the gradient of the following scalar potential

$$\phi_M(r,\theta)=\begin{cases}\sum_{l=0}^{\infty}A_l\left(\frac{r}{R}\right)^l P_l(\cos\theta)\\ \sum_{l=0}^{\infty}B_l\left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta)\end{cases} \tag{20.6.2}$$

where we have imposed the conditions that the field not blow up at the origin and that it go to zero at infinity. We then determine the constant coefficients in (20.6.2) by imposing the boundary conditions that we must have on the field across $r = R$. These boundary conditions are discussed in the next section.

20.7 Boundary Conditions on the Magnetic Field

Before leaving magnetostatics, we discuss the boundary conditions that must be true across any thin interface in magnetostatics. We know that $\nabla\cdot\mathbf{B}(\mathbf{r})=0$, so we see that we must have no change in the normal component of the magnetic field across a thin interface, that is

$$B_{2n}=B_{1n} \tag{20.7.1}$$

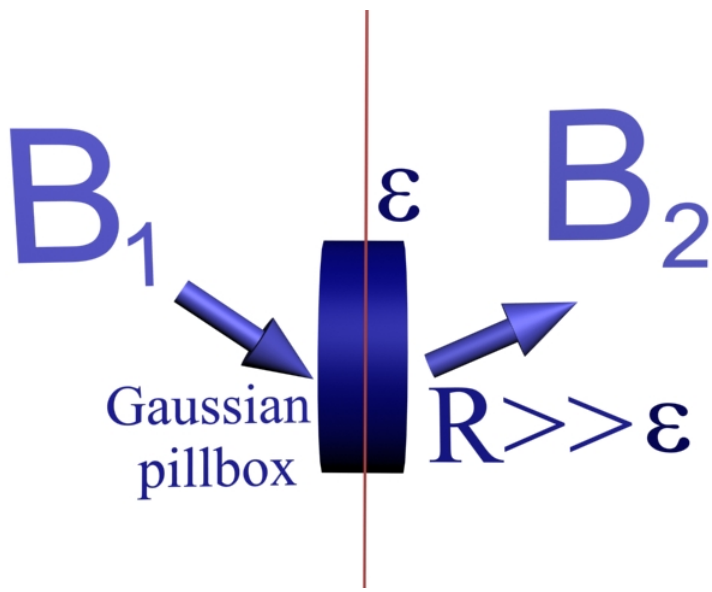

Figure 20-7: Boundary condition on the normal component of B across an interface

To make absolutely sure we understand where (20.7.1) comes from, consider the figure above. We Gauss's Law to the pillbox shown, making sure that the radius R of the pillbox is arbitrarily large compared to its height ε. The area of the sides of the pillbox is therefore given by $2\pi R\varepsilon$ and the area of the ends is given by πR^2. By making ε small enough compared to R, we can insure that any flux through the sides of the pillbox due to tangential components of **B** do not contribute to the integral over the surface area of the pillbox. Only the normal components of **B** contribute to the integral, and we easily obtain (20.7.1).

What about the tangential components of **B**? These can have a jump if there is a current sheet in the interface. To see this, take an amperean loop that spans the interface, with width ε perpendicular to the interface and length l tangential to the interface, with a loop normal that is parallel to the current sheet direction, as shown in the figure below. If we use Ampere's Law with the displacement term included, and integrate around this loop, we have

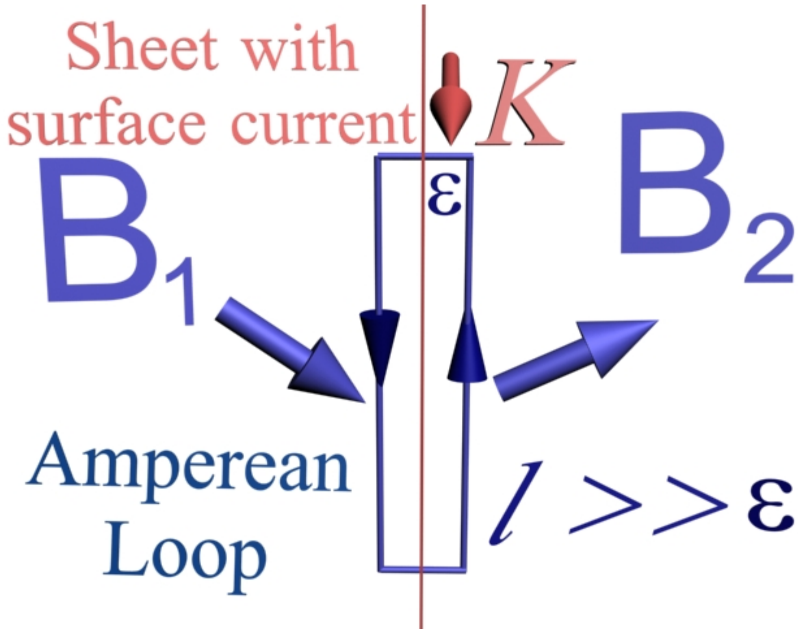

Figure 20-8: Boundary condition on the tangential component of B across an interface.

$$\oint \mathbf{B}\cdot d\,\mathbf{l} = \mu_o \int_{\text{open surface}} \mathbf{J}\cdot\hat{\mathbf{n}}\,da + \mu_o\varepsilon_o \int_{\text{open surface}} \frac{\partial \mathbf{E}}{\partial t}da \qquad (20.7.2)$$

If we make the width ε very small compared to the length l, the only component that will enter into the left hand side of (20.7.2) will be the tangential **B**, since any normal component will be multiplied by $\varepsilon << l$. And the magnitude of that component will be $B_t\, l$. If we look at the right hand side of (20.7.2), the first term will give $\mu_o\, l\,\kappa$, independent of ε, whereas the second term involves an area $l\varepsilon$. Again if we make $\varepsilon << l$ we can make the area integral of $\partial\mathbf{E}/\partial t$ insignificant. Therefore we will always have for the magnetic field, even in a time varying situation, that

$$\mathbf{B}_{2t} - \mathbf{B}_{1t} = \mu_o \boldsymbol{\kappa}\times\hat{\mathbf{n}} \qquad (20.7.3)$$

20.8 Biot-Savart as the Relativistic Transformation of the Rest Frame E Field

Before leaving magnetostatics, let us give a heuristic derivation of where the Biot-Savart Law comes from. If we think of our current source $I d\mathbf{l}$ as due to a moving point charge dq with velocity **v**, then $Id\mathbf{l}$, which has units of charge times velocity, becomes $dq\mathbf{v}$, and Biot-Savart becomes

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l}\times\mathbf{R}}{R^3} = \frac{\mu_0 dq}{4\pi}\frac{\mathbf{v}\times\mathbf{R}}{R^3} = \mu_0\varepsilon_o\mathbf{v}\times\left[\frac{dq}{4\pi\varepsilon_o}\frac{\mathbf{R}}{R^3}\right] = \frac{1}{c^2}\mathbf{v}\times\left[\frac{dq}{4\pi\varepsilon_o}\frac{\mathbf{R}}{R^3}\right] \quad (20.8.1)$$

We have written the last form in (20.8.1) so that we can discuss it in the context of (15.5.11), which we reproduce here.

$$\bar{B}_{\parallel} = B_{\parallel} \qquad \bar{\mathbf{B}}_{perp} = \gamma\left(\mathbf{B}_{perp} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right) \quad (20.8.2)$$

This tells us how to get the magnetic field in a frame moving at velocity **v** with respect to the laboratory frame. Let us suppose that in the laboratory frame we have a charge dq at rest. Then in that frame we have

$$d\mathbf{E} = \frac{dq}{4\pi\varepsilon_o}\frac{\mathrm{R}}{R^3} \qquad d\bar{\mathbf{B}} = 0 \quad (20.8.3)$$

We now use (20.8.2) to find the magnetic field in a frame moving at velocity $-\mathbf{v}$ with respect to the laboratory frame, where we assume that $\mathrm{v} << c$

$$\bar{\mathbf{B}} = \gamma\left(\mathbf{B}_{perp} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right) = \left(-\frac{1}{c^2}(-\mathbf{v})\times\frac{dq}{4\pi\varepsilon_o}\frac{\mathrm{R}}{R^3}\right) = \frac{1}{c^2}\mathbf{v}\times\left[\frac{dq}{4\pi\varepsilon_o}\frac{\mathrm{R}}{R^3}\right] \quad (20.8.4)$$

In this frame the charge dq appears to be moving at velocity $+\mathbf{v}$, and we see that the magnetic field is just that specified by Biot-Savart in the form of the last term in (20.8.1). Thus the magnetic field of a moving point charge is just the relativistic transformation of the Coulomb field in its rest frame to the frame in which the charge is moving.

20.9 Visualizations of the Magnetic Field of a Charge Moving at Constant Velocity

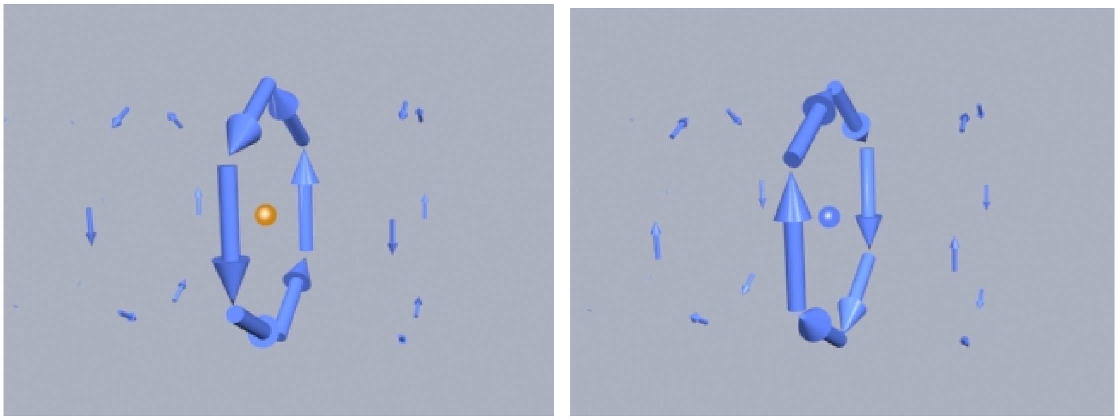

Figure 20-9: The Magnetic Field of a Positive and Negative Charge at Constant Speed

In Figure 20-9 we show the magnetic field of a positive and negative charge moving at constant speed. Movies of this can be found at

http://web.mit.edu/viz/EM/visualizations/magnetostatics/MagneticFieldConfigurations/

21 Magnetic Force on a Moving Charge and on a Current Element

21.1 Learning Objectives

We look at the properties of the $q\mathbf{v}\times\mathbf{B}$ force and the corresponding form for a current carrying wire segment, $Id\mathbf{l}\times\mathbf{B}$.

21.2 $q\mathbf{E}$ and $q\mathbf{v}\times\mathbf{B}$ as the Result of Electric and Magnetic Pressure

The Lorentz force on a moving charge is given by

$$\mathbf{F} = q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) \tag{21.2.1}$$

There are a number of things to be said about this equation, the first being that non-relativistically, the combination $\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$ is the electric field as seen in the rest frame of the charge. We can see this by looking back at equation (15.5.10) for how $\mathbf{E}$ transforms, which we reproduce here.

$$\overline{E}_{\|} = E_{\|} \qquad \overline{\mathbf{E}}_{\perp} = \gamma\left(\mathbf{E}_{\perp}+\mathbf{v}\times\mathbf{B}\right) \tag{21.2.2}$$

There will be a number of occasions where we will identify $\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$ as the electric field in the rest frame of the charge q or of the moving current carrying element $d\mathbf{l}$.

Also, if we consider the Maxwell Stress Tensor, we see that the $q\mathbf{v}\times\mathbf{B}$ magnetic force can be understood as due to pressure and tension in the total magnetic field, just as $q\mathbf{E}$ can be understood as due to pressure and tension from the total electric field. In Figure 21-1 we show frames of movies that illustrate this for the electric field, and in Figure 21-2 we show frames of movies that illustrate this for the magnetic field, and we also give links to the respective movies. If we look at these movies with an eye towards the pressures and tensions given by the Maxwell Stress Tensor, we can get a qualitative feel for why these forces, both electric and magnetic, are in the directions that they are.

Figure 21-1: The qE Force as Due to Maxwell Stresses and Tensions

The figures are frames from movies which can be found at
http://web.mit.edu/viz/EM/visualizations/electrostatics/ForcesOnCharges/forceq/forceq.htm
and
http://web.mit.edu/viz/EM/visualizations/electrostatics/ForcesOnCharges/force_in_efield/force_in_efield.htm

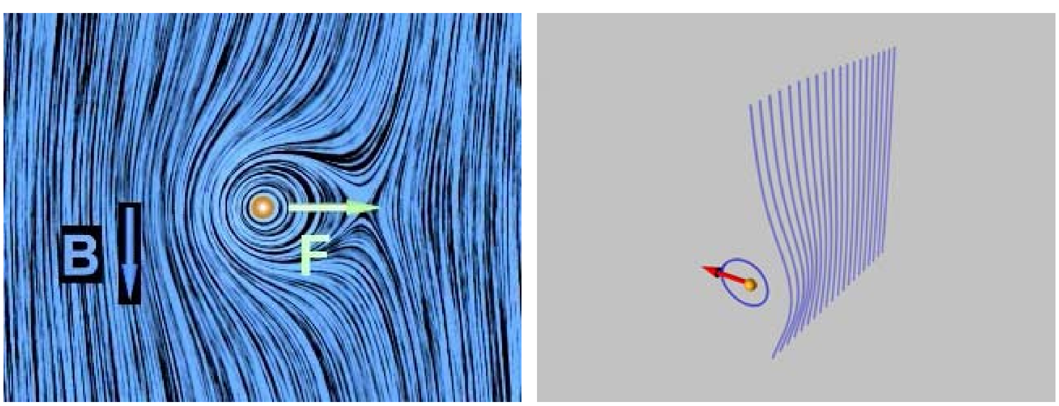

Figure 21-2: The $q\mathbf{v}\times\mathbf{B}$ Force as Due to Maxwell Stresses and Tensions

The figures are frames from movies which can be found at
http://web.mit.edu/viz/EM/visualizations/magnetostatics/ForceOnCurrents/forcemovingq/forcemovingq.htm
and

http://web.mit.edu/viz/EM/visualizations/magnetostatics/ForceOnCurrents/force_in_bfield/force_in_bfield.htm

21.3 Where Does the Momentum Go?

When a charge is moving in cyclotron motion in a magnetic field, its momentum is continually changing. How is momentum conserved in this situation? To answer this, we show in Figure 21-3 a charge entering an external magnetic field which is non-zero only over the circular segment shown in the Figure. The charge's velocity is such that it makes exactly a quarter of a cyclotron revolution before exiting the region where the external field is non-zero. We show only those magnetic field lines which are just outside the circle of revolution of the charge. The field shown is the sum of the external field and the field of the moving charge.

It is clear that the momentum change of the particle over this sequence is absorbed by the current segments which are producing the external field, contained within the solid circular segments at the top and bottom of the figures. If you look at the stress tensor near the currents producing the external field, you can clearly see a force which is pushing those current elements in a direction such as to conserve total momentum. The field is the conduit of the momentum exchange between the charge and the currents producing the external field, but it stores no momentum itself.

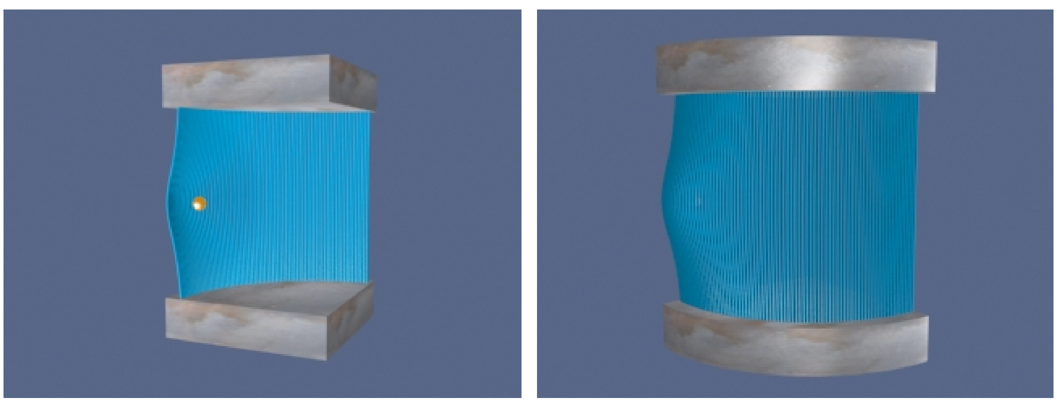

Figure 21-3: A Charge in a External Magnetic Field Transfering Momentum to the Sources of the External Field

The figures are frames from movies which can be found at
http://web.mit.edu/viz/EM/visualizations/magnetostatics/ForceOnCurrents/MovingQinMagnet/MovingQinMagnetFront.htm
and
http://web.mit.edu/viz/EM/visualizations/magnetostatics/ForceOnCurrents/MovingQinMagnet/MovingQinMagnetBack.htm

21.4 The $Id\mathbf{l}\times\mathbf{B}$ Force

Suppose we have n charges per unit volume moving at speed $\mathbf{v}$ in an external $\mathbf{B}$ field. Then the force per unit volume is the force on one times the number of charges per unit volume, or

$$\mathbf{F}_{\text{unit volume}} = qn\,\mathbf{v}\times\mathbf{B} = \mathbf{J}\times\mathbf{B} \tag{21.4.1}$$

where we have used the fact that $\mathbf{J}=qn\,\mathbf{v}$ to get the last form in (21.4.1). To get the total force on a given volume, we integrate over volume.

$$\mathbf{F} = \int \mathbf{J}\times\mathbf{B}\,d^3x \rightarrow \oint Id\mathbf{l}\times\mathbf{B} \tag{21.4.2}$$

where the last form in (21.4.2) is the total force on a current carrying wire in an external field. Thus we see that $Id\mathbf{l}\times\mathbf{B}$ is the force on a current carrying wire segment.

21.5 Force between Parallel Current Carrying Wires

It is easy to see using $Id\mathbf{l}\times\mathbf{B}$ that if we have two parallel current carrying wires, the magnetic force is attractive if the currents in the wires are in the same direction, and repulsive if the currents are in opposite directions. We can also see this from looking at the Maxwell Stress Tensor. If the current in the wires is in the same direction, the field from the two wires subtracts between the wires, and if the current in the wires is in opposite directions, the field adds between the wires. Thus the wires are either pulled together or forced apart by the pressures and tensions associated with the field. This is shown in the two movies in Figure 21-4.

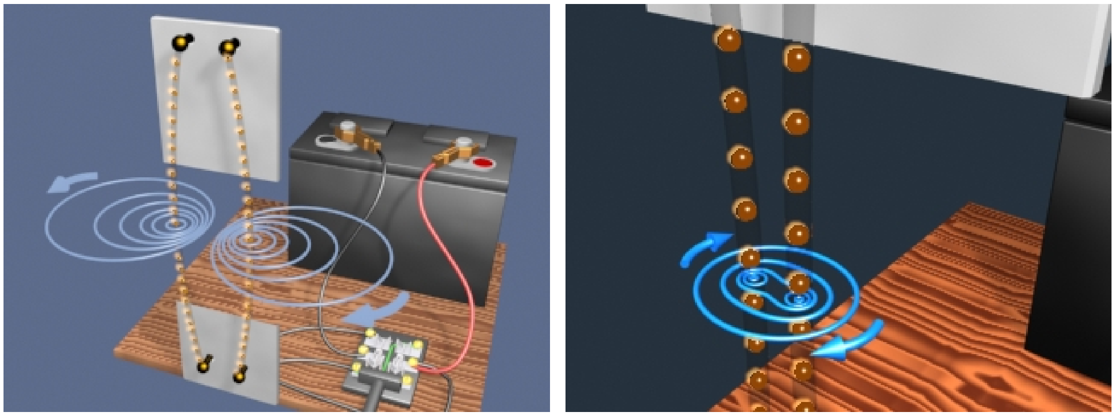

Figure 21-4: Magnetic fields between Parallel Current Carrying Wires

The figures are frames from movies which can be found at
http://web.mit.edu/viz/EM/visualizations/magnetostatics/ForceOnCurrents/SeriesWires/SeriesWires.htm
and

http://web.mit.edu/viz/EM/visualizations/magnetostatics/ForceOnCurrents/ParallelWires/ParallelWires.htm

21.6 Forces on Electric and Magnetic Dipoles in External Fields

21.6.1 The General Derivation

We consider the total force on an isolated and finite distribution of charges $\rho(\mathbf{r})$ and currents $\mathbf{J}(\mathbf{r})$ sitting in the electric field $\mathbf{E}^{ext}(\mathbf{r})$ and magnetic field $\mathbf{B}^{ext}(\mathbf{r})$ due to some external distribution of charges and currents. If the extent of our isolated distrition of charges is d, we will assume that any characteristic scale of variation L in the external fields is such that $d << L$. Then the force on our isolated distribution of charges and currents is given by

$$\mathbf{F} = \int_{volume\ V} \left[\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\right] d^3x = \int_{volume\ V} \left[\rho\mathbf{E}^{ext} + \mathbf{J}\times\mathbf{B}^{ext}\right] d^3x \tag{21.6.1}$$

We suppose that our isolated distribution of charges and currents are centered at the origin, and we expand our external fields in a Taylor series about the origin.

$$E_i^{ext}(\mathbf{r}) = E_i^{ext}(0) + x_j \frac{\partial}{\partial x_j} E_i^{ext}(0) + ... = E_i^{ext}(0) + \mathbf{r}\cdot\nabla E_i^{ext}(0) + ... \tag{21.6.2}$$

where $\frac{\partial}{\partial x_j} E_i^{ext}(0)$ is the gradient of the i$^{\text{th}}$ component of $\mathbf{E}$ with respect to x_j, evaluated at the origin, and similarly for the magnetic field. With this expansion, (21.6.1) becomes to first zeroth order in d/L

$$\mathbf{F} = \mathbf{E}^{ext}(0)\int_V \rho d^3x + \left(\int_V \mathbf{J} d^3x\right)\times\mathbf{B}^{ext}(0) = Q\mathbf{E}^{ext}(0) \tag{21.6.3}$$

since in magnetostatics we have $\int_V \mathbf{J} d^3x = 0$, and where we are using

$$Q = \int_V \rho(\mathbf{r}) d^3x \tag{21.6.4}$$

If we consider the first order terms in d/L in (21.6.1), using the Taylor series expansion (21.6.2), we have

$$\mathbf{F} = \left(\int_V \rho\,\mathbf{r}\, d^3x\right)\cdot\nabla\mathbf{E}^{ext}(0) + \left(\int_V \mathbf{J}\times\left[\mathbf{r}\cdot\nabla\mathbf{B}^{ext}(0)\right] d^3x\right) \tag{21.6.5}$$

and after some work (21.6.5) can be written as

$$\mathbf{F} = \mathbf{p} \cdot \nabla \mathbf{E}^{ext}(0) + \mathbf{m} \cdot \nabla \mathbf{B}^{ext}(0) \tag{21.6.6}$$

where

$$\mathbf{p} = \int_V \mathbf{r} \rho(\mathbf{r})\, d^3x \tag{21.6.7}$$

$$\mathbf{m} = \frac{1}{2}\int_V \mathbf{r} \times \mathbf{J}(\mathbf{r})\, d^3x' = \frac{I}{2}\oint \mathbf{r} \times d\mathbf{l} \tag{21.6.8}$$

Thus there is no force on an electric and magnetic dipole sitting in external fields unless the external fields have gradients, and if they do have gradients then the forces on the dipoles are given by (21.6.6).

21.6.2 A Specific Example of a $\mathbf{m} \cdot \nabla \mathbf{B}^{ext}(0)$ Force

It is easy to see why an electric dipole will field a net force in an inhomogeous electric field, but is not so obvious why a magnetic dipole will field a force in an inhomogeneous magnetic field. We give here an example of how this happens, using the case of two circular loops of current having the same axis, as shown in Figure 21-5.

Suppose we have two current loops sharing the same axis. Suppose furthermore that the sense of the current is the same in both loops as in Figure 21-5. Then these loops attract one another. This is a specific example of the forces on magnetic dipoles in non-uniform fields. To see why this is so, consider the magnetic field $\mathbf{B}_1$ of the lower loop as seen at the location of the upper loop. The presence of this **B** field will cause a force on the upper loop because of the $d\mathbf{F} = i\, d\mathbf{l}$ x $\mathbf{B}_1$ force. If we draw $d\mathbf{l}$ and $\mathbf{B}_1$ carefully, and take their cross-product, we get the result shown in the figure, that is a $d\mathbf{F}$ that has a radial component and a downward component. The radial component will cancel out when we integrate all away around the upper loop to find the net force, but the downward component of the force will not. Thus the upper loop feels a force *attracting* it to the lower loop.

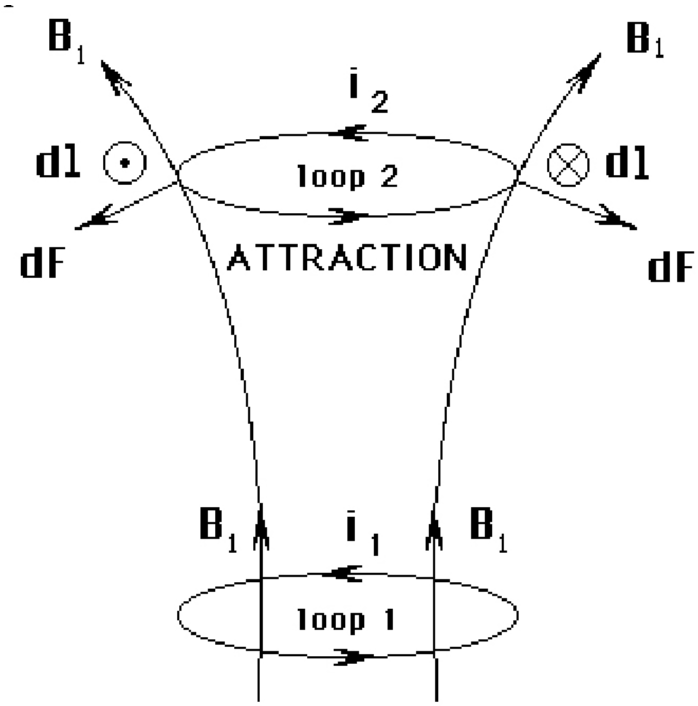

Figure 21-5: Co-axial current loops with currents in the same direction

If now we reverse the direction of current in the upper loop, so that now the currents in the two loops are in opposite senses, we find that the loops are repelled by each other, because the $d\mathbf{F}$'s have reversed from the situation above (see Figure 21-6).

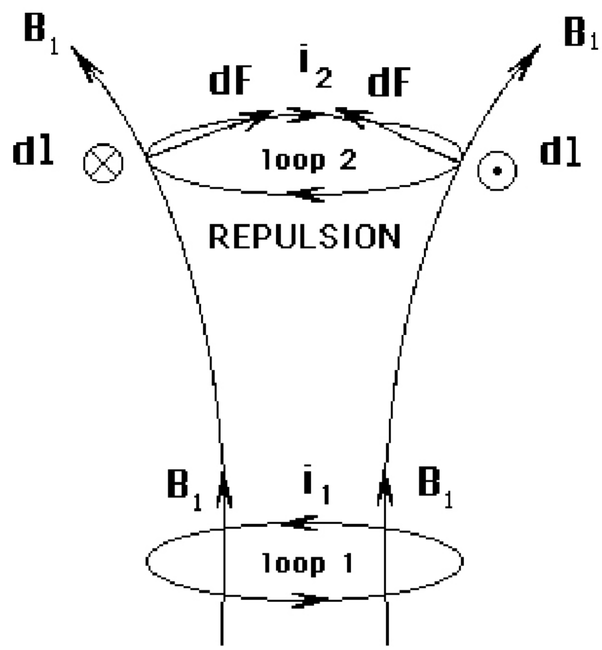

Figure 21-6: Co-axial current loops with currents in opposite directions

Note that this attraction or repulsion depending on the relative sense of the currents explains why north poles of permanent magnets attract the south poles of other magnetics and repel their north poles. We know that permanent magnets are caused by circulating atomic currents in the materials making up the magnet. The north pole of a permanent magnet is right-handed with respect to the its atomic currents--that is, if you curl the fingers of your right hand in the direction of atomic current flow, your thumb will point in the direction of the north pole of the permanent magnet.

In Figure 21-7, I show two permanent magnets, both with their north poles up, and the sense of their atomic currents for this orientation (the atomic currents of course flow on the surface or*inside* the magnets). The magnets in this orientation attract one another, for the same reason that the two co-axial current loops in Figure 21-5 attract one another--the currents are in the same sense, and the resulting *i d***l** x **B** force on the atomic currents in one magnet due to the presence of the magnetic field of the other magnet results in attraction. We loosely say that the south pole of the top magnet "attracts" the adjacent north pole below.

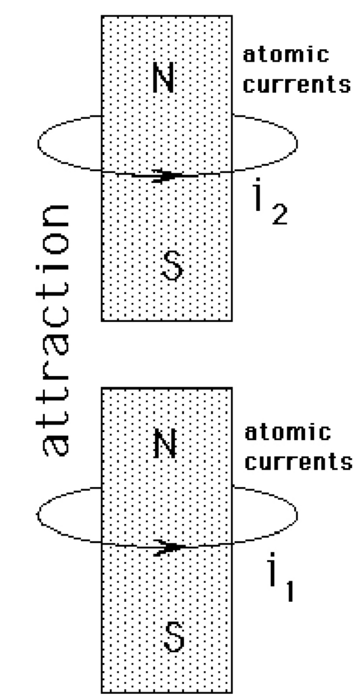

Figure 21-7: Two bar magnets attracting

If we turn the top magnet upside down, so that now the north poles of the two magnets are adjacent, we have reversed the sense of the currents, and therefore we get repulsion, for the same reasons the two current loops repel one another in Figure 21-6. We loosely say that the north pole of the top magnet "repels" the adjacent north pole below. All of these phenomena are simply the result of q**v x B** forces, which are the same as *i* d**l** x **B** forces.

21.7 Torques on Electric and Magnetic Dipoles in External Fields

21.7.1 The General Derivation

Now we consider the torque on an isolated and finite distribution of charges $\rho(\mathbf{r})$ and currents $\mathbf{J}(\mathbf{r})$ sitting in external fields. The torque is given by

$$\boldsymbol{\tau} = \int_V \mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right]d^3x = \int_V \mathbf{r}\times\left[\rho\mathbf{E}^{ext}+\mathbf{J}\times\mathbf{B}^{ext}\right]d^3x \tag{21.7.1}$$

And after some manipulation this can be written as

$$\boldsymbol{\tau} = \mathbf{p}\times\mathbf{E}_{ext}(0)+\mathbf{m}\times\mathbf{B}_{ext}(0) \tag{21.7.2}$$

21.7.2 A Specific Example of a $\mathbf{m}\times\mathbf{B}_{ext}(0)$ Torque

Although it is easy to see where the torque on an electric dipole in an external electric field comes from, it is not so obvious where the torque on a magnetic dipole in an external magnetic field comes from, so we consider this topic further. Suppose we have a rectangular loop of wire with sides of length *a* and *b*, carrying current *i* (see Figure 21-8). It is free to rotate about the axis indicated in the figure. The normal to the plane of the rectangular loop of wire is **n**, where we take **n** to be right-handed with respect to the direction of the current flow (if you curl the fingers of your right hand in the direction of current flow, then your thumb is in the direction of **n**). The normal **n** makes an angle θ with a uniform external magnetic field $\mathbf{B}_{ext}$, as indicated. If we define the magnetic dipole moment **m** of the loop to be $iA\mathbf{n}$, where $A = ab$ is the area of the loop, then according to (21.7.2), the loop will feel a net torque $\boldsymbol{\tau}$ due to magnetic forces with $\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}_{ext}$. This torque will tend to align **m** with $\mathbf{B}_{ext}$.

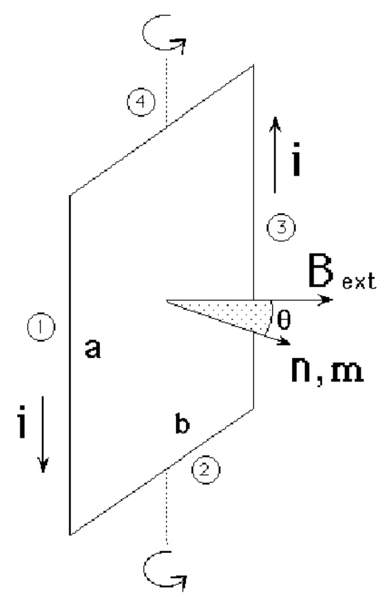

Figure 21-8: A rectangular loop of current in an external magnetic field

To demonstrate that $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}_{ext}$ is the correct expression, consider the forces on each of the four sides of the loop, using the expression for the force on a segment of wire given by $d\mathbf{F} = Id\mathbf{l} \times \mathbf{B}$. The directions of these four forces are indicated in Figure 21-9 below.

In each case, these forces are perpendicular both to $\mathbf{B}_{\text{ext}}$ and to the direction of the current in the segment. The magnitude of the force $\mathbf{F}_2$ on side 2 (of length *b*) is $i\, b\, B_{ext} \cos\theta$, and is equal to the magnitude of the force $\mathbf{F}_4$ on side 4, although opposite in direction. These two forces tend to expand the loop, but taken together contribute nothing to the net force on the loop, and moreover they have zero moment arm through the center of the loop, and therefore contribute no net torque.

The forces $\mathbf{F}_1$ and $\mathbf{F}_3$ also have a common magnitude, $i\, a\, B_{ext}$, and are oppositely directed, so again they contribute nothing to the net force on the loop. However, they *do* contribute to a net torque on the loop, and it is obvious from studying the figure that the two forces tend to rotate the loop in a direction that tends to bring **m** into alignment with $\mathbf{B}_{\text{ext}}$. The forces both have a moment arm of $(b/2) \sin\theta$, and so the total torque is

$$\left|\boldsymbol{\tau}\right| = 2\left(iaB_{ext}\right)\frac{b}{2}\sin\theta = iab\sin\theta = \left|\mathbf{m}\times\mathbf{B}\right| \tag{21.7.3}$$

which is the result we desired. In vector form, $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}_{ext}$.

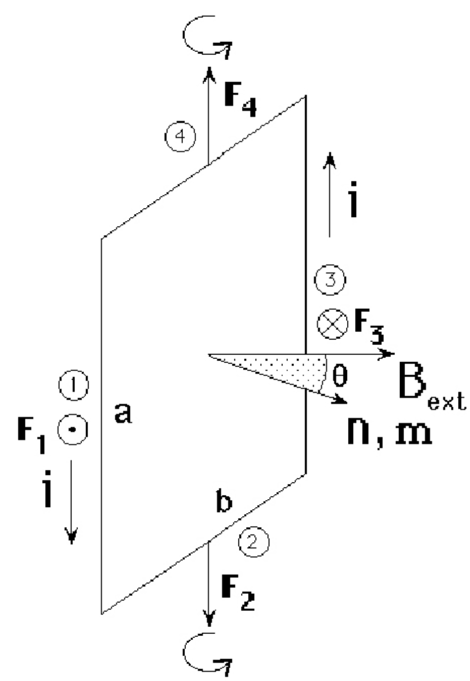

Figure 21-9: The forces on a rectangular loop in an external magnetic field

21.8 Free Dipoles Always Attract

Finally, we point out that when dipoles free to rotate and translate are allowed to interact, the combination of the torques and forces are such that the dipoles always attract. For example the torque will cause to magnetic dipoles to align so that their currents are in the same sense, and then the force of attraction in that configuration will pull them together. An example of this behavior is shown in Figure 21-10.

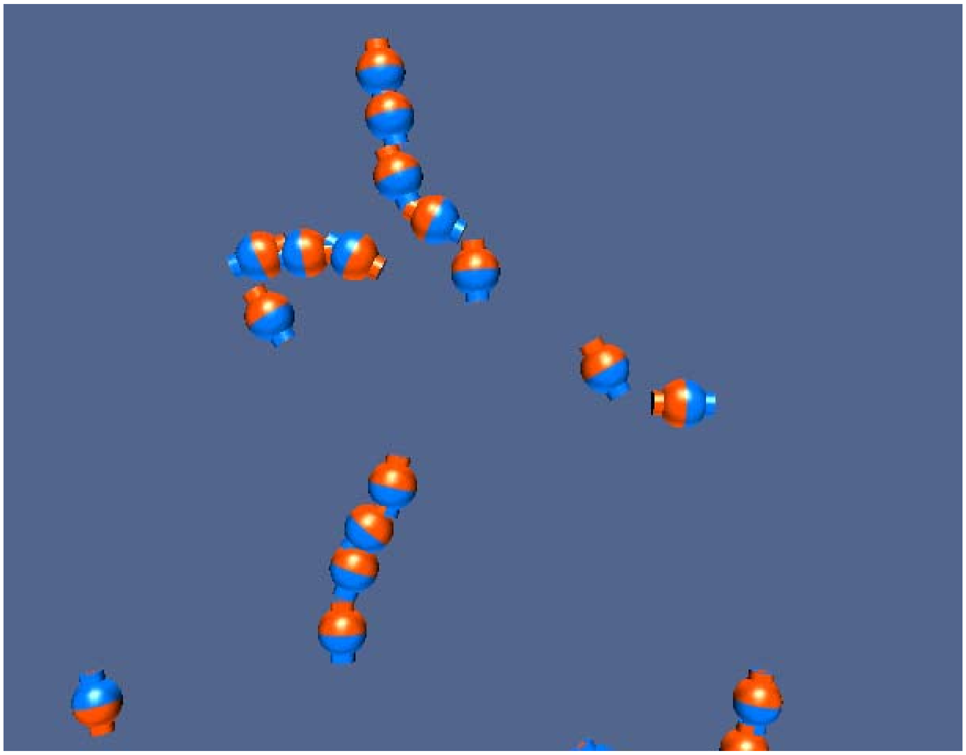

Figure 21-10: Interacting Dipoles Attracting

The application from which this frame is taken can be found at
http://web.mit.edu/viz/EM/visualizations/electrostatics/ForcesTorquesOnDipoles/DipolesShock/DipolesShock.htm

22 Creating and Destroying Electromagnetic Energy and Angular Momentum

22.1 Learning Objectives

We begin our considerations of Faraday's Law, and consider it first in the context of the creation and destruction of magnetic fields, energy, and angular momentum. In later sections we will consider other uses, e.g. with respect to inductance in circuits, but first we look at the most fundamental implication of Faraday's Law.

22.2 Faraday's Law

Faraday's Law in differential form is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{22.2.1}$$

This law is the basis for how we understand why it takes energy to create magnetic fields. The process by which we put energy into magnetic fields, and retrieve it, is fundamental enough that it bears emphasis, and it is a good illustration of the conservation laws we have developed. We have not discussed the energy in magnetic fields up until now because calculating the energy to create magnetic fields is an intrinsically time dependent process.

You may argue that calculating the energy to assemble electric charges is also an intrinsically time dependent process, and we used statics there and got an answer. The difference is that we must have an electric field to work against to evaluate the work done--magnetic fields do no work--and in electrostatics we know the electric fields we are working against to a first approximation. In magnetostatics there are no electric fields in the absolutely static limit, and up to now we have not known how to calculate the electric field produced by slowly varying magnetic fields, which is what we must work against to produce them.

So we are finally in a position to consider the forces that arise, and the work we must do against those forces, when we try to create a magnetic field. The best way to convince us intuitively that magnetic fields require energy to create is to set up a situation where *we* must do the work to create them (rather than let a battery do the work, for example). This is the basis of the example in Section

22.2.1 E x B drift of monopoles in crossed E and B fields

Before we get to the quantitative details, let us first look at a visualization of this process. Before we get to the visualization we need to discuss how we animate the magnetic field lines in this movie. Here is how it is done.

For an electric charge with velocity **v**, mass *m*, and electric charge *q*, the non-relativistic equation of motion in constant **E** and **B** fields is

$$\frac{d}{dt} m\mathbf{v} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{22.2.2}$$

If we define the $\mathbf{E} \times \mathbf{B}$ drift velocity for electric monopoles to be

$$\mathbf{V}_{d,E} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{22.2.3}$$

and make the substitution

$$\mathbf{v} = \mathbf{v}' + \mathbf{V}_{d,E} \tag{22.2.4}$$

then (22.2.2) becomes (assuming **E** and **B** are perpendicular and constant)

$$\frac{d}{dt} m\mathbf{v}' = q\, \mathbf{v}' \times \mathbf{B} \tag{22.2.5}$$

The motion of the electric charge thus reduces to a gyration about the magnetic field line superimposed on the steady drift velocity given by (22.2.3). This expression for the drift velocity is only physically meaningful if the right-hand side is less than the speed of light. This assumption is equivalent to the requirement that the energy density in the electric field be less than that in the magnetic field.

For a hypothetical magnetic monopole of velocity **v**, mass *m*, and magnetic charge q_m, the non-relativistic equation of motion[i] is

(22.2.7) $$\frac{d}{dt} m\mathbf{v} = q_m (\mathbf{B} - \mathbf{v} \times \mathbf{E} / c^2) \tag{22.2.6}$$

If we define the $\mathbf{E} \times \mathbf{B}$ drift velocity for magnetic monopoles to be

$$\mathbf{V}_{d,B} = c^2 \frac{\mathbf{E} \times \mathbf{B}}{E^2} \tag{22.2.7}$$

and make the substitution analogous to (22.2.3), then we recover (22.2.5) with **B** replaced by $-\mathbf{E}/c^2$. That is, the motion of the hypothetical magnetic monopole reduces to a gyration about the electric field line superimposed on a steady drift velocity given by (22.2.7). This expression for the drift velocity is only physically meaningful if it is less than the speed of light. This assumption is equivalent to the requirement that the energy density in the magnetic field be less than that in the electric field. Note that these drift velocities are independent of both the charge and the mass of the monopoles.

In situations where **E** and **B** are not independent of space and time, the drift velocities given above are still approximate solutions to the full motion of the monopoles as long as the radius and period of gyration are small compared to the characteristic length and time scales of the variation in **E** and **B**. There are other drift velocities that depend on both the sign of the charge and the magnitude of its gyroradius, but these can be made arbitrarily small if the gyroradius of the monopole is made arbitrarily small. The gyroradius depends on the kinetic energy of the charge as seen in a frame moving with the drift velocities. When we say that we are considering "low-energy" test monopoles in what follows, we mean that we take the kinetic energy (and thus the gyroradii) of the monopoles in a frame moving with the drift velocity to be as small as we desire.

The definition use to construct our electric field line motions is equivalent to taking the local velocity of an electric field line in electro-quasi-statics to be the drift velocity of low energy test magnetic monopoles spread along that field line. Similarly, the definition we use to construct our magnetic field line motions below is equivalent to taking the local velocity of a magnetic field line in magneto-quasi-statics to be the drift velocity of low energy test electric charges spread along that field line. These choices are thus physically based in terms of test particle motion, and have the advantage that the local motion of the field lines is in the direction of the Poynting vector.

22.2.2 A visualization of the creation and destruction process

Before we get to the quantitative details, let us consider a movie of this process. In Figure 22-1, we show one frame of a movie showing the creation process. You can find a link to this movie at

http://web.mit.edu/viz/EM/visualizations/faraday/CreatingMagneticEnergy/

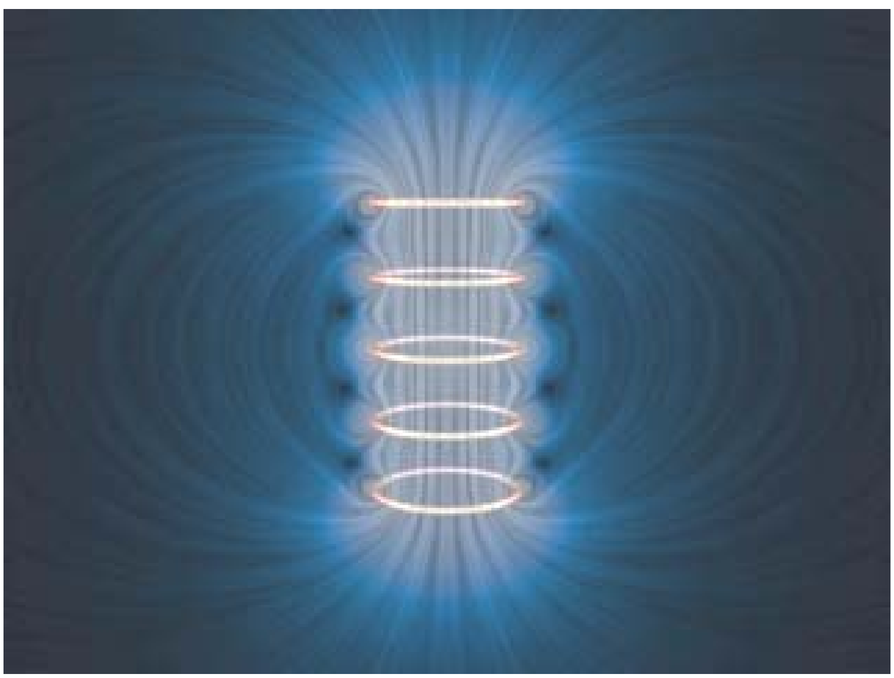

Figure 22-1: One frame of a movie showing the creation of magnetic fields

In the movie, we have five rings that carry a number of free positive charges that are not moving. Since there is no current, there is no magnetic field. Now suppose a set of external agents come along (one for each charge) and simultaneously spin up the charges counterclockwise as seen from above, at the same time and at the same rate, in a manner that has been pre-arranged. Once the charges on the rings start to accelerate, there is a magnetic field in the space between the rings, mostly parallel to their common axis, which is stronger inside the rings than outside. This is the solenoid configuration we shall consider in quantitative detail below in the next section.

As the magnetic flux through the rings grows, Faraday's Law tells us that there is an electric field induced by the time-changing magnetic field. This electric field is circulating clockwise as seen from above. The force on the charges due to this electric field is thus opposite the direction the external agents are trying to spin the rings up in (counterclockwise), and thus the agents have to do additional work to spin up the rings because they are charged. This is the source of the energy that is appearing in the magnetic field between the rings-the work done by the agents against the "back emf".

Over the time when the magnetic field is increasing in the animation, the agents moving the charges to a higher speed against the induced electric field are continually doing work. The electromagnetic energy that they are creating at the place where they are doing work (the path along which the charges move) flows both inward and outward. The direction of the flow of this energy is shown by the animated texture patterns. This is the electromagnetic energy flow that increases the strength of the magnetic field in the space between the rings as each positive charge is accelerated to a higher and higher velocity.

In the case of the destruction of magnetic fields, suppose we have the same five rings as above, this time carrying a number of free positive charges that are moving counter-clockwise. This current results in a magnetic field that is strong inside the rings and weak outside. Now suppose a set of external agents come along (one for each charge) and simultaneously spin down the charges as seen from above, at the same time and at the same rate, in a manner that has been pre-arranged. Once the charges on the rings start to deccelerate, the magnetic field begins to decrease in intensity.

As the magnetic flux through the rings decreases, Faraday's Law tells us that there is an electric field induced by the time-changing magnetic field. This electric field is circulating clockwise as seen from above. The force on the charges due to this electric field is thus opposite the direction the external agents are trying to spin the rings down in (counter-clockwise), and thus work is done on those agents.

As the strength of the magnetic field decreases, the magnetic energy flows from the field back to the path along which the charges move, and is now provided to the agents trying to spin down the moving charges. The energy provided to those agents as they destroy the magnetic field is exactly the amount of energy that they put into creating the magnetic field in the first place, neglecting radiative losses. This is a totally reversible process if we neglect such losses. That is, the amount of energy the agents put into creating the magnetic field is exactly returned to the agents as the field is destroyed.

22.3 The spinning cylinder of charge

22.3.1 The fields of a spinning cylinder

Now let us do the quantitative calculation that confirms the qualitative picture above. Suppose we create a magnetic field in the following way. We have a long cylindrical shell of non-conducting material which carries a surface charge fixed in place (glued down) of σ Coulombs per square meter. The length of the cylinder is *L*, which is much greater than its radius *R*. The cylinder is suspended in a manner such that it is free to revolve about its axis, without friction. Initially it is at rest, and there is no magnetic field. We come along and spin the cylinder up until the speed of the surface of the cylinder is V_o. After spinning it up, we will have a magnetic field inside the cylinder and outside the cylinder the field will be zero. The field inside the cylinder will be given by

$$\mathbf{B}_o = \mu_o \kappa_o \,\hat{\mathbf{z}} = \mu_o \sigma V_o \,\hat{\mathbf{z}} \tag{22.3.1}$$

This is just our standard magnetostatic formula for the field inside a solenoid with surface current κ_o. Let's calculate the amount of work we have to do to create this magnetic field.

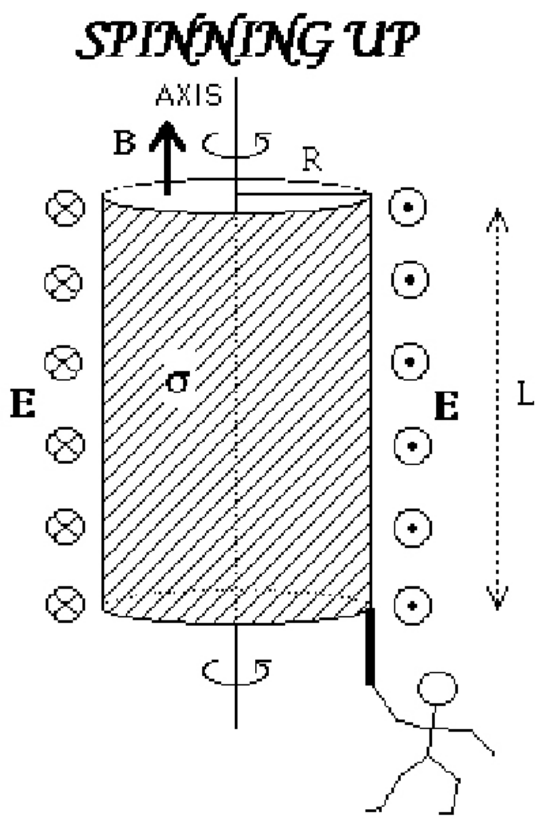

Figure 22-2: Spinning up a cylindrical shell of charge

Imagine we are in the middle of this process, and have gotten the cylinder up to some speed *V(t)* at time *t*, where *V(t)* is increasing with time and is less than the final speed V_o. At that point, there will be a magnetic field inside the cylinder that is also increasing with time, which is ***approximately*** given by

$$\mathbf{B}(\mathbf{r},t) \approx \mu_o \kappa(t)\,\hat{\mathbf{z}} = \mu_o \sigma V(t)\,\hat{\mathbf{z}} \tag{22.3.2}$$

You should immediately object that this is now a time varying situation, so the static solution is no longer correct. However we have seen in our studies of the various regions around an isolated time varying set of charges and currents (see Section 9.3) that it is ok to use the static solutions as long as the time *T* over which we spin the cylinder up significantly is much longer than the time that it takes light to cross the cylinder, R/c. If this is true, and we assume it is, and if we are within the Near Zone, that is, $r << cT$ then equation (22.3.2) is a good approximation to the actual magnetic field.

Now, $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$, with **B** in the z-direction, and let us assume that the electric field is a combination of an azimuthal electric field in addition to our cylindrical radial field associated with the charge density σ. That is, we assume that

$$\mathbf{E}(\mathbf{r},t) = \begin{cases} E_\phi \hat{\boldsymbol{\phi}} & r < R \\ \dfrac{\sigma}{\varepsilon_o}\hat{\mathbf{r}} + E_\phi \hat{\boldsymbol{\phi}} & r > R \end{cases} \tag{22.3.3}$$

If we apply Faraday's Law to a circle of cylindrical radius r about the axis of the cylinder, we find that the line integral of the electric field yields

$$\oint \mathbf{E}(\mathbf{r},t)\cdot d\mathbf{l} = 2\pi r E_\phi \tag{22.3.4}$$

The magnetic flux through the imaginary circle at time *t* is given by

$$\int_{surface} \mathbf{B}(\mathbf{r},t)\cdot\hat{\mathbf{n}}\,da = \begin{cases} \pi r^2 \sigma V(t) & r < R \\ \pi R^2 \sigma V(t) & r > R \end{cases} \tag{22.3.5}$$

Thus we find that that part of the electric field "induced" by the time-changing magnetic field is

$$E_\phi \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \begin{cases} \dfrac{\mu_o r\,\sigma}{2}\dfrac{dV(t)}{dt} & r < R \\ \dfrac{\mu_o R^2 \sigma}{2r}\dfrac{dV(t)}{dt} & r > R \end{cases} \tag{22.3.6}$$

We frequently refer to that part of the electric field associated purely with time-changing magnetic fields as *induced* fields, thus the origin of the terms "inductance", "inductors", and so on.

The sense of the electric field is shown in Figure 22-2. When we are trying to spin up the cylinder, the “induced” electric field will cause forces on the (glued on) charges on the cylinder walls which are such as to *resist* us spinning up the cylinder. This is why we have to do additional work to spin up a charged cylinder, because we have to do work to overcome the forces associated with the induced electric fields, which are in a direction such as to resist change. This is an example of Lenz's Law--the reaction of the system to change is always such as to resist the change.

22.3.2 The work done to spin up the cylinder

How much energy does it take us to do this? We must exert a force in the $\hat{\boldsymbol{\phi}}$ direction to increase the speed of the cylinder. In the following, we ignore the radial part of the electric field, since we do no work against it. To increase the speed of a little bit of charge $+dq$ on the cylinder, we must provide a force $\mathbf{F}_{me} = -dq\,\mathbf{E}$. That is, *we* must provide an additional force in the azimuthal direction that the charge is moving that *balances* the retarding force due to the induced electric field (plus a little teeny bit more, to actually increase the speed of the charge, but we can neglect that little teeny bit). Thus the rate at which we do work, $\frac{dW}{dt} = \mathbf{F}_{me} \cdot \mathbf{V}$, is positive, since $\mathbf{F}_{me} = -dq\,\mathbf{E}$ is in the direction of $\boldsymbol{V}$. The work to increase the speed of a little $+dq$ is thus

$$\frac{dW}{dt} = \mathbf{F}_{me} \cdot \mathbf{V} = -dqV(t)E_{\phi} = +dqV(t)\left[\frac{\mu_o R\sigma}{2}\frac{dV(t)}{dt}\right] \qquad (22.3.7)$$

$dW/dt = +\, dq\, V(\mu_o \sigma R/2)dV/dt$, where we have used our expression in (22.3.6) for E_{ϕ}. Since the total charge on the cylinder is $Q = \sigma\left[2\pi rL\right]$, the total rate at which we do work to spin the cylinder up is just found by replacing the dq in (22.3.7) by Q, giving

$$\frac{dW}{dt} = \sigma\left[2\pi rL\right]V(t)\left[\frac{\mu_o R\sigma}{2}\frac{dV(t)}{dt}\right] \qquad (22.3.8)$$

which after some manipulation and using $B = \mu_o \sigma V(t)$ can be written as

$$\frac{dW}{dt} = \pi R^2 L \frac{d}{dt}\left[\frac{B^2}{2\mu_o}\right] \qquad (22.3.9)$$

To get the total energy to spin up the cylinder, we just integrate this expression with respect to time to obtain $\pi R^2 L\left[\frac{B_o^2}{2\mu_o}\right]$. Since $\pi R^2 L$ is the volume of our cylinder, this implies an energy density in the magnetic field of $\frac{B_o^2}{2\mu_o}$. It is clear where the energy to create this field came from--*us!*

Moreover, this is process is totally reversible--we can get the energy right back out of the magnetic field. To do this, we just grab hold of the spinning cylinder and spin it down. When we try to spin the cylinder down, we are trying to decrease the magnetic flux, and thus we will have a induced electric field that will try to *keep the cylinder spinning*, that is it will reverse direction from the situation above. This is exactly what we expect from Lenz's Law--the induced electric field is in a direction so as to keep things the same. The rate at which we do work, $\frac{dW}{dt} = \mathbf{F}_{me} \cdot \mathbf{V}$, will now be negative, since $\mathbf{F}_{me} = -dq\,\mathbf{E}$ reverses sign with the electric field. That means that work is being done on us, and we can use that work to take a free ride, or whatever.

In any case, we are getting energy back out of the process as we spin down the cylinder, and thereby destroying the magnetic fields. With a little thought, it is clear that we will get back exactly the amount of energy we put in the first place to create the magnetic field. Of course we are neglecting radiation losses when we make this statement, but from our discussions in Section 9.5, equation (9.5.7), we know that the energy loss compared to the stored energy is of order $\left(R / cT\right)^3$, and we can make this as small as we desire by simply doing things more and more slowly.

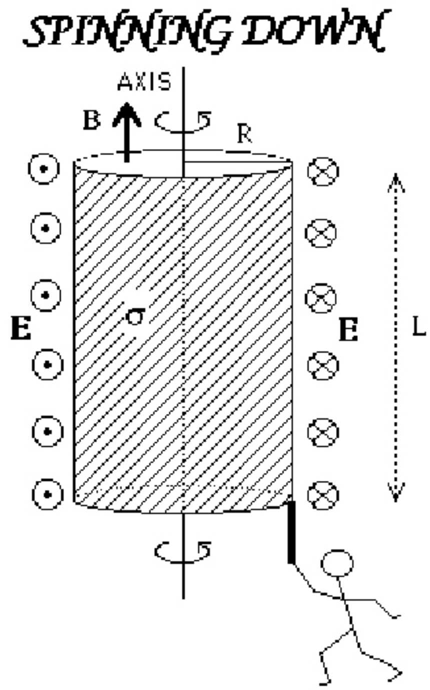

Figure 22-3: Spinning down a cylindrical shell of charge

What we have presented above, based on forces, is an intuitive approach to understanding the origin of energy in magnetic fields. It is makes manifest that to create a magnetic field you must do work against the induced electric fields that are associated with the increasing magnetic flux. Conversely, when you destroy a magnetic field, work is done on you by the induced electric field associated with the decreasing magnetic flux.

22.3.3 Energy flow in spinning up the cylinder

Now, consider this entire process from the point of view of the conservation of energy law given in (4.4.2) in integral form, that is,

$$\frac{\partial}{\partial t}\int_{volume}\left[\frac{1}{2}\varepsilon_o E^2+\frac{B^2}{2\mu_o}\right]d^3x+\int_{surface}\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right)\cdot\hat{\mathbf{n}}\,da=\int_{volume}\left[-\mathbf{E}\cdot\mathbf{J}\right]d^3x \qquad (22.3.10)$$

When we have a surface current, the term

$$\int_{volume}\left[-\mathbf{E}\cdot\mathbf{J}\right]d^3x=\int_{surface}\left[-\mathbf{E}\cdot\boldsymbol{\kappa}\right]da \qquad (22.3.11)$$

where $\boldsymbol{\kappa}=\sigma V\hat{\phi}$. This is the term we evaluated above. We are doing work and therefore depositing energy at the circumference of the cylinder. That energy, deposited in a thin cylindrical shell of radius R, then flows inward toward the center of the cylinder at a rate given by the Poynting vector $\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}$. We can easily see that the total Poynting flux calculated just inside the surface of the cylinder r a little less that R is given by

$$\int_{surface}\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right)\cdot\hat{\mathbf{n}}\,da=\int_{surface}\left(-\frac{R\sigma}{2}\frac{dV(t)}{dt}B\,\hat{\mathbf{r}}\right)\cdot\hat{\mathbf{r}}\,da=-\pi R^2L\frac{d}{dt}\left[\frac{B^2}{2\mu_o}\right] \qquad (22.3.12)$$

Thus we create the energy at $r = R$, and it flows inward to reside in the magnetic field for $r < R$.

22.3.4 Electromagnetic angular momentum for the spinning cylinder

In you think a minute you realize that we should also be creating electromagnetic angular momemtum when we spin up the cylinder, because we have to apply an additional torque to spin up the cylinder because of the "back" emf. But if you look at the density of electromagnetic angular momentum after we are finished, $\mathbf{r}\times\left[\varepsilon_o\mathbf{E}\times\mathbf{B}\right]$, it is zero everywhere. In actual fact you can indeed calculate the creation rate of angular momentum at the shell and the flux of angular momentum outward for $r > R$, and it is *not* zero, but it flows to infinity and is stored in the fringing fields at infinity, which are not accessible to us because we have assumed an infinitely long cylinder. To see the electromagnetic angular momemtum we put into spinning up a distribution of charge, we must take a finite, not infinitely long body, which we do in the next section for a sphere.

22.4 The Spinning Sphere of Charge

22.4.1 The fields of a spinning sphere

We now carry out the same calculations as above except we look at a more realistic situation, which will allow us to see where the electromagnetic angular momentum is stored. Instead of an infinitely long spinning cylinder of charge, we look at a spherical shell of radius R that carries a uniform surface charge σ. Its total charge Q is $4\pi R^2\sigma$, and its Coulomb electric field is

$$\mathbf{E}_{coulomb}=\begin{cases}0 & r<R\\ \dfrac{Q}{4\pi\varepsilon_o r^2}\hat{\mathbf{r}} & r>R\end{cases} \qquad (22.4.1)$$

We begin spinning the sphere at a angular velocity $\omega(t)$ with $\omega R << c$. The motion of the charge glued onto the surface of the spinning sphere results in a surface current

$$\boldsymbol{\kappa}(t)=\sigma\,\omega(t)\,R\sin\theta\,\hat{\boldsymbol{\phi}}=\kappa(t)\sin\theta\,\hat{\boldsymbol{\phi}} \qquad (22.4.2)$$

where $\kappa(t) = \sigma\,\omega(t)\,R$.

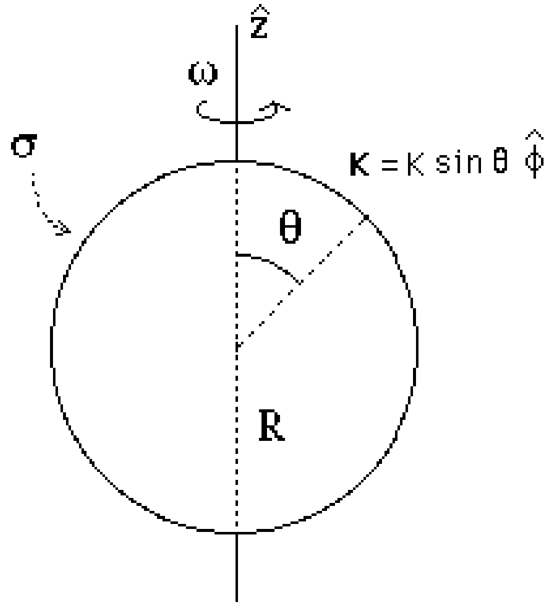

Figure 22-4: A spinning sphere of charge with surface current κ

We can use the quasi-static approximation to get a good approximation to the time dependent solution for **B** (good for variations in $\kappa(t)$ with time scales $T \approx \dfrac{\kappa}{d\kappa/dt} \gg \dfrac{R}{c}$) If we define

$$m(t) = \frac{4\pi\, R^3}{3}\kappa(t) \qquad B(t) = \frac{2\mu_o}{3}\kappa(t) \tag{22.4.3}$$

so that

$$B(t) = \frac{\mu_o m(t)}{2\pi R^3} \tag{22.4.4}$$

then our quasi-static solution for **B** can be shown to be (see Griffiths Example 5.11 page 236-237)

$$\mathbf{B}(\mathbf{r},t) = \begin{cases} \dfrac{\mu_o m(t)}{4\pi\, r^3}\left(2\cos\theta\;\hat{\mathbf{r}} + \sin\theta\;\hat{\boldsymbol{\theta}}\right) & (r > R) \\ \hat{\mathbf{z}}\;B(t) & (r < R) \end{cases} \tag{22.4.5}$$

We thus have from (22.4.5) that

$$\frac{d\mathbf{B}(\mathbf{r},t)}{dt} = \begin{cases} \dfrac{\mu_o}{4\pi\, r^3}\left(2\cos\theta\;\hat{\mathbf{r}} + \sin\theta\;\hat{\boldsymbol{\theta}}\right)\dfrac{dm(t)}{dt} & (r > R) \\ \hat{\mathbf{z}}\;\dfrac{d}{dt}B(t) & (r < R) \end{cases} \tag{22.4.6}$$

Given (22.4.6), we can find the induction electric field everywhere in space, as follows.

For $r < R$, take we take a circle whose normal is along the z-axis, whose center is located a distance $r\cos\theta$ up the z-axis, and whose radius is $r\sin\theta$. If we apply Faraday's Law in integral form to that circle, we have

$$2\pi r\sin\theta E_\phi = -\pi\left(r\sin\theta\right)^2 \frac{dB}{dt} \tag{22.4.7}$$

or

$$\mathbf{E}_{\text{induction}} = -\hat{\boldsymbol{\phi}}\frac{r\sin\theta}{2}\frac{dB}{dt} = -\hat{\boldsymbol{\phi}}\frac{r\mu_o}{3}\frac{d\kappa}{dt}\sin\theta = -\hat{\boldsymbol{\phi}}\frac{r\mu_o\sin\theta}{4\pi R^3}\frac{dm}{dt} \tag{22.4.8}$$

where we have used (22.4.3).

For $r > R$, if we assume $\mathbf{E}_{induction} = E_\phi\hat{\boldsymbol{\phi}}$, then

$$\nabla\times\mathbf{E} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta E_\phi\right) - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial r}\left(rE_\phi\right) \tag{22.4.9}$$

Comparing this expression to (22.4.6), we see that from Faraday's Law we have for $r > R$

$$\mathbf{E}_\phi = -\frac{\mu_o\sin\theta}{4\pi r^2}\frac{dm}{dt} \tag{22.4.10}$$

Our complete electric field, Coulomb plus induction field, is thus given by

$$\mathbf{E} = \mathbf{E}_{coulomb} + \mathbf{E}_{induction} = \begin{cases} -\hat{\boldsymbol{\phi}}\dfrac{r\mu_o\sin\theta}{4\pi R^3}\dfrac{dm}{dt} & r < R \\ \dfrac{Q}{4\pi\varepsilon_o r^2}\hat{\mathbf{r}} - \hat{\boldsymbol{\phi}}\dfrac{\mu_o\sin\theta}{4\pi r^2}\dfrac{dm}{dt} & r > R \end{cases} \tag{22.4.11}$$

It is instructive to compare the induction term for $r > R$ to the Coulomb term

$$\frac{E_{induction}}{E_{coulomb}} = \frac{4\pi\varepsilon_o r^2}{Q}\frac{\mu_o\sin\theta}{4\pi r^2}\frac{dm}{dt} = \frac{\sin\theta}{Qc^2}\frac{d}{dt}\left[\frac{4\pi R^3}{3}\omega R\frac{Q}{4\pi R^2}\right] \tag{22.4.12}$$

$$\frac{E_{induction}}{E_{coulomb}} = \frac{\sin\theta R^2}{3c^2}\frac{d\omega}{dt} = \frac{\omega\sin\theta R^2}{3c^2}\frac{1}{\omega}\frac{d\omega}{dt} \approx \frac{\omega R^2}{3c^2 T} \approx \frac{1}{3}\left(\frac{\omega R}{c}\right)\left(\frac{R}{cT}\right) \qquad (22.4.13)$$

To get the final form in (22.4.13) above we have used the time scale T for changes in the angular velocity defined by

$$\frac{1}{T} = \frac{1}{\omega}\frac{d\omega}{dt} \qquad (22.4.14)$$

Our final result in (22.4.13) above shows that the ratio is the product of two terms, both of which we are assuming to be small, so the ratio of the induction field to the Coulomb is second order small in small quantities.

22.4.2 The total magnetic energy of the spinning sphere

For our purposes below, we want to calculate the magnetic energy outside of R and inside of R . Outside of the sphere

$$\int_{r>R}\left[\frac{B^2}{2\mu_o}\right]d^3x = 2\pi\int_{-1}^{1} d\left(\cos\theta\right)\int_R^\infty\left[\frac{B^2}{2\mu_o}\right]r^2 dr \qquad (22.4.15)$$

and using (22.4.5)

$$\begin{aligned}\int_{r>R}\left[\frac{B^2}{2\mu_o}\right]d^3x &= 2\pi\int_{-1}^{1} d\left(\cos\theta\right)\int_R^\infty\left[\frac{1}{2\mu_o}\left(\frac{\mu_o m}{4\pi\, r^3}\right)^2\left(4\cos^2\theta+\sin^2\theta\right)\right]r^2 dr \\ &= 2\pi\int_{-1}^{1}\left(3\cos^2\theta+1\right)d\left(\cos\theta\right)\int_R^\infty\left[\frac{\mu_o m^2}{32\pi^2}\frac{1}{r^4}\right]dr \qquad (22.4.16) \\ &= \frac{\mu_o m^2}{12\pi R^3}\end{aligned}$$

In contrast, inside the sphere,

$$\int_{r<R}\left[\frac{B^2}{2\mu_o}\right]d^3x = \frac{B^2}{2\mu_o}\frac{4\pi R^3}{3} = \frac{2\pi R^3}{3\mu_o}\left[\frac{2\mu_o}{3}\kappa(t)\right]^2 = \frac{\mu_o m^2}{6\pi R^3} \qquad (22.4.17)$$

so the magnetic energy inside the sphere is twice that outside the sphere. The total energy we have at time t in the magnetic field is given by the sum of the two terms we calculated above, or $\frac{\mu_o m^2}{6\pi R^3}+\frac{\mu_o m^2}{12\pi R^3}$, which is $\frac{\mu_o m^2}{4\pi R^3}$.

22.4.3 The creation rate of magnetic energy

The total rate at which electromagnetic energy is being created as the sphere is being spun up, $\int\limits_{\text{all space}} -\mathbf{J}\cdot\mathbf{E}\, d^3x$, is given by

$$\int\limits_{\text{all space}} -\mathbf{J}\cdot\mathbf{E}\, d^3x = \int\limits_{\text{all space}} -\left[\hat{\boldsymbol{\phi}}\kappa(t)\sin\theta\delta(r-R)\right]\cdot\left[-\hat{\boldsymbol{\phi}}\frac{\mu_o\sin\theta}{4\pi r^2}\frac{dm}{dt}\right]d^3x$$
$$=\frac{\mu_o\kappa(t)}{2}\frac{dm}{dt}\int_{-1}^{1}\sin^2\theta\, d\left(\cos\theta\right) = \frac{2\mu_o\kappa(t)}{3}\frac{dm}{dt} = \frac{\mu_o m}{2\pi R^3}\frac{dm}{dt} = \frac{d}{dt}\left[\frac{\mu_o m^2}{4\pi R^3}\right] \quad (22.4.18)$$

so from (22.4.18) we are doing work at just the rate needed to increase the total magnetic energy density we have at a given time. You might worry that we have not taken into account the energy required to create the inductive electric field, but if we estimate how big that energy is, using (22.4.10), we have

$$\frac{1}{2}\varepsilon_o\mathbf{E}_\phi^2 \approx \frac{1}{2}\varepsilon_o\left(\frac{\mu_o}{4\pi R^2}\frac{m}{T}\right)^2 \approx \frac{1}{2}\varepsilon_o\left(\frac{R^3}{4\pi R^2}\frac{B}{T}\right)^2 \approx \left(\frac{R}{cT}\right)^2\frac{B^2}{2\mu_o} \quad (22.4.19)$$

So we can neglect the energy in the induction electric field compared to that in the magnetic field, since it is second order small. If you are spinning up the sphere, it is you who are creating this energy by the additional work you must do to offset the force associated with the induction electric field.

22.4.4 The flow of energy in the spinning sphere

The energy is being created at $r = R$, and we can see if flowing away from its creation site by using the Poynting vector to calculate the flux of electromagnetic energy $\int\limits_{surface}\left[\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right]\cdot\hat{\mathbf{r}}\, da$ through a spherical surface of radius r for r a little greater than R and also for r a little smaller than R. First let's do this for r a little greater than R. There we have

$$\mathbf{E}\times\mathbf{B} = \left[\frac{Q}{4\pi\varepsilon_o r^2}\hat{\mathbf{r}} - \hat{\boldsymbol{\phi}}\frac{\mu_o\sin\theta}{4\pi r^2}\frac{dm}{dt}\right]\times\left[\frac{\mu_o m(t)}{4\pi r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)\right]$$
$$=\frac{\mu_o m(t)}{4\pi r^3}\left[\frac{Q}{4\pi\varepsilon_o r^2}\sin\theta\left(\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}\right) - \frac{\mu_o\sin\theta}{4\pi r^2}\frac{dm}{dt}2\cos\theta\left(\hat{\boldsymbol{\phi}}\times\hat{\mathbf{r}}\right) - \frac{\mu_o\sin^2\theta}{4\pi r^2}\frac{dm}{dt}\left(\hat{\boldsymbol{\phi}}\times\hat{\boldsymbol{\theta}}\right)\right]$$
$$==\frac{\mu_o m(t)}{4\pi r^3}\left[\frac{Q}{4\pi\varepsilon_o r^2}\sin\theta\,\hat{\boldsymbol{\phi}} - \frac{\mu_o 2\sin\theta\cos\theta}{4\pi r^2}\frac{dm}{dt}\hat{\boldsymbol{\theta}} + \frac{\mu_o\sin^2\theta}{4\pi r^2}\frac{dm}{dt}\hat{\mathbf{r}}\right]$$
(22.4.20)

So that the energy moving away from the sphere at a distance a little greater than R is

$$\int_{surface} \left[\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right]\cdot\hat{\mathbf{r}}\,da = \int_{surface} \left[\frac{1}{\mu_o}\frac{\mu_o m(t)}{4\pi\ R^3}\frac{\mu_o \sin^2\theta}{4\pi R^2}\frac{dm}{dt}\right]da = \frac{\mu_o m(t)}{6\pi\ R^3}\frac{dm}{dt} = \frac{d}{dt}\left[\frac{\mu_o m^2(t)}{12\pi\ R^3}\right] \tag{22.4.21}$$

This is exactly the energy flow outward we need to increase the magnetic energy outside the sphere at a given time. Now let us consider the energy flow across a sphere of radius $r < R$. There we have

$$\mathbf{E}\times\mathbf{B} = \left[-\hat{\boldsymbol{\phi}}\frac{r\mu_o \sin\theta}{3}\frac{d\kappa}{dt}\right]\times\left[\hat{\mathbf{z}}\ B(t)\right] = -\frac{r\mu_o}{3}\frac{d\kappa}{dt}\sin\theta B(t)\left(\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}}\right)$$

$$\mathbf{E}\times\mathbf{B} = -\frac{r\left(\mu_o\right)^2}{9}\frac{d\kappa^2}{dt}\sin\theta\left(\hat{\boldsymbol{\phi}}\times\left(\hat{\mathbf{r}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta\right)\right) = -\frac{d}{dt}\frac{r\left(\mu_o\right)^2 m^2}{\left(4\pi R^3\right)^2}\sin\theta\left(\hat{\boldsymbol{\theta}}\cos\theta+\hat{\mathbf{r}}\sin\theta\right) \tag{22.4.22}$$

$$\text{So}\quad \int_{surface}\left[\frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\right]\cdot\hat{\mathbf{r}}\,da = -2\pi\frac{\mu_o}{\left(4\pi\right)^2 R^3}\frac{dm^2}{dt}\int_{-1}^{1} d(\cos\theta)\sin^2\theta = -\frac{d}{dt}\frac{\mu_o m^2}{6\pi R^3}$$

The minus sign in this equation means the energy flow is inward, and it is exactly the amount we need to account for the rate at which magnetic energy is building up in the interior (see (22.4.17)).

So all of this makes sense, we are creating energy at the shell where we are doing work, and it is flowing out from where we create it at exactly the rate that we need for the build up of magnetic energy inside and outside of the sphere.

Unfortunately, to solve the problem in this relatively simple form, we have had to assume we are doing everything really slowly compared to the speed of light transit time across the sphere, so in none of our terms above do we explicitly see fields propagating at the speed of light. Our solutions just change instantaneously in time everywhere in space, and that is because we have essentially assumed that c is infinite to get the tractable expressions above. We can however solve the full problem without making any assumptions as to how T compares to R/c, but the solutions are much more complicated (see http://web.mit.edu/viz/spin/) When we do this, we obtain solutions in which the energy created at $r = R$ propagates inward and outward at the speed of light, reaching the center of the sphere at a time R/c after it was created at the cylinder walls. We show one frame of a visualization of this process, in which you can just see the inward and outward propagation of fields at the speed of light, in Figure 22-5. The full movie for Figure 22-5 can be found at

http://web.mit.edu/viz/spin/visualizations/l=1/slow/slow.htm

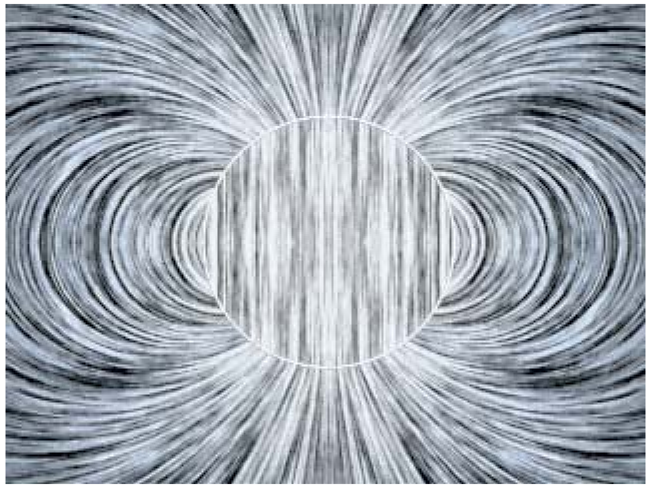

Figure 22-5: One frame of a movie showing the creation of magnetic energy

22.4.5 The electromagnetic angular momentum of the spinning sphere

Let us now look at the conservation of electromagnetic angular momentum, which we could not treat in the cylindrical case because it was stored in the fringing fields at infinity. The form that this conservation law takes is (4.5.7), which we reproduce below

$$\frac{d}{dt}\int_V \mathbf{r}\times\left[\varepsilon_o \mathbf{E}\times\mathbf{B}\right] d^3x + \int_S \left(-\mathbf{r}\times\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\right) da = -\int_V \mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right] d^3x \qquad (22.4.23)$$

If we compute the total electromagnetic angular momentum of this spinning charge configuration, it is

$$\int_{\text{all space}} \mathbf{r}\times\left[\varepsilon_o \mathbf{E}\times\mathbf{B}\right] d^3x \qquad (22.4.24)$$

and we have

$$\mathbf{r}\times\left(\mathbf{E}\times\mathbf{B}\right) = \begin{cases} \mathbf{r}\times\left[-\dfrac{d}{dt}\dfrac{r\left(\mu_o\right)^2 m^2}{\left(4\pi R^3\right)^2}\sin\theta\left(\hat{\boldsymbol{\theta}}\cos\theta+\hat{\mathbf{r}}\sin\theta\right)\right] & r < R \\ \dfrac{\mu_o m(t)}{4\pi\, r^3}\mathbf{r}\times\left[\dfrac{Q}{4\pi\varepsilon_o r^2}\sin\theta\,\hat{\boldsymbol{\phi}} - \dfrac{\mu_o 2\sin\theta\cos\theta}{4\pi r^2}\dfrac{dm}{dt}\hat{\boldsymbol{\theta}} + \dfrac{\mu_o \sin^2\theta}{4\pi r^2}\dfrac{dm}{dt}\hat{\mathbf{r}}\right] & r > R \end{cases}$$

(22.4.25)

$$\mathbf{r}\times\left(\mathbf{E}\times\mathbf{B}\right)=\begin{cases}\left[-\dfrac{d}{dt}\dfrac{r^2\left(\mu_o\right)^2 m^2}{\left(4\pi R^3\right)^2}\sin\theta\left(\hat{\boldsymbol{\phi}}\cos\theta\right)\right] & r<R\\ -\dfrac{\mu_o m(t)}{4\pi\, r^3}\left[\dfrac{Q}{4\pi\varepsilon_o r}\sin\theta\,\hat{\boldsymbol{\theta}}-\dfrac{\mu_o 2\sin\theta\cos\theta}{4\pi r}\dfrac{dm}{dt}\hat{\boldsymbol{\phi}}\right] & r>R\end{cases} \tag{22.4.26}$$

We can ignore the *dm/dt* terms in (22.4.26) for two reasons: (1) these terms integrate to zero because of the $\hat{\boldsymbol{\phi}}$ dependence; and (b) these terms are small compared to the others. So we have

$$\mathbf{r}\times\left(\varepsilon_o\mathbf{E}\times\mathbf{B}\right)=\begin{cases}0 & r<R\\ -\dfrac{m(t)}{4\pi\, r^3c^2}\dfrac{Q}{4\pi\varepsilon_o r}\sin\theta\,\hat{\boldsymbol{\theta}}= & r>R\end{cases} \tag{22.4.27}$$

Using $\hat{\boldsymbol{\theta}}=-\hat{\mathbf{z}}\sin\theta+\hat{\mathbf{x}}\cos\theta\cos\phi+\hat{\mathbf{y}}\cos\theta\sin\phi$, and realizing in advance that the *x* and *y* components will average to zero because of the $\cos\phi$ and $\sin\phi$ terms, we have

$$\mathbf{r}\times\left(\varepsilon_o\mathbf{E}\times\mathbf{B}\right)=\begin{cases}0 & r<R\\ \dfrac{m(t)}{4\pi\, r^3c^2}\dfrac{Q\sin^2\theta}{4\pi\varepsilon_o r}\hat{\mathbf{z}} & r>R\end{cases} \tag{22.4.28}$$

The total electromagnetic angular momentum is located entirely outside the sphere and is given by

$$\begin{aligned}\int\mathbf{r}\times\left(\varepsilon_o\mathbf{E}\times\mathbf{B}\right)d^3x&=\hat{\mathbf{z}}\frac{m(t)Q}{\left(8\pi\varepsilon_o\right)c^2}\int_R^\infty\frac{dr}{r^2}\int_{-1}^{1}d(\cos\theta)\sin^2\theta\\ &=\hat{\mathbf{z}}\frac{mQ}{\left(6\pi\varepsilon_o\right)R\,c^2}=\hat{\mathbf{z}}\frac{4\pi R^3\sigma\,\omega RQ}{3\left(6\pi\varepsilon_o\right)R\,c^2}\end{aligned} \tag{22.4.29}$$

$$\int\mathbf{r}\times\left(\varepsilon_o\mathbf{E}\times\mathbf{B}\right)d^3x=\hat{\mathbf{z}}\frac{\omega RQ^2}{18\pi\varepsilon_o\,c^2} \tag{22.4.30}$$

22.4.6 The creation rate of electromagnetic angular momentum

The total rate at which electromagnetic angular momentum is being created as the sphere is being spun up is according to (22.4.23)

$$\int_{\text{all space}}-\mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right]d^3x=\int_{surface}-\mathbf{r}\times\sigma\mathbf{E}\,da=\int_{surface}\left(\mathbf{r}\times\hat{\boldsymbol{\phi}}\right)\sigma\left(\frac{\mu_o\sin\theta}{4\pi R^2}\frac{dm}{dt}\right)da \tag{22.4.31}$$

so that

$$\int_{\text{all space}} -\mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right]d^3x = -\int_{surface}\hat{\boldsymbol{\theta}}\left(\frac{\sigma\mu_o\sin\theta}{4\pi R}\frac{dm}{dt}\right)da$$
$$= \hat{\mathbf{z}}\left(\frac{\sigma\mu_o}{4\pi R}\frac{dm}{dt}\right)\int_{surface}\sin^2\theta\,da = \hat{\mathbf{z}}\left(\frac{\sigma\mu_o}{4\pi R}\frac{dm}{dt}\right)2\pi R^2\frac{4}{3} \tag{22.4.32}$$

$$\int_{\text{all space}} -\mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right]d^3x = \hat{\mathbf{z}}\frac{d}{dt}\left(\frac{2\sigma\mu_o R\left(4\pi R^3\kappa\right)}{9}\right)$$
$$= \hat{\mathbf{z}}\frac{d}{dt}\left(\frac{2\sigma\mu_o 4\pi R^4\omega R\sigma}{9}\right) = \hat{\mathbf{z}}\frac{d}{dt}\left(\left(\frac{Q}{4\pi R^2}\right)^2\frac{8\mu_o\pi R^5\omega}{9}\right) \tag{22.4.33}$$

$$\int_{\text{all space}} -\mathbf{r}\times\left[\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right]d^3x = \hat{\mathbf{z}}\frac{d}{dt}\left(\frac{Q^2\omega R}{18\pi\varepsilon_o c^2}\right) \tag{22.4.34}$$

In the first of the equations above we have used (22.4.11), in the second we have ignored a term that looks like $\hat{\mathbf{r}}\sin\theta\cos\theta$ because it will integrate to zero when we do the theta integration, and in (22.4.34) we have used the definitions of m and κ. This is the rate we want to see, because when integrated over time it gives the total electromagnetic angular momentum created, as given in (22.4.30). If you are spinning up the sphere, it is you who are creating this angular momentum by the additional torque you must impose to overcome “back” force due to the induction electric field.

22.4.7 The flux of electromagnetic angular momentum

Finally, let us calculate the flux of electromagnetic angular momentum, $\int_{surface}\left[-\mathbf{r}\times\overleftrightarrow{\mathbf{T}}\right]\cdot\hat{\mathbf{r}}\,da$, through a sphere of radius r.

$$\int_{surface}\left[-\mathbf{r}\times\overleftrightarrow{\mathbf{T}}\right]\cdot\hat{\mathbf{r}}\,da = \int_{surface}\left[-\mathbf{r}\times\left(\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{r}}\right)\right]da = \int_{surface}\left[-\mathbf{r}\times\left(T_{rr}\hat{\mathbf{r}}+T_{r\theta}\hat{\boldsymbol{\theta}}+T_{r\phi}\hat{\boldsymbol{\phi}}\right)\right]da \tag{22.4.35}$$

For a sphere with r a little less than R, this flow will be zero, as we expect, because there is no electromagnetic angular momentum in the interior of the sphere when we are finishing spinning up the sphere. For r a little greater than R, we have

$$\int_{surface} \left[-\mathbf{r}\times T_{r\phi}\hat{\boldsymbol{\phi}}\right] da = \hat{\boldsymbol{\theta}} \int_{surface} RT_{r\phi}\, da = -\hat{\boldsymbol{\theta}}\varepsilon_o \int_{surface} R\frac{Q}{4\pi\varepsilon_o R^2}\frac{\mu_o \sin\theta}{4\pi R^2}\frac{dm}{dt} da$$
$$= -\hat{\boldsymbol{\theta}}\varepsilon_o \frac{d}{dt}\int_{surface} R\frac{Q}{4\pi\varepsilon_o R^2}\frac{\mu_o \sin\theta}{4\pi R^2}\frac{4\pi R^3\sigma\omega R}{3} da = +\hat{\mathbf{z}}\frac{d}{dt}\frac{Q^2\omega R}{18\pi c^2\varepsilon_o} \tag{22.4.36}$$

This is exactly the rate of flow that we want, because when integrated over time it gives us the electromagnetic angular momentum stored outside the sphere when the sphere is fully spun up.

23 The classical model of the electron

23.1 Learning Objectives

We extend the results we have obtained above for the magnetic energy and angular momentum of a spinning shell of charge to the case of a spherical shell of charge with a linear velocity. We compute the linear electromagnetic momentum involved in that motion. Finally, put this all in the context of the attempt in the early 1900's to make a purely electromagnetic model of the electron.

23.2 The momentum and angular momentum of a shell of charge

23.2.1 Angular momentum

In the previous section, we derived the total electromagnetic angular momentum of a spinning uniformly charged shell of radius *R* and total charge *Q*, spinning at an angular speed $\omega\hat{\mathbf{z}}$. The total angular momentum (all of which is contained in the region outside of the shell) is from (22.4.30)

$$\mathbf{L}_E = \int \mathbf{r}\times\left(\varepsilon_o \mathbf{E}\times\mathbf{B}\right) d^3x = \hat{\mathbf{z}}\frac{\omega R Q^2}{18\pi\varepsilon_o\, c^2} \tag{23.2.1}$$

The amount of energy that it took to assemble the charges in this shell, U_E, is given by

$$U_E = \frac{1}{2}\int \rho\phi d^3x = \frac{1}{2}\int \sigma\,\phi da = \frac{Q^2}{8\pi\varepsilon_o R} \tag{23.2.2}$$

Let us define the "electromagnetic inertial mass", m_E, of this charge configuration by the equation

$$m_E c^2 = U_E \Rightarrow m_E = \frac{Q^2}{8\pi\varepsilon_o R c^2} \tag{23.2.3}$$

With this definition, we see that the angular momentu in (23.2.1) can be written as

$$\mathbf{L}_E = \hat{\mathbf{z}}\frac{\omega R Q^2 R}{18\pi\varepsilon_o\, c^2 R} = \hat{\mathbf{z}}\left(\frac{4}{9}m_E R^2\right)\omega = I_E\boldsymbol{\omega} \qquad (23.2.4)$$

where the “electromagnetic moment of intertia” is defined by

$$I_E = \frac{4}{9}m_E R^2 \qquad (23.2.5)$$

What about the magnetic energy of the spinning spherical shell? In (22.4.18) found that this magnetic energy is given by

$$U_B = \frac{\mu_o m^2}{4\pi R^3} = \frac{2}{9}m_E\omega^2 R^2 = \frac{2}{9}\frac{\omega^2 R^2}{c^2}U_E \qquad (23.2.6)$$

so that we have $U_B << U_E$ if $\omega R << c$. Moreover, we can also write (23.2.6) as

$$U_B = \frac{2}{9}R^2 m_E\omega^2 = \frac{1}{2}I_E\omega^2 \qquad (23.2.7)$$

At this point, bells start going off in our head’s, and the same bells went off in the heads of physicists in the early 1900’s. From mechanics we remember that a thin shell of mass m has a moment of inertia given by $I = \frac{2}{5}mR^2$. When it is rotating it has rotational kinetic energy of $\frac{1}{2}I\omega^2$ and angular momentum $I\omega$. That looks an awful like our expressions in (23.2.4) and (23.2.7). Maybe mass is just charge?

23.2.2 Linear momentum

Let see if we can get some kind of similar result for linear momentum. We move the spherical shell at velocity $V << c$ in the z direction. Assuming for the moment that the velocity V is constant, we know that in the rest frame of the shell, there is only an electric field. How can we use that fact to find the magnetic field in the laboratory frame, neglecting terms of order $(V/c)^2$? If we return to equation (14.3.15), the fact that the magnetic field in the barred frame is zero means that

$$\bar{B}_x = B_x = 0 \qquad \bar{B}_y = \gamma\left(B_y + \frac{\mathrm{v}}{c^2}E_z\right) = 0 \qquad \bar{B}_z = \gamma(B_z - \frac{\mathrm{v}}{c^2}E_y) = 0 \qquad (23.2.8)$$

This set of equations can be written as

$$\mathbf{B} = \mathbf{V} \times \mathbf{E} / c^2 \tag{23.2.9}$$

We also have from (14.3.8) that

$$E_x = \overline{E}_x \qquad E_y = \gamma\left(\overline{E}_y + \mathrm{V}\,\overline{B}_z\right) \qquad E_z = \gamma\left(\overline{E}_z - V\overline{B}_y\right) \tag{23.2.10}$$

Since the magnetic field in the barred frame is zero, we thus see from (23.2.10) that the electric field in the laboratory frame is just the Coubomb field in the rest frame of the sphere, neglecting terms of order $(V/c)^2$.

Now, let us consider a time in the lab frame where the spherical shell is just at the origin. Then the total amound of linear momentum in the fields of the sphere at that time is given by

$$\mathbf{P}_E = \int \varepsilon_o \mathbf{E} \times \mathbf{B}\, d^3x = \int \varepsilon_o \mathbf{E} \times \left[\mathbf{V} \times \mathbf{E}\right] / c^2 d^3x \tag{23.2.11}$$

We only need to do this integral over the exterior of the sphere since the electric field vanishes in the interior, so we have

$$\mathbf{P}_E = \frac{V}{c^2}\int \varepsilon_o \left(\frac{Q}{4\pi\varepsilon_o r^2}\right)^2 \hat{\mathbf{r}} \times \left[\hat{\mathbf{z}} \times \hat{\mathbf{r}}\right] r^2 da = \frac{V}{c^2}\int \varepsilon_o \left(\frac{Q}{4\pi\varepsilon_o r^2}\right)^2 \hat{\mathbf{r}} \times \left[\sin\theta\,\hat{\boldsymbol{\phi}}\right] r^2 da \tag{23.2.12}$$

$$\mathbf{P}_E = -\frac{V}{c^2}\int \varepsilon_o \left(\frac{Q}{4\pi\varepsilon_o r^2}\right)^2 \hat{\boldsymbol{\theta}} \sin\theta\, r^2 da = \hat{\mathbf{z}}\frac{2\pi V \varepsilon_o}{c^2}\left(\frac{Q}{4\pi\varepsilon_o}\right)^2 \int_{-1}^{1} \sin^2\theta d\left(\cos\theta\right) \int_R^\infty \frac{dr}{r^2} \tag{23.2.13}$$

$$\mathbf{P}_E = \hat{\mathbf{z}}\frac{4}{3}V\left(\frac{Q^2}{8\pi\varepsilon_o c^2 R}\right) = \frac{4}{3} m_E \mathbf{V} \tag{23.2.14}$$

So we see that the moving spherical shell has a net linear electromagnetic momentum associated with it, which looks something like its “electromagnetic mass” times its velocity, except for the funny fact of $\frac{4}{3}$.

23.3 The self force on an accelerating set of charges

Where did this linear electromagnetic momentum stored in the fields of the moving shell come from? We are going to make the following statement, and then give an example. Suppose you have an electrostatic problem where you have assembled a set of charges and in doing this you had to expend energy U_E. We assume that the charges are held in place by some rigid framework, and that if we try to move them they will move as a unit. Now suppose you come along and "kick" this distribution of charges, applying a force to it of $\mathbf{F}_{me}$. Then to total force on the charge will be given in part[6] by

$$\mathbf{F} = \mathbf{F}_{me} - \xi \frac{U_E}{c^2}\frac{d\mathbf{V}}{dt} = \mathbf{F}_{me} - \xi m_E \frac{d\mathbf{V}}{dt} \tag{23.3.1}$$

where ξ is a dimensionless factor of order unity and $m_E = U_E / c^2$. What this means is that a set of charges such as this resists being moved. That is, it has an inertia, and the order of magnitude of the "back reaction" to being moved is given in (23.3.1).

23.3.1 The electromagnetic inertia of a capacitor

The statement above in (23.3.1) is a generally true. Here we want to give a specific example with very idealized geometry to give you some feel for why this inertial "back reaction" exists for a charged object. Consider a capacitor oriented as shown in Figure 0-1, and moving upward. The plates of the capacitor have area *A*, and the distance between the plates is *d*. The right plate carries a charge per unit area of $+\sigma$, and the left plate carries a charge per unit area $-\sigma$. When the capacitor is at rest, the electric field between the plates is $\mathbf{E}_o = -\frac{\sigma}{\varepsilon_o}\hat{\mathbf{x}}$. The total electrostatic energy in the capacitor is thus

$$U_E = Ad\left(\frac{1}{2}\varepsilon_o E_o^2\right) = Ad\left(\frac{1}{2}\varepsilon_o\left(\frac{Q}{\varepsilon_o A}\right)^2\right) = \frac{d\,Q^2}{2\varepsilon_o A} \tag{23.3.2}$$

Now suppose we have gotten the capacitor up to speed $V\,\hat{\mathbf{y}}$. Then the upward motion of the positive sheet of charge on the right will correspond to a current sheet with current per unit length $\sigma V\,\hat{\mathbf{y}}$, and the upward motion of the negative sheet of charge on the left will correspond to a current sheet with current per unit length $-\sigma V\,\hat{\mathbf{y}}$. As a result of these current sheets, there will be a magnetic field between the sheets of current given by

$$\mathbf{B} = \mu_o \sigma V\,\hat{\mathbf{z}} \tag{23.3.3}$$

Now suppose we try to increase the speed of the sheet. This will result in an increase in the magnetic field given in (23.3.3), and therefore to the presence of an induced electric field. If we take a loop in the *xy* plane centered on the x axis with width *2x* in the *x*-direction and length *L* in the *y* direction, and integrate around that loop, the Faraday's Law gives, assuming $x < d/2$

[6] We say in part because we are neglecting the "radiation reaction" force, we considered above in Section 17.8, and which is proportional to the time derivative of the acceleration.

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\left[L2xB\right] = -\frac{d}{dt}\left[2xL\mu_o\sigma V\right] = -2xL\mu_o\sigma\frac{dV}{dt} \qquad (23.3.4)$$

A little thought will convince you that this induced electric field is in the *y*-direction and an antisymmetic function of *y*, with the induced field for $|x| < d/2$ given by

$$\mathbf{E}_{induced} = -x\mu_o\sigma\frac{dV}{dt}\hat{\mathbf{y}} = -x\mu_o\frac{Q}{A}\frac{dV}{dt}\hat{\mathbf{y}} \qquad (23.3.5)$$

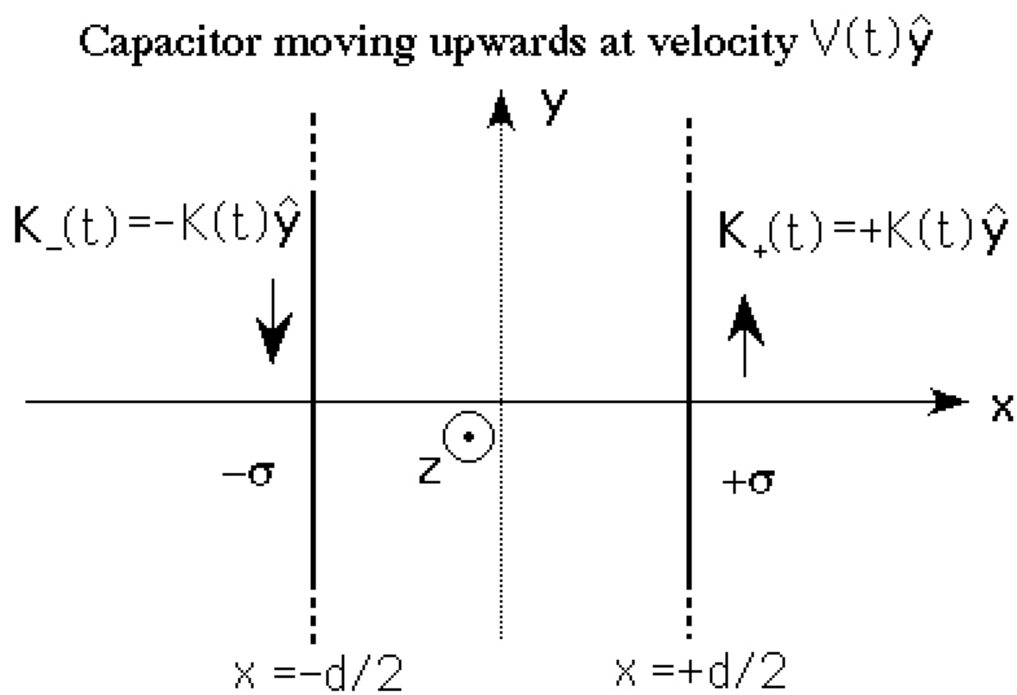

Figure 0-1: A Capacitor Moving Upward

This induced field will exext a downward force on the positive sheet at $x = d/2$ and a downward force on the negative sheet at $x = -d/2$ (because the induced electric field has switched diretion there), and therefore if we try to accelerate the capacitor the induced electric field resists that acceleration with a force given by

$$\mathbf{F}_{back} = -\hat{\mathbf{y}}2Q\frac{d}{2}\mu_o\frac{Q}{A}\frac{dV}{dt} = -\frac{dQ^2}{\varepsilon_o Ac^2}\frac{dV}{dt}\hat{\mathbf{y}} = -\frac{2U_E}{c^2}\frac{dV}{dt}\hat{\mathbf{y}} = -2m_E\frac{d\mathbf{V}}{dt} \qquad (23.3.6)$$

where we have used (23.3.2).

Equation (23.3.6) is the form given in (23.3.1), as we asserted that it must be. In this highly idealized geometry we see why this back reaction force arises. If we try to move a set of charges, we will create currents which will generate magnetic fields proportional to velocity. If we try to accelerate the set of charges, we will thus have a time changing magnetic field which will be associated with induced electric fields which are proportion to the time rate of change of the magnetic fields, or to the acceleration. These induced electric fields will exert self forces in a direction such as to resist any attempt to move the charges. That is why there is an additional electromagnetic momentum (23.2.14) in our moving spherical shell, and where it comes from. It as usually comes from me, the agent who got the shell to move, who had to exert an addition force because the shell was charged, and that force is the source of the additional momentum in the shell.

It is important to note that this process is *reversible*. That is, if an object is charged we have to put extra energy into getting it moving, but when we stop it we get all of that energy back. This is very different from the radiation reaction force, which is always a drain of energy. This is clear from out moving capacitor example above. When we try to stop this moving capacitor, we will be reducing the amount of current flowing and thus the magnetic field those currents produce, and thus there will be an associated induction electric field which will now try to kept the capacitor moving. That additional inertia due the charging means as the capacitor slows down we can use the induced electric field to do work on us, and the work that is done on us is exactly the additional work we had to put in to get the capacitor up to speed because of the charge.

23.3.2 The classical model of the electron

What do we mean when we talk about the inertial mass of an uncharged object? What we mean is that if we appy a known force $\mathbf{F}_{me}$ to the object, the resultant acceleration will be in the same direction as the force, and if we measure the acceleration *a* of the object due to this force, then the inertial mass of the object is given by

$$m_o = \frac{F_{me}}{a} \tag{23.3.7}$$

where we are using m_o to denote the mass of the uncharged object. We will always get this same ratio, regardless of the magnitude of the force applied. That is, if we double F_{me}, we will observe double the acceleration *a*, giving the same interial mass as before.

If we now charge up this object, and measure its inertial mass after we have charged it up, then comparing (23.3.7) to (23.3.1), we see that the inertial mass after we have charged it up will increase, because the back reaction will have decreased the acceleration we observe for the same external force applied, $\mathbf{F}_{me}$. That is, we always have that

$$\mathbf{F} = m_o\mathbf{a} = \mathbf{F}_{me} - \xi \frac{U_E}{c^2}\mathbf{a} \tag{23.3.8}$$

where the last expression comes from (23.3.1). If we solve (23.3.8) for **a** in terms of the applied force, we have

$$\left(m_o + \xi \frac{U_E}{c^2}\right)\mathbf{a} = \mathbf{F}_{me} \tag{23.3.9}$$

Or an inertial mass *m* of

$$m = \frac{F_{me}}{a} = m_o + \xi \frac{U_E}{c^2} \tag{23.3.10}$$

Thus when we charge up a neutral object and then measure its inertial mass, we see an increase in that mass.

The great excitement in the early 1900's was that when physicists realized that a charged object had properties that looked exactly like "ordinary" matter, e.g. momentum, angular momentum, and inertia, they suddenly realized that perhaps there was no "ordinary" matter at all, but that everything was simply electromagnetic in character. We know the mass of the electron m_e and its charge e, and if it were a ball of charge of radius r_c given by

$$r_e = \frac{e^2}{4\pi\varepsilon_o m_e c^2} \tag{23.3.11}$$

then the expression in (23.3.10) would account for all of its inertial mass, and we would not need the "neutral" inertial mass m_o.

The reason this does not work is that of course you need something besides electromagnetic fields to hold together an electron. If we only had electric fields, for example, the charge making up an electron would fly off to inifinty due to mutual repulsion. So there must be something else that is holding the electron together, and those stresses would also contribute to the inertia. The strange factors we get above (like the factor of 4/3 in (23.2.14) for one thing means that we can not construct a four momentum for a purely electrostatic electron that transforms correctly under Lorentz transformations, another indication that we are leaving out something significant.

In any case, with the advent of quantum mechanics, all of these classical models went by the boards. But it is worth pointing out that the length scale defined by (23.3.11) appears repeatedly in classical electromagnetic radiation calculations. For example, as you will show in Assignment 10, the square of this distance is proportional to the Thompson cross-section for an electron, that is its cross-section for scattering energy out of an incident electromagnetic wave.

24 Moving Magnets, Einstein, and Faraday

24.1 Learning Objectives

We first review the example that Einstein gave in the first paragraph of his 1905 paper on special relativity, and try to explain what motivated him to focus on this phenomena in one of his most famous papers. This involves thinking about the magnetic field of a moving magnetic and what electric field is associated with it, and conversely the force that will be seen by a conductor moving in the magnetic field of a stationary magnet.

24.2 What did Einstein mean?

On the Electrodynamics of Moving Bodies
by A. Einstein

(Translated from "Zur Elektrodynamik bewegter Korper, " Annalen der Physik, 17, 1905)

It is known that Maxwell's electrodynamics- as usually understood at the present time- when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomena here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighborhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But, if the magnet is stationary and the conductor in motion, no electric field arises in the neighborhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise- assuming equality of relative motion in the two cases discussed- to electric currents of the same path and intensity as those produced by the electric forces in the former case.

Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium," suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest....We will raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body.....

Let us try to understand in detail what Einstein meant by the example he gives above about moving magnets versus stationary magnets. First, he talks about the electric field of magnets in motion. Let us investigate the electric fields of moving magnets.

24.2.1 The electric field of a magnet moving at constant velocity

Suppose we have a magnet whose dipole moment is along the *x*-direction moving in the *x*-direction at constant velocity $\mathbf{V} = V\,\hat{\mathbf{x}}$ as seen in the laboratory frame. Suppose **E** and **B** are the fields of the magnet as seen in the laboratory frame. If we look at (14.3.8) for the transformation of fields between co-moving frames, we have

$$\overline{E}_x = E_x \qquad \overline{E}_y = \gamma\left(E_y - V\,B_z\right) \qquad \overline{E}_z = \gamma\left(E_z + VB_y\right) \tag{24.2.1}$$

where the "barred" frame is the rest frame of the magnet. If we neglect terms of order $\left(\mathrm{V}/c\right)^2$ equation (24.2.1) can be written as

$$\overline{\mathbf{E}} = \mathbf{E} + \mathbf{V}\times\mathbf{B} \tag{24.2.2}$$

But in the rest frame of the magnet, the electric field $\overline{\mathbf{E}}$ is zero, so that we must have the following relationship between the electric and magnetic fields of the magnet as seen in the laboratory frame, where the magnet is moving:

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B} \tag{24.2.3}$$

Now how does the magnetic field in the laboratory frame relate to the dipolar magnetic field we have in the rest frame of the magnet? If we look at the transformations (14.3.15) and consider them when we are going from the rest frame of the magnet to the laboratory frame (this reverses the sign of the velocity and changes a bar to an unbar and vice versa), we have

$$B_x = \overline{B}_x \qquad B_y = \gamma\left(\overline{B}_y - \frac{\mathrm{V}}{c^2}\overline{E}_z\right) \qquad B_z = \gamma(\overline{B}_z + \frac{\mathrm{V}}{c^2}\overline{E}_y) \tag{24.2.4}$$

Since we know that the electric field $\overline{\mathbf{E}}$ in the rest frame of the magnet is zero, this means that to first order in $\left(\mathrm{V}/c\right)$, we have that the magnetic field in the laboratory frame is the same as the magnetic field in the rest frame of the magnet, which is dipolar. Therefore we have from this fact and (24.2.3)

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B}_{dipole} \tag{24.2.5}$$

In Figure 24-1 we show this "motional electric field" for the case where the magnet moves to the right. In Figure 24-2 we show the electric field where the magnet moves to the left. In both cases the electric field moves in circles about the direction of motion of the magnet, with the sense of the circulation of the electric field reversing from in front of the magnet to behind the magnet.

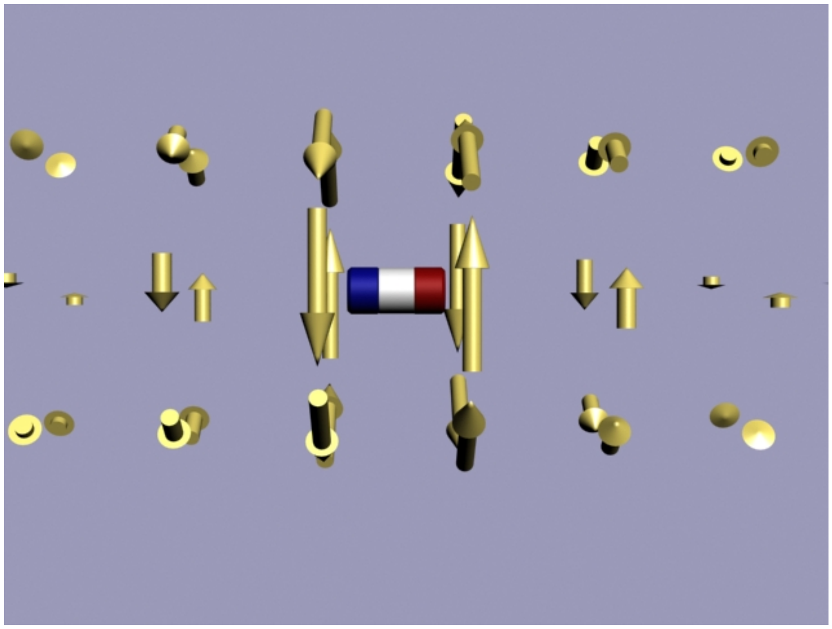

Figure 24-1: The E field of a magnet moving to the right (red is the north pole)

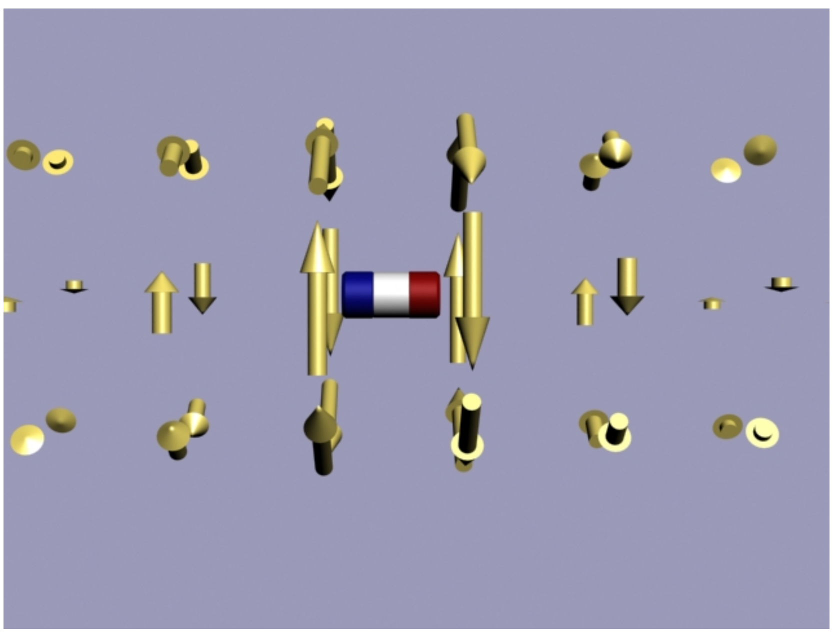

Figure 24-2: The E field of a magnetic moving to the left (red is the north pole)

24.2.2 "… the magnet is in motion and the conductor at rest…"

Now let us return to Einstein's example above. He first considers the situation where the magnet is in motion and the conductor is at rest. If we actually do this experiment, as in Figure 24-3, which shows one frame of a movie of the experiment[7], we find that the current in the coil is left-handed with respect to the magnetic dipole vector (pointing to the right in the figure) when the magnet is moving toward the coil, and left-handed with respect to the magnetic dipole vector when the magnet is being pulled away from the coil. This is in keeping with the direction of the electric field shown in Figure 24-1 and in Figure 24-2, and this is what Einstein means when he says that "…there arises in the neighborhood of the [moving] magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated…". That is, there is a current in the loop because of the electric field of the moving magnet.

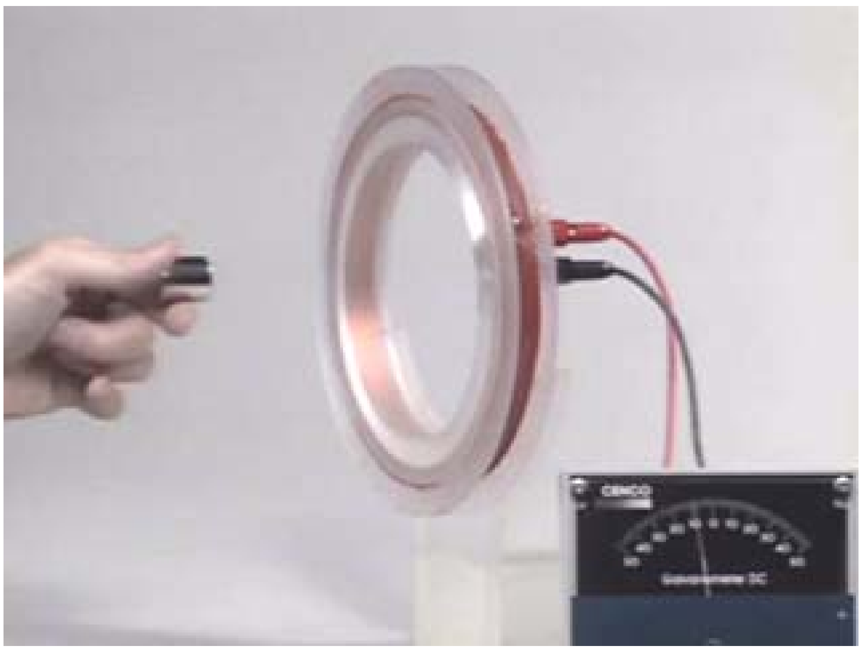

Figure 24-3: A magnet moving toward and away from a stationary loop of wire.

24.2.3 "…the magnet is stationary and the conductor in motion…"

On the other hand, suppose the magnet is at rest and the conductor is moving. Then there is no electric field as seen in the conductor. But there is a $\mathbf{V}\times\mathbf{B}$ force on the charges in the coil, because they are now moving along with the coil, and a little thought shows that this force is in the same direction and has the same magnitude as the electric field given in (24.2.5) (remember we had the magnet moving toward the coil at velocity $\mathbf{V} = V\,\hat{\mathbf{x}}$, so this means that the coil is moving toward the magnet at velocity $-\mathbf{V}$). Thus in either case we get the same current in the coil, but in one case the observer would say that the charges in the coil feel a force producing a current because of the electric field they see, and in the other an observer would say that the charges feel a force producing a current because of the $\mathbf{V}\times\mathbf{B}$ force they see. Regardless of what the source of the force

[7] http://web.mit.edu/viz/EM/visualizations/faraday/faradaysLaw/

is, we see a current as a result. Let us look at the mathematical form of Faraday's Law and put these qualitative ideas into quantitative form.

24.3 Differential and Integral Forms of Faraday's Law

24.3.1 Faraday's Law in Differential Form

Faraday's Law in differential form is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{24.3.1}$$

We can begin the process of writing Faraday's Law in integral form by integrating both sides of (24.3.1) over any open surface *S(t)* and converting the integral of the curl of **E** to a line integral of **E** over the bounding contour *C(t)* using Stokes Theorem, giving

$$\int_{S(t)} \left(\nabla \times \mathbf{E}\right) \cdot \hat{\mathbf{n}} da = \oint_{C(t)} \mathbf{E} \cdot d\mathbf{l} - \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\mathbf{n}} da \tag{24.3.2}$$

We would like to move the $\frac{\partial}{\partial t}$ term out from under the integral sign to become a $\frac{d}{dt}$ in front of the integral sign, but we have to be careful about doing this because we frequently apply Faraday's Law in integral form to moving circuits. We have to pause for a bit to prove the following mathematical theorem.

24.3.2 A Mathematical Theorem

The theorem we now prove has nothing specifically to do with electromagnetism, it is a general theorem about the flux through moving open surfaces. Consider the following problem. You are given a vector field $\mathbf{F}(\mathbf{r}, t)$ which is a function of space and time. You are also given an open surface *S* with associated bounding contour *C,* and this surface is moving in space, with each element of the surface moving at some specified velocity $\mathbf{v}(\mathbf{r}, t)$ (the **r** here of course must lie on the open surface). All of these things are given. Suppose you now compute the flux of **F** through *S* at any given instant of time *t*, in the usual way, that is

$$\Phi_F(t) = \int_{S(t)} \mathbf{F}(\mathbf{r}, t) \cdot \hat{\mathbf{n}}\, da \tag{24.3.3}$$

where we have used the notation *S(t)* to indicate that the surface is changing in time, as well as **F**. Note that $\Phi_F(t)$ depends only on time--we have integrated over space. Now, here is the question we want to answer. What is the time derivative of $\Phi_F(t)$? Let's start out by giving the answer, and then we will show how it comes about.

$$\frac{d\Phi_F(t)}{dt} = \int_{S(t)} \frac{\partial \mathbf{F}(\mathbf{r},t)}{\partial t} \cdot \hat{\mathbf{n}}\, da - \oint_{C(t)} \left[\mathbf{v} \times \mathbf{F}(\mathbf{r},t) \right] \cdot \mathbf{dl} + \int_{S(t)} \left[\nabla \cdot \mathbf{F}(\mathbf{r},t) \right] \mathbf{v} \cdot \hat{\mathbf{n}}\, da \qquad (24.3.4)$$

The first term on the right of this equation is due to the intrinsic time variability of **F**. The other two terms on the right, a line integral around the bounding contour *C(t)*, and another surface integral over *S(t)*, arise purely because of the motion of *S* (that is, they disappear when $\mathbf{v}(\mathbf{r},t)$ is zero everywhere).

To prove (24.3.4), let us start out with the definition of the derivative of a function:

$$\frac{d\Phi_F(t)}{dt} = \lim_{\Delta t \to 0} \frac{\Phi_F(t+\Delta t) - \Phi_F(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\int_{S(t+\Delta t)} \mathbf{F}(\mathbf{r}, t+\Delta t) \cdot \hat{\mathbf{n}}\, da - \int_{S(t)} \mathbf{F}(\mathbf{r},t) \cdot \hat{\mathbf{n}}\, da \right]$$
(24.3.5)

We use a Taylor series expansion of **F** about *t*,

$$\mathbf{F}(\mathbf{r}, t+\Delta t) = \mathbf{F}(\mathbf{r},t) + \Delta t \frac{\partial \mathbf{F}(\mathbf{r},t)}{\partial t} + ... \qquad (24.3.6)$$

to write the first term on the right of equation (24.3.5) as two terms, namely

$$\frac{d\Phi_F(t)}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\Delta t \int_{S(t+\Delta t)} \frac{\partial \mathbf{F}(\mathbf{r},t)}{\partial t} \cdot \hat{\mathbf{n}}\, da + \int_{S(t+\Delta t)} \mathbf{F}(\mathbf{r},t) \cdot \hat{\mathbf{n}}\, da - \int_{S(t)} \mathbf{F}(\mathbf{r},t) \cdot \hat{\mathbf{n}}\, da \right] \qquad (24.3.7)$$

Now the next part is tricky. The divergence theorem is just as good for time varying functions as for functions which do not vary in time, and it says the following. Pick any closed volume *V'* bounded by the closed surface *S'*. ***This is different from the surface S above*** (for one thing, *S'* is closed, and *S* is open). Now, the divergence theorem states that at any time *t*, we have

$$\int_{V'} \left[\nabla \cdot \mathbf{F}(\mathbf{r}',t) \right] d^3x' = \oint_{S'} \mathbf{F}(\mathbf{r},t) \cdot \hat{\mathbf{n}}'\, da' \qquad (24.3.8)$$

This equation is true for any volume *V'* and any given instant of time, and we are going to apply it at time *t* to the following volume (brace yourself). The volume *V'* at time *t* which we are going to use in (24.3.8) is *the volume swept out by our original open surface S, as it moves through space between time t and time t+Δt.*

This at first seems like a peculiar volume to use at time *t*, since it depends on things that happen in the future, but it is a perfectly well defined volume at time *t*, we just need to know what is going to happen with *S(t)* in the future to define this volume of space at *t*, but that is given. Figure 24-4 shows this closed surface and the associated volume.

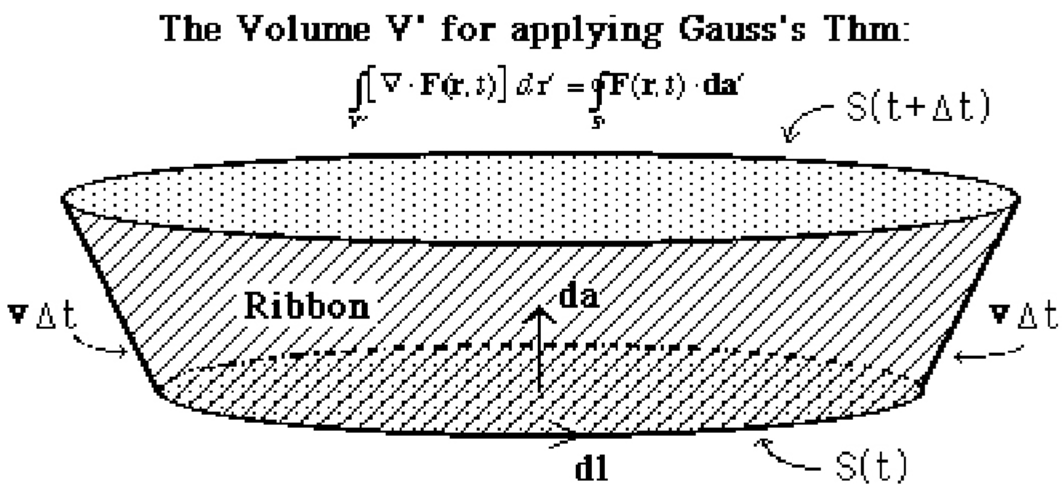

Figure 24-4: The closed surface at time *t* we use for applying the divergence theorem

The infinitesimal line element **dl** and the infinitesimal area element $\mathbf{da} = \hat{\mathbf{n}}\, da$ shown in Figure 24-4 are associated with the open surface *S*. They must be right-handed with respect to one another, which is why **da** must be up if **dl** is counterclockwise. The vector $\mathbf{da} = \hat{\mathbf{n}}\, da$ is *not* the infinitesimal area element **da'** associated with the closed surface *S'*--that vector must always point away from the volume of interest, namely *V'*. So **da'** is anti-parallel to **da** on the bottom of the volume shown above, and parallel to **da** on the top of the volume shown above.

Applying Gauss's Theorem to the volume *V'* gives

$$\int_{V'} \left[\nabla \cdot \mathbf{F}(\mathbf{r},t)\right] d^3x' = \oint_{S'} \mathbf{F}(\mathbf{r},t)\cdot \mathbf{da}' = \oint_{S(t+\Delta t)} \mathbf{F}(\mathbf{r},t)\cdot \mathbf{da}' + \oint_{S(t)} \mathbf{F}(\mathbf{r},t)\cdot \mathbf{da}' + \oint_{\text{ribbon}} \mathbf{F}(\mathbf{r},t)\cdot \mathbf{da}' \tag{24.3.9}$$

where we have gone all around the closed surface *S'* containing *V'*, including the "ribbon" of area that is swept out by the moving contour *C(t)* between times *t* and *t+Δt*. If we use the fact that $\mathbf{da}' = \mathbf{da}$ on the surface *S(t+Δt)* and $\mathbf{da}' = -\mathbf{da}$ on the surface *S(t)*, we have

$$\int_{V'} \left[\nabla \cdot \mathbf{F}(\mathbf{r},t)\right] d^3x' = \oint_{S'} \mathbf{F}(\mathbf{r},t)\cdot \mathbf{da}' = \oint_{S(t+\Delta t)} \mathbf{F}(\mathbf{r},t)\cdot \mathbf{da} - \oint_{S(t)} \mathbf{F}(\mathbf{r},t)\cdot \mathbf{da} + \oint_{\text{ribbon}} \mathbf{F}(\mathbf{r},t)\cdot \mathbf{da}' \tag{24.3.10}$$

Now what is d^3x' for this volume *V'*? It is pretty easy to see that $d^3x' = \mathbf{v}\cdot \mathbf{da}\ \Delta t$. This is dimensionally correct, and has the right behavior (if **v** is perpendicular to **da** at some point on *S* there is no increase in differential volume d^3x' at that point between time *t* and *t+Δt*). In addition, it is also pretty obvious that the infinitesimal area element on the surface of the "ribbon" is given by $\mathbf{da}'_{\text{ribbon}} = \mathbf{dl} \times \mathbf{v}\,\Delta t$. To see this clearly, consider Figure 24-5**Error! Reference source not found.**, which is a blowup of an area of detail of Figure 24-4 . This sketch shows the geometry of the situation. This form $\mathbf{da}'_{\text{ribbon}} = \mathbf{dl} \times \mathbf{v}\,\Delta t$ has the right dependence--if **v** is parallel to **dl**, at some point, the contour *C* is moving parallel to itself at that point, and there is no area swept out by *C* at that point between *t* and *t+Δt*.

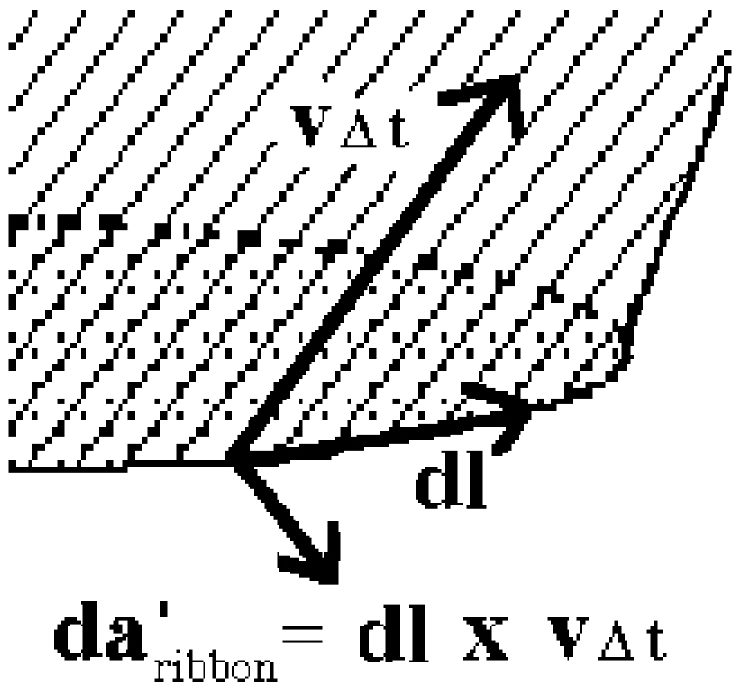

Figure 24-5: The infinitesimal area element for the ribbon

So, using these two forms for d^3x' and $\mathbf{da}'_{\text{ribbon}}$ in equation (24.3.10), we can convert the volume integral on the left hand side into a surface integral over *S(t)*, and the area integral over the ribbon on the right hand side to a line integral over *C(t)*. We certainly make some error in doing this, but the corrections will be of order *Δt*, and we already have a *Δt* in these terms, so that the corrections will vanish as we go to the limit of *Δt = 0*. Thus, we have that

$$\int_{S(t)} \left[\nabla\cdot\mathbf{F}(\mathbf{r},t)\right]\,\mathbf{v}\cdot\mathbf{da}\;\Delta t = \oint_{S(t+\Delta t)} \mathbf{F}(\mathbf{r},t)\cdot\mathbf{da} - \oint_{S(t)} \mathbf{F}(\mathbf{r},t)\cdot\mathbf{da} + \oint_{C(t)} \mathbf{F}(\mathbf{r},t)\cdot\left(\mathbf{dl}\;\mathbf{x}\;\mathbf{v}\right)\Delta t \tag{24.3.11}$$

which we can rewrite as

$$\oint_{S(t+\Delta t)} \mathbf{F}(\mathbf{r},t)\cdot\mathbf{da} - \oint_{S(t)} \mathbf{F}(\mathbf{r},t)\cdot\mathbf{da} = -\Delta t \oint_{C(t)} \mathbf{F}(\mathbf{r},t)\cdot\left(\mathbf{dl}\times\mathbf{v}\right) + \Delta t \int_{S(t)} \left[\nabla\cdot\mathbf{F}(\mathbf{r},t)\right]\mathbf{v}\cdot\mathbf{da}$$

(24.3.12)

Thus we can write (24.3.7) as

$$\frac{d\Phi_{\mathrm{F}}(t)}{dt} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\Delta t \int_{\mathrm{S}(t+\Delta t)} \frac{\partial \mathbf{F}(\mathbf{r},t)}{\partial t}\cdot\mathbf{da} - \Delta t \oint_{C(t)} \mathbf{F}(\mathbf{r},t)\cdot\left(\mathbf{dl}\times\mathbf{v}\right) + \Delta t \int_{S(t)} \left[\nabla\cdot\mathbf{F}(\mathbf{r},t)\right]\mathbf{v}\cdot\mathbf{da}\right]$$

(24.3.13)

If we take the limit, we have

$$\frac{d\Phi_{\mathrm{F}}(t)}{dt} = \int_{S(t)} \frac{\partial \mathbf{F}(\mathbf{r},t)}{\partial t}\cdot\mathbf{da} - \oint_{C(t)} \mathbf{F}(\mathbf{r},t)\cdot\left(\mathbf{dl}\ \mathbf{x}\ \mathbf{v}\right) + \int_{S(t)} \left[\nabla\cdot\mathbf{F}(\mathbf{r},t)\right]\mathbf{v}\cdot\mathbf{da} \qquad (24.3.14)$$

If we use the vector identity $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{C}\times\mathbf{A})\cdot\mathbf{B}$ in the second term on the right, we can also write this as

$$\frac{d\Phi_{\mathrm{F}}(t)}{dt} = \int_{\mathrm{S(t)}} \frac{\partial \mathbf{F}(\mathbf{r},t)}{\partial t}\cdot\mathbf{da} - \oint_{C(t)} \left[\mathbf{v}\times\mathbf{F}(\mathbf{r},t)\right]\cdot\mathbf{dl} + \int_{S(t)} \left[\nabla\cdot\mathbf{F}(\mathbf{r},t)\right]\mathbf{v}\cdot\mathbf{da} \qquad (24.3.15)$$

which is the result we were after. What this equation says is that the flux of **F** through the moving open surface can change in three ways. First, there can be changes due to the innate time dependence of **F** (first term on the right in (24.3.15)). But also the flux can change because flux is lost out of the boundary of the surface as it moves along (second term on the right). And finally, the flux can change because the surface sweeps across sources of **F**, that is regions where the divergence of F is non-zero.

24.3.3 Faraday's Law in Integral Form

We are now ready to write Faraday's Law in integral form. We apply (24.3.15) to the magnetic field, using the fact that we always have $\nabla\cdot\mathbf{B} = 0$

$$\frac{d\Phi_{B}(t)}{dt} = \int_{S(t)} \frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}\cdot\hat{\mathbf{n}}\,da - \oint_{C(t)} \left[\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right]\cdot d\mathbf{l} \qquad (24.3.16)$$

Using this equation, (24.3.2) becomes

$$\oint_{C(t)} \left[\mathbf{E}+\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right]\cdot d\mathbf{l} = -\frac{d}{dt}\int_{S(t)} \mathbf{B}\cdot\hat{\mathbf{n}}\,da \qquad (24.3.17)$$

But we have just seen this term on the left before. Non-relativistically it is the electric field *as seen in the rest frame of* $d\mathbf{l}$. Equation (24.3.17) is the correct form for the integral version of Faraday's Law. If we define $\overline{\mathbf{E}} = \mathbf{E} + \mathbf{v} \times \mathbf{B}(\mathbf{r}, t)$, then we have

$$\oint_{C(t)} \overline{\mathbf{E}} \cdot d\mathbf{l} = -\frac{d\Phi_B(t)}{dt} \tag{24.3.18}$$

24.4 Faraday's Law Applied to Circuits with Moving Conductors

We consider several examples that demonstrate how the form of Faraday's Law given in (24.3.18) applies to moving conductors.

24.4.1 The Falling Magnet/Falling Loop

To see how our final form of Faraday's Law removes the distinction between whether the magnet is moving or the conductor is moving, consider the following problem. We have a magnet with dipole moment $\mathbf{m} = m\hat{\mathbf{z}}$ and mass M, and associated magnetic field $\mathbf{B}_{dipole}$. The dipole moment of the magnet is always up, and the magnet is constrained to move only on the z-axis, but it is allowed to move freely up and down on that axis (see Figure 24-6). Let $Z(t)$ be the location of the magnet at time t. The z-axis is

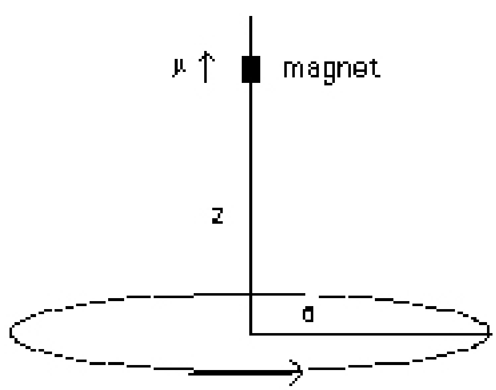

Figure 24-6: A magnet falling on the axis of a conducting loop

also the axis of a circular stationary loop of radius a, resistance R, and inductance L, fixed in place at $z = 0$. The magnet moves downward under the influence of gravity due to the force $-Mg\hat{\mathbf{z}}$. There will be a current I induced in the loop as the magnet falls, because of the changing magnetic flux through the loop, and that current will produce a magnetic field $\mathbf{B}_{loop}(\mathbf{r}, t)$. The falling magnet will feel a force due to the current it induces in the loop due to the magnetic field $\mathbf{B}_{loop}$, given by $\mathbf{m}\cdot\nabla\mathbf{B}_{loop}$ (see Griffiths equation (6.3) page 258). The equation of motion is thus

$$M\frac{d^2Z}{dt^2} = -Mg + m\frac{dB_z^{loop}}{dz} \tag{24.4.1}$$

We need another equation to solve the problem, which we get from Faraday's Law. Faraday's Law (24.3.18) applied to the loop is

$$\oint \overline{\mathbf{E}}\cdot\mathbf{dl} = -\frac{d}{dt}\int \mathbf{B}\cdot\mathbf{dA} = -\frac{d}{dt}\int\left[\mathbf{B}_{loop}+\mathbf{B}_{dipole}\right]\cdot\hat{\mathbf{n}}\,da \tag{24.4.2}$$

From the definition of the self-inductance of the loop L, we have $LI = \int \mathbf{B}_{loop}\cdot\hat{\mathbf{n}}\,da$, so

$$\oint \overline{\mathbf{E}}\cdot\mathbf{dl} = -L\frac{dI}{dt} - \frac{d}{dt}\int\left[\mathbf{B}_{dipole}\right]\cdot\hat{\mathbf{n}}da \tag{24.4.3}$$

If ρ is the resistivity of the loop material, Ohm's Law in microscopic form is $\overline{\mathbf{E}} = \rho\mathbf{J}$, and if $A_{\text{cross section}}$ is the area of the cross section of the wire, then $J = I / A_{\text{cross section}}$ and

$$\oint \overline{\mathbf{E}}\cdot\mathbf{dl} = \oint \rho\,\mathbf{J}\cdot\mathbf{dl} = \frac{I}{A_{\text{cross section}}}\oint \rho\,dl = I\left[\frac{2\pi a\rho}{A_{\text{cross section}}}\right] = IR \tag{24.4.4}$$

so that Faraday's Law can be written as

$$IR = -L\frac{dI}{dt} - \frac{d}{dt}\int \mathbf{B}_{dipole}\cdot\mathbf{dA} \tag{24.4.5}$$

Here is the crucial point. If we were to apply Faraday's Law to the situation where the magnet is at rest and the ring is falling, (24.4.5) ***does not change.*** It does not change because in getting to (24.4.4), we used Ohm's Law in microscopic form (see Section 28.2, (28.2.8), and the proper electric field to use in Ohm's Law is always the electric field in the rest frame of the conductor. That is what causes charges to move, and that is where we want to evaluate the electric field and not in any other frame. It is clear then that our equations of motion (24.4.5) and (24.4.1) depend only on the relative position of the magnet and the loop and the rate at which that is changing, and not on whether the loop is moving or the magnet is moving, or whether we are in a frame where both move. The resultant relative motion of the two is the same, regardless of the inertial frame in which we describe it.

24.4.2 A Circuit with a Sliding Bar

We consider another problem with a moving conductor. In Figure 24-7 we have a highly conducting cylindrical bar has mass *M*. It moves in the $+x$-direction along two frictionless horizontal rails separated by a distance *W*, as shown in the sketch. The rails are connected on the far right by a resistor of resistance *R*, as shown. The resistance of the bar, the rails, and the external resistor is R_{total}. For $t < 0$, the bar is in the region

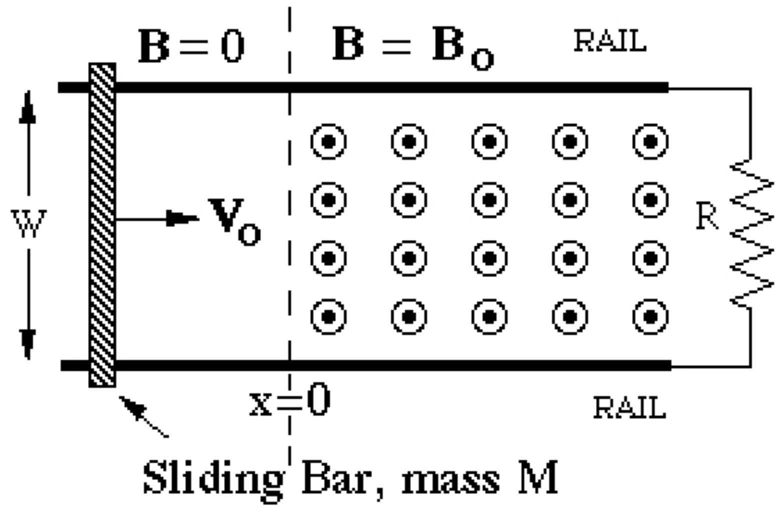

Figure 24-7: The sliding bar circuit

$x < 0$, sliding at a constant speed V_0 through a region with no magnetic field (see sketch). At time $t = 0$, when the bar is at location $x = 0$, the bar enters a region containing magnetic field $\mathbf{B}_o$, which is directed out of paper. After this time, suppose the bar has speed $V(t)$.

Faraday's Law tells us that $\oint \overline{\mathbf{E}} \cdot \mathbf{dl}$ around the circuit is equal to the negative of the time rate of change of the magnetic flux. Once the bar enters the field the flux is

$$\Phi(t) = \int_{surface} \mathbf{B} \cdot \hat{\mathbf{n}}\, da = B_o (L - X(t))W \quad \text{where} \quad \frac{dX(t)}{dt} = V(t) \tag{24.4.6}$$

In the surface integral, we have taken $\hat{\mathbf{n}}$ to be out of the page, which means that the direction of the contour integral $d\mathbf{l}$ is positive counterclockwise. So we have

$$\begin{aligned} &\oint \overline{\mathbf{E}} \cdot \mathbf{dl} = IR_{\text{total}} = -\frac{d\Phi}{dt} = +B_o V(t) W \\ &\Rightarrow I = \frac{B_o V(t) W}{R_{\text{total}}} \end{aligned} \tag{24.4.7}$$

where the plus sign in this equation means that the induced electric field and the resultant current is counterclockwise around the circuit. This means that the current in the bar is down in the drawing above.

Again, the crucial point here is that when we do the evaluation of $\oint \bar{\mathbf{E}} \cdot \mathbf{dl}$, we can always use Ohm's Law in the form $\bar{\mathbf{E}} = \rho \mathbf{J}$, and we will always end up with the total resistance of the resistors in the circuit, even though some are moving and some are not. This is because in Faraday's Law we are always evaluating the electric field in the rest frame of the circuit element.

If we ask about the force $\boldsymbol{F}$ on the bar at time $t > 0$, we see that the total force on the bar is the force per unit length $\mathbf{I} \times \mathbf{B}_o$ times the length W, and the direction is in the $-\hat{\mathbf{x}}$ direction, so we have

$$\mathbf{F} = -\hat{\mathbf{x}} WIB_o = -\hat{\mathbf{x}} V(t) \frac{\left(B_o W\right)^2}{R_{\text{total}}} \tag{24.4.8}$$

Given this force $\boldsymbol{F}$ on the bar, the differential equation for $V(t)$ for $t > 0$, is

$$\begin{aligned} & M \frac{dV}{dt} = -V(t) \frac{\left(B_o W\right)^2}{R_{\text{total}}} \\ & \Rightarrow V(t) = V_o e^{-t/\tau} \\ & \text{where} \quad \tau = \frac{MR_{\text{total}}}{B_o^2 W^2} \text{ is the e-folding time} \end{aligned} \tag{24.4.9}$$

The time dependence of I is given by (see (24.4.7))

$$I(t) = \frac{B_o W}{R_{\text{total}}} V_o e^{-t/\tau} \tag{24.4.10}$$

Thus we easily have

$$\begin{aligned} & \frac{d}{dt}\left(\frac{1}{2} M V^2\right) = MV \frac{dV}{dt} = \frac{M}{\tau} V_o^2 e^{-2t/\tau} \\ & = \frac{M}{\tau} I^2 \left(\frac{R_{\text{total}}}{B_o W}\right)^2 = I^2 \frac{M B_o^2 W^2}{M R_{\text{total}}} \left(\frac{R_{\text{total}}}{B_o W}\right)^2 = I^2 R_{\text{total}} \end{aligned} \tag{24.4.11}$$

Thus the Joule heating rate at any time is equal to the rate at which the moving bar is losing kinetic energy, as we would expect from the conservation of energy.

24.4.3 A Loop Falling Out of a Magnetic Field

We consider one last example of Faraday's Law applied to moving conductors.

A loop of mass M, resistance R, inductance L, height H, and width W sits in a magnetic field given by $\mathbf{B} = \hat{\mathbf{x}} \begin{cases} B_o & z \geq 0 \\ 0 & z < 0 \end{cases}$. At $t = 0$ the loop is at rest and its mid-point is at $z = 0$, as shown in Figure 24-8, and the current in the loop is zero at $t = 0$. The acceleration of gravity is downward at g.

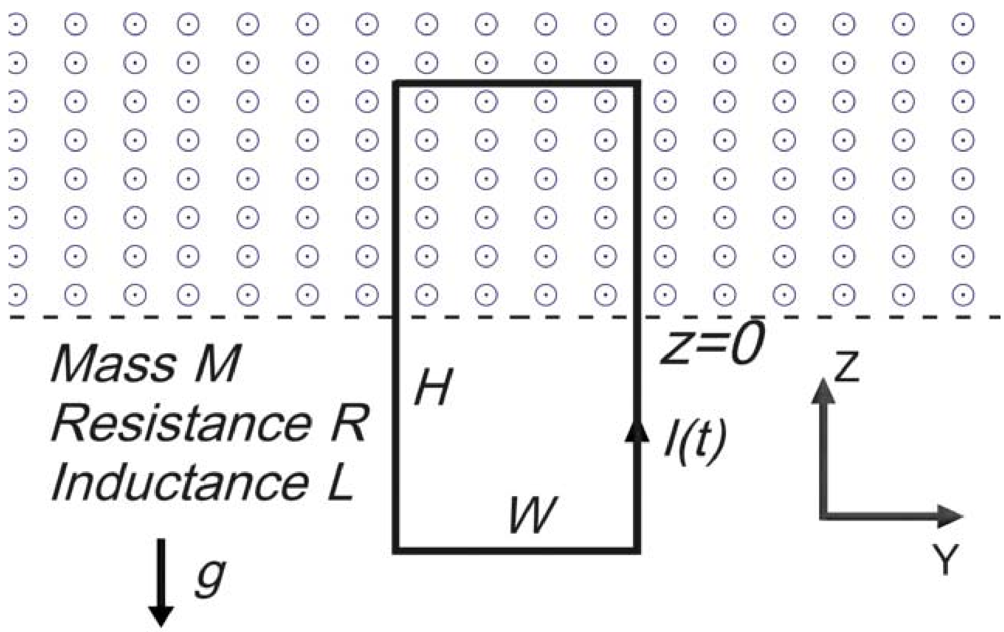

Figure 24-8: A Loop Falling Out of a Magnetic Field

The two differential equations that determine the subsequent behavior of the loop are as follows, where we take the direction of positive current to be counterclockwise, as shown in the figure. From Faraday's Law we have

$$\begin{aligned} IR &= -L\frac{dI}{dt} - \frac{d}{dt}W\left[B_o(H/2 + z(t))\right] \\ \Rightarrow IR &= -L\frac{dI}{dt} - WB_o\mathrm{v}(t) \end{aligned} \tag{24.4.12}$$

and the equation of motion is simply

$$M\frac{d}{dt}\mathrm{v}(t) = -Mg + IWB_o \tag{24.4.13}$$

If we multiply the first equation by I and the second equation by v, we have

$$I^2 R = -LI\frac{dI}{dt} - WIB_o \text{v}(t) = -\frac{d}{dt}\frac{1}{2}LI^2 - WIB_o \text{v}(t) \qquad (24.4.14)$$

and

$$M\text{v}\frac{d}{dt}\text{v}(t) = -Mg\text{v} + IWB_o\text{v} = \frac{d}{dt}\frac{1}{2}M\text{v}^2 \qquad (24.4.15)$$

Putting these two together gives conservation of energy

$$\frac{d}{dt}\left[\frac{1}{2}M\text{v}^2 + Mgz + \frac{1}{2}LI^2\right] = -I^2 R \qquad (24.4.16)$$

Let solve the case where the resistance of the loop is zero. Assuming that the loop never falls out of the magnetic field, (24.4.13) is

$$\begin{aligned} &M\frac{d}{dt}\text{v}(t) = -Mg + IWB_o \\ &\Rightarrow M\frac{d^2}{dt^2}\text{v}(t) = WB_o\frac{dI}{dt} \end{aligned} \qquad (24.4.17)$$

and (24.4.14) is

$$\begin{aligned} &0 = -L\frac{dI}{dt} - WB_o\text{v}(t) \\ &\Rightarrow \frac{dI}{dt} = -\frac{WB_o\text{v}(t)}{L} \end{aligned} \qquad (24.4.18)$$

Thus we have

$$\begin{aligned} &M\frac{d^2}{dt^2}\text{v}(t) = WB\frac{dI}{dt} = -\frac{W^2B_o^2}{L}\text{v}(t) \\ &\Rightarrow \frac{d^2}{dt^2}\text{v}(t) + \frac{W^2B_o^2}{ML}\text{v}(t) = 0 \end{aligned} \qquad (24.4.19)$$

And therefore our solution for the velocity is

$$\text{v}(t) = -\frac{g}{\omega}\sin(\omega t) \quad \text{where} \quad \omega^2 = \frac{W^2B_o^2}{ML} \qquad (24.4.20)$$

where we have picked the sin so that v is 0 at $t = 0$. This implies that

$$z(t) = \frac{g}{\omega^2}\left(\cos\left(\omega t\right) - 1\right) \tag{24.4.21}$$

where we have picked the integration constant so that z is 0 at t =0
Finally, we determine the constant A from the fact that we have to satisfy at $t = 0$ the equation

$$M\frac{d}{dt}\mathrm{v}(t) = -Mg + IWB_o \tag{24.4.22}$$

This means that we must have $M\frac{d}{dt}\mathrm{v}(0) = MA\omega\cos\left(0\right) = -Mg \Rightarrow A = -\frac{g}{\omega}$

so

$$\mathrm{v}(t) = -\frac{g}{\omega}\sin\left(\omega t\right) \quad \text{where} \quad \omega^2 = \frac{W^2 B_o^2}{ML} \quad \text{and} \quad z(t) = \frac{g}{\omega^2}\left(\cos\left(\omega t\right) - 1\right) \tag{24.4.23}$$

25 EMF's and Faraday's Law in Circuits

25.1 Learning Objectives

We now consider the concept of electromotive force in a circut. We have seen above that a given observer may think that the motion of charges is driven either by a $q\mathbf{E}$ or a $q\mathbf{v}\times\mathbf{B}$ force in the observer's rest frame, or some combination of both, but that this always reduces to the electric field in the rest frame of the circuit element. There may also be or some other force entirely (see our example of the battery below), and that leads us to consider the general concept of an EMF. We then turn our attention to the typical circuit elements: batteries, resistors, capacitors, and inductors. We pay special attention to inductors, since there are a huge number of misconceptions about the "voltage drop" across an inductor. We will only consider single loop circuits here. In the beginning we will only consider circuits with batteries with resistors. Then we will add inductors, and finally capacitors. ***Unlike in the situation above, in this Section we assume that all parts of the circuit are at rest.***

25.2 The electromotive force

Suppose we have a current flowing in a closed circuit. To have a flow of current in our single loop circuit, there must be at every point in the circuit some force per unit charge **f** on the charge carriers which causes them to move. For the moment we consider a circuit containing only batteries and resistors. Ohm's Law states the relation between the force per unit charge **f** at any point in the circuit and the current density **J** at that point is $\mathbf{J} = \sigma_c\mathbf{f}$. This is the same relation we have derived in Section 28.2, except that there we considered only the force per unit charge due to an electric field **E,** *whereas here we* *consider the force per unit charge due to any force.* The constant σ_C is called the conductivity (we use the symbol σ_C to distinguish this quantity from surface charge σ). Why do we have this particular relationship? The classical model is given in Section 28.2, and we do not repeat those arguments here, other than that we point out that those arguments easily generalize from the electric field to *any* force per unit charge **f.**

Now, the electromotive force (or emf) of our single loop circuit is denoted by the symbol $\mathcal{E}$, and is defined by the equation

$$\mathcal{E} = \oint \mathbf{f} \cdot d\mathbf{l} \tag{25.2.1}$$

where the integral is around the complete circuit, and at every point in the circuit **f** is the force per unit charge that is felt by the charge carriers located at that point (the same **f** that we were dealing with above). The terminology here is poor, since an "electromotive force" is not a force at all, but instead is a closed line integral of a force per unit charge. Note that the units of emf are Joules/coulomb, or Volts.

In any case, it is the emf $\mathcal{E}$ defined in (25.2.1) that determines how much current will flow in a circuit, by the following argument. The crucial step in this argument is understanding the following point.

For a single loop circuit, the current *I* is to an good approximation the same in all parts of the circuit

Why is this so? Basically, although the current will start out at *t=0* being unequal in different parts of the circuit, those inequalities mean that charge is piling up somewhere. The accumulating charge at the pile up will quickly produce an electric field, and *this electric field is always in the sense so as to even out the inequalities in the current*. We give a semi-qualitative example of this evening-out below in Section 25.8, but for the moment we simply accept it.

The upshot is that in a very short time electric fields will be set up around the circuit along with various pockets of accumulated charge, all arranged so as to made the current the same in every part of the circuit. This will be true all around the circuit as long as we long as we are considering time changes in the current that are long compared to the speed of light transit time across the circuit.

Given this, let's figure out how the resultant current I is related to the emf $\mathcal{E}$. We have

$$\mathcal{E} = \oint \mathbf{f} \cdot d\mathbf{l} = \oint \frac{\mathbf{J}}{\sigma_c} \cdot d\mathbf{l} = \oint \frac{I}{A\sigma_c} dl = I \oint \frac{dl}{A\sigma_c} \tag{25.2.2}$$

where A is the cross-sectional area of the circuit at any point, and σ_C its conductivity, and both of these quantities can vary at different points in the circuit. However, the current I

does not vary around the circuit, which allows us to take it out of the integral in equation (25.2.2). We have also assumed that the current at any point is uniformly distributed across the cross-sectional area *A*. This is not a crucial assumption, and we can relax that easily. Thus if we define the total resistance R_T of our one loop circuit as

$$R_T = \oint \frac{dl}{A\sigma_c} \tag{25.2.3}$$

then we have the relationship between the emf, the current *I*, and the total resistance of the circuit R_T :

$$I = \frac{\mathcal{E}}{R_T} \tag{25.2.4}$$

From equation (25.2.4), the units of the resistance are clearly *volt/amp* which we define as an *ohm*. Thus we see that the conductivity σ_C must have units of $ohm^{-1}\ m^{-1}$. Sometimes we will refer to the resistivity of a material. The resistivity is just the inverse of the conductivity, and has units of *ohm meters*.

25.3 An example of an EMF: batteries

Let's apply these ideas to a concrete example--a simple circuit consisting of a battery connected by highly conducting wires to a resistor (a conductor with low conductivity σ_C). Batteries provide one of the most familiar examples of a source of electromagnetic energy. Batteries are in many ways like capacitors, with one fundamental difference. There are chemical people inside batteries that do work, and provide an "electromotive force", as follows. The positive terminal of a battery carries positive charge, just like the positive plate of a (charged) capacitor, and similarly for the negative terminal (see sketch next page). There is therefore an internal electric field $\mathbf{E}_B$ in the battery, going from the positive terminal to the negative terminal, and the positive terminal of the battery is at a higher electric potential than the negative terminal. When the battery is placed in a circuit, say with a resistor, there is then a path for positive charge to flow from higher to lower potential (through the resistor--see below), and charge will do just that when the circuit is established. Up to this point, we have a situation that looks very much like a capacitor that is discharging through a resistor.

But the terminals of the battery do *not* discharge. The cartoon essence of a battery is the following. Suppose a charge $+dq$ leaves the positive terminal, flows through the resistor, and then arrives at the negative terminal. When that charge arrives at the negative terminal, a chemical person picks it up and, and applying a force per unit charge $\mathbf{f}_S$, moves it *against* the internal electric field of the battery, and deposits it on the positive terminal again. No matter how rapidly charge flows off the positive terminal through the external circuit and arrives at the bottom plate, the chemical people manage to keep up, transferring the incoming charge on the negative terminal to the positive terminal, as fast as it arrives. Clearly our "chemical people" are doing work in this process, just as we do work in charging a capacitor, because they are moving positive charge against the electric field of the battery by applying the force $\mathbf{f}_S$.

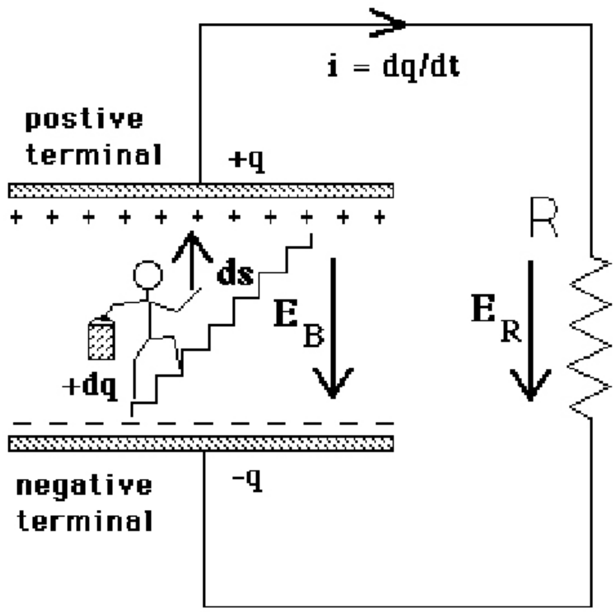

Figure 25-1: A cartoon view of a battery in a circuit

However, if we look at the complete circuit, there are really two forces per unit charge involved in driving current around the circuit: the "source" of the emf, $\mathbf{f}_S$, which is ordinarily confined to one portion of the loop (inside the battery here), and the electrostatic force per unit charge $\mathbf{E}$ whose function is to smooth out the flow and communicate the influence of the source to distant parts of the circuit (*Griffiths*, page 292). The total force per unit charge, $\mathbf{f}$, that is the $\mathbf{f}$ that appears in equation (25.2.1) above, is therefore given by the sum

$$\mathbf{f} = \mathbf{f}_S + \mathbf{E} \tag{25.3.1}$$

In the case of the circuit here, a battery and a resistor, $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ because this is an electrostatic field. Furthermore, $\mathbf{f}_S$ vanishes outside of the battery, so that we have simply that $\mathcal{E} = \int_{\text{bottom battery}}^{\text{top battery}} \mathbf{f}_s \cdot d\mathbf{l}$. This is just the work done per unit charge by the chemical people in moving the charge against the electric field of the battery from the bottom plate to the top plate. The *rate* at which they are doing work is *dq/dt* times this work per unit charge, or

$$P_{\text{rate at which work done by battery}} = I\mathcal{E} \tag{25.3.2}$$

and this is the rate that the battery is providing energy to the circuit.

The origin of the electromotive force in a battery is the internal mechanism (the "chemical people") that transports charge carriers in a direction opposite to that in which the electric field would move them. In ordinary batteries, it is chemical energy that makes the charge carriers move against the internal field of the battery. That is, a positive charge will move towards higher electric potential if in so doing it can engage in a chemical reaction that will yield more energy than it costs to move against the electric field. The electromotive force in a battery depends on atomic properties. The values of potential differences between the battery terminals lie in the range of volts because the binding energies of the outer electrons of atoms are in the range of several electron volts, and it is essentially the differences in these binding energies that determine the voltage of the battery.

The actual details of all this are complicated, so we will not go beyond the cartoon level. Purcell (*Electricity and Magnetism, Berkeley Physics Course, Volume 2*, McGraw-Hill, 1965) has a excellent discussion of batteries in Chapter 4 of that volume.

25.4 The resistance of a resistor

Finally, let's finish up our discussion of the simple battery and resistor circuit shown in Figure 25-1. Suppose the resistor in our circuit consists of a conducting cylinder of length L and cross-sectional area A , with conductivity . Since the only force per unit charge in the resistor is the electric field in the resistor, $\mathbf{E}_R$ (the battery "source" $\mathbf{f}_S$ is zero there), the current density in the resistor, $\mathbf{J}_R$, is $\sigma_c \mathbf{E}_R$ by the microscopic form of Ohm's Law. Let $\Delta V_R = -\int_{\text{bottom R}}^{\text{top R}} \mathbf{E}_R \cdot d\mathbf{l}$ be the potential difference from the bottom to the top of the resistor (see Figure 25-1). Then

$$I / A = J_R = \sigma_c E_R = \sigma_c \Delta V_R / L \tag{25.4.1}$$

Solving equation (25.4.1) for I in terms of ΔV_R, we obtain

$$I = \Delta V_R / R \quad \text{with} \quad R = \frac{L}{\sigma_c A} \tag{25.4.2}$$

This is the macroscopic form of Ohm's Law with which we are most familiar. The quantity R is the resistance of the resistor, in ohms, and is a function of both the fundamental properties of the material, via σ_c and of the shape of the material, via A and L.

Suppose now that the conductivity in the connecting wires and in the battery is so much larger than σ_c that we can take them to be infinite for all practical purpose. Then equation (25.2.3) becomes $R_T = R$ That is, the total resistance of our circuit is to a good approximation just the resistance of our (low conductivity) resistor. If we consider equation (25.2.4) (with $R_T = R$), we obtain $I = \mathcal{E}/R = \Delta V_R/R$, and therefore $\mathcal{E} = \Delta V_R$.

Since $\oint \mathbf{E}\cdot\mathbf{dl}=0$, this only can be true if inside the battery we have approximately that $\mathbf{f}_s=-\mathbf{E}_B$, where $\mathbf{E}_B$ is the electric field in the battery. Note that since **J** and **E** are anti-parallel in the battery, the creation rate of electromagnetic energy, $-\mathbf{E}\cdot\mathbf{J}$, is positive there, and therefore electromagnetic energy is being created in the battery.

25.5 Joule heating

Since $\mathbf{J}_R$ and $\mathbf{E}_R$ are parallel in the resistor, the creation rate of electromagnetic energy, $-\mathbf{E}\cdot\mathbf{J}$, is negative, and therefore electromagnetic energy must be being destroyed in the resistor. Where is this energy going? Well, the electric field is doing work on the charges at the rate of $qn\mathbf{E_R}\cdot\mathbf{v}_e$, or $+\mathbf{E}_\text{R}\cdot\mathbf{J}_\text{R}$, per unit volume, just as we would expect (if electromagnetic energy is disappearing, it must be going some place). That work done by the field in the steady state is transmitted from the charges to the lattice via collisions, i.e., to random thermal energy. The total rate at which this heating takes place is $+\mathbf{E}_\text{R}\cdot\mathbf{J}_\text{R}$ times the volume *AL*, or *EJAL*, or *VI*. Thus the heating rate P_heating, in joules per second, is
(25.6.1)

$$P_\text{heating}=\Delta V_R I=I^2R \tag{25.5.1}$$

This is the familiar form of the Joule Heating Law. If we furthermore use the fact that $\mathcal{E}=\Delta V_R$, we have that the rate at which the battery is doing work is equal to the rate at which energy is appearing as heat in the resistor. Thus electromagnetic energy is being created in the battery at the same rate at which it is being destroyed in the resistor.

25.6 Self-inductance and simple circuits with "one-loop" inductors:

The addition of time-changing magnetic fields to simple circuits means that the closed line integral of the electric field around a circuit is no longer zero. Instead, we have, for any open surface

$$\oint_C \mathbf{E}\cdot\mathbf{dl}=-\int_S\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}\cdot\hat{\mathbf{n}}\,da \tag{25.6.1}$$

Any circuit in which the current changes with time will have time-changing magnetic fields, and therefore associated “induced” electric fields, *which are due to the time changing currents, not to the time changing magnetic field (association is not causation).* How do we solve simple circuits taking such effects into account? We discuss here a consistent way to understand the consequences of introducing time-changing magnetic fields into circuit theory--that is, *self-inductance.*

As soon as we introduce time-changing currents, and thus time changing magnetic-fields, the electric potential difference between two points in our circuit is not longer well-defined. When the line integral of the electric field around a closed loop is no longer zero, the potential difference between points *a* and *b*, say, is no longer independent of the path used to get from *a* to *b*. That is, the electric field is no longer a conservative field, and the electric potential is no longer an appropriate concept (that is, **E** can no longer be written as the negative gradient of a scalar potential). However, we can still write down in a straightforward fashion the differential equation for *I(t)* that determines the time-behavior of the current in the circuit.

To show how to do this, consider the circuit shown in Figure 25-2. We have a battery, a resistor, a switch *S* that is closed at $t = 0$, and a "one-loop inductor". It will become clear what the consequences of this "inductance" are as we proceed. For $t > 0$, current will flow in the direction shown (from the positive terminal of the battery to the negative, as usual). What is the equation that governs the behavior of our current *I* for $t > 0$?

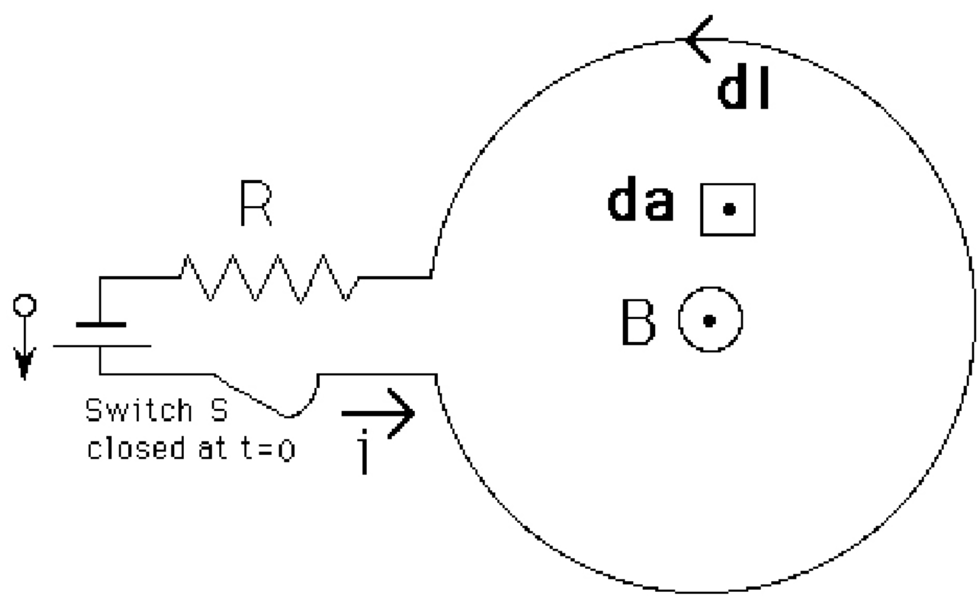

Figure 25-2: A simple circuit with battery, resistor, and a one-loop inductor

To investigate this, we apply Faraday's Law to the open surface bounded by our circuit, where we take $d\mathbf{a} = \hat{\mathbf{n}}\,da$ to be out of the page, and thus $d\mathbf{l}$ is counter-clockwise, as shown. First, what is the integral of the electric field around this circuit? That is, what is the left-hand side of (25.6.1)? Well, there is an electric field in the battery in the direction of $d\mathbf{l}$ that we have chosen, we are moving against that electric field, so that $\int \mathbf{E}\cdot d\mathbf{l}$ is negative. Thus the contribution of the battery to our integral is $-\mathcal{E}$ (see the discussion in Section 25.3 above). Then there is an electric field in the resistor, in the direction of the current, so when we move through the resistor in that direction, $\int \mathbf{E}\cdot d\mathbf{l}$ is positive, and that contribution to our integral is $+I\,R$. What about when we move through our “one-loop inductor”? There is no electric field in this loop if the resistance of the wire making up the loop is zero, so there is no contribution to $\int \mathbf{E}\cdot d\mathbf{l}$ from this part of the circuit. This may bother you, and we talk at length about it below. So, going totally around the closed loop, we have

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\mathcal{E} + I\,R \tag{25.6.2}$$

Now what is the right hand side of (25.6.1). Since we have assumed in this Section that the circuit is not moving, we can take the partial with respect to time outside of the surface integral and then we simply have the time derivative the magnetic flux through the loop. What is the magnetic flux through the open surface? First of all, we arrange the geometry so that the part of the circuit which includes the battery, the switch, and the resistor, makes only a small contribution to the magnetic flux as compared to the (much larger area) of the open surface which consititutes our “one-loop inductor”. Second, we know that the sign of the magnetic flux is positive in that part of the circuit because current flowing counter-clockwise will produce a **B** field out of the paper, which is the same direction of $\hat{\mathbf{n}}\,da$, so that $\mathbf{B}\cdot\hat{\mathbf{n}}\,da$ is positive. Note that our magnetic field here is the *self* magnetic field—that is the magnetic field produced by the current flowing in the circuit, and not by an currents external to this circuit.

We also know that at any point in space, **B** is proportional to the current *I*, since it can be computed from the Biot-Savart Law, that is,

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_o I(t)}{4\pi}\oint \frac{d\mathbf{l}'\times(\mathbf{r}-\mathbf{r}')}{\left|(\mathbf{r}-\mathbf{r}')\right|^3} \tag{25.6.3}$$

You may immediately object that the Biot-Savart Law is only good in time-independent situations, but in fact, as we have seen before when considering radiation, as long as the current is varying on time scales *T* long compared to the speed of light travel time across the circuit and we are within a distance *cT* of the currents, then (25.6.3) is an excellent approximation to the time dependent magnet field. If we look at (25.6.3), although for a general point in space it involves a very complicated integral over the circuit, it is clear that $\mathbf{B}(\mathbf{r},t)$ is everywhere propostional to *I(t)*. That is, if we double the current, **B** at any point in space will also double. It then follows that the magnetic flux itself must be proportional to *I*, because it is the surface integral of **B**, and **B** is everywhere proportional to *I*. That is,

$$\Phi(t) = \int_{S(t)} \mathbf{B}(\mathbf{r},t)\cdot\hat{\mathbf{n}}da = \int_S \left\{\frac{\mu_o I(t)}{4\pi}\oint \frac{d\mathbf{l}'\times(\mathbf{r}-\mathbf{r}')}{\left|(\mathbf{r}-\mathbf{r}')\right|^3}\right\}\cdot\hat{\mathbf{n}}\,da = I(t)\int_S \left\{\frac{\mu_o}{4\pi}\oint \frac{d\mathbf{l}'\times(\mathbf{r}-\mathbf{r}')}{\left|(\mathbf{r}-\mathbf{r}')\right|^3}\right\}\cdot\hat{\mathbf{n}}\,da \tag{25.6.4}$$

or

$$\Phi(t) = LI(t) \qquad L = \frac{\mu_o}{4\pi}\int_S \left\{ \oint \frac{d\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{\left|(\mathbf{r}-\mathbf{r}')\right|^3} \right\} \cdot \hat{\mathbf{n}}\, da \tag{25.6.5}$$

So the magnetic flux is a constant *L* times the current. Note that *L* is a constant in the sense that it stays the same as long as we do not change the geometry of the circuit. If we change the geometry of the circuit (for example we halve the radius of the circle in our Figure above), we will change L, but for a given geometry, L does not change. Even though it may be terrifically difficult to do the integrals in (25.6.5), once we have done it for a given circuit geometry we know *L*, and *L* is a constant for that geometry. The quantity L is called the self-inductance of the circuit, or simply the inductance. From the definition in (25.6.5), you can show that the dimensions of *L* are μ_o times a length. In Assignment 10, you show that a lower limit to the inductance of a single loop of wire is proportional to μ_o times its radius.

Regardless of how hard or easy it is to compute *L*, it is a constant for a given circuit geometry and now we can write down the equation that governs the time evolution of *I*. If $\Phi(t) = LI(t)$, then $d\Phi(t)/dt = LdI(t)/dt$, and equation (25.6.1) becomes

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\mathcal{E} + I\,R = -L\frac{dI}{dt} \tag{25.6.6}$$

If we divide (25.6.6) by *L* and rearrange terms, we find that the equation that determines the time dependence of *I* is

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{\mathcal{E}}{L} \tag{25.6.7}$$

The solution to this equation given our initial conditions is

$$I(t) = \frac{\mathcal{E}}{R}(1 - e^{-tR/L}) \tag{25.6.8}$$

This solution reduces to what we expect for large times, that is $I = \frac{\mathcal{E}}{R}$, but it also shows a continuous rise of the current from 0 initially to this final value, with a characteristic time τ_L defined by

$$\tau_L = \frac{L}{R} \tag{25.6.9}$$

This time constant is known as the inductive time constant. This is the effect of having a non-zero inductance in a circuit, that is, of taking into account the "induced" electric fields which always appear when there are time changing **B** fields. And this is what we expect—the reaction of the system is to try to keep things the same, that is to delay the build-up of current (or its decay, if we already have current flowing in the circuit).

25.7 Kirchhoff's Second Law modified for inductors: a warning

We can write the governing equation for $I(t)$ from above as

$$\mathcal{E} - I\,R - L\frac{dI}{dt} = \sum_{\mathrm{i}} \Delta V_i = 0 \qquad (25.7.1)$$

where we have now cast it in a form that "looks like" a version of Kirchhoff's Second Law, a rule that is often quoted in elementary electromagnetism texts. Kirchhoff's Second Law states that the sum of the potential drops around a circuit is zero. In a circuit with no inductance, this is just a statement that the line integral of the electric field around the circuit is zero, which is certainly true if there is no time variation. However, in circuits with currents that vary in time, this "Law" is no longer true.

Unfortunately, many elementary texts choose to approach circuits with inductance by preserving "Kirchhoff's Second Law", or the loop theorem, by specifying the "potential drop" across an inductor to be $-LdI/dt$ if the inductor is traversed in the direction of the current. Use of this formalism will give the correct equations. However, the continued use of Kirchhoff's Second Law with inductors is misleading at best, for the following reasons.

Kirchhoff's Second Law was originally based on the fact that the integral of **E** around a closed loop was zero. With time-changing currents and thus time-changing self-magnetic fields, *this is no longer true* (the **E** field is no longer conservative), and thus the sum of the "potential drops" around the circuit, if we take that to mean the *negative* of the closed loop integral of **E**, is ***no longer zero***--in fact it is LdI/dt (this is equation (25.6.6) with the sign reversed).

The continued use of Kirchhoff's Second Law in this way gives the right equations, but it confuses the physics. In particular, saying that there is a "potential drop" across the inductor of $-LdI/dt$ implies that there is an electric field in the inductor such that the integral of **E** through the inductor is equal to $-LdI/dt$. **This is not always, or even usually, true**. For example, suppose in our "one-loop" inductor above that the wire making up the loop has negligible resistance compared to the resistance R. The integral of **E** through our "one-loop" inductor above is then **very small**, **NOT** $-LdI/dt$. Why is it very small? Well, to repeat our assertion above

For a single loop circuit, the current *I* is to an good approximation the same in all parts of the circuit

This is just as valid in a circuit with inductance. Again, although the current may start out at *t=0* being unequal in different parts of the circuit, those inequalities mean that charge is piling up somewhere. The accumulating charge at the pile-up will quickly produce an electric field, and this *electric field is always in the sense so as to even out the inequalities in the current*. In this particular case, if the conductivity of the wires making up our one-loop inductor is very large, then there will be a very small electric field in those wires, because it takes only a small electric field to drive any current you need. The amount of current needed is determined in part by the larger resistance in other parts of the circuit, and it is the charge accumulation at the ends of those low conductivity resistors that cancel out the field in the inductor and enhance it in the resistor, so as to maintain constant current in the circuit. We return to this point below in Section 25.8.

One final point, to confuse the issue further. If you have ever put the probes of a voltmeter across an the terminals of an inductor (with very small resistance) in a circuit, what you measured on the meter of the voltmeter was a "voltage drop" of $-LdI/dt$. But that is not because there is an electric field in the inductor! It is because putting the voltmeter in the circuit will result in a time changing magnetic flux through the voltmeter circuit, consisting of the inductor, the voltmeter leads, and the large internal resistor in the voltmeter. A current will flow in the voltmeter circuit because there will be an electric field in the large internal resistance of the voltmeter, with a potential drop across *that* resistor of $-LdI/dt$, by Faraday's Law applied to the voltmeter circuit, and that is what the voltmeter will read. The voltmeter as usual gives you a measure of the potential drop across its own internal resistance, but this is *not* a measure of the potential drop across the inductor. It is a measure of the time rate of change of magnetic flux in the voltmeter circuit! As before, there is only a very small electric field in the inductor if it has a very small resistance compared to other resistances in the circuit.

25.8 How can the electric field in an inductor be zero?

Students are always confused about the electric field in inductors, in part because of the kinds of problems they have seen. Quite often in simple problems with time varying magnetic fields, there is an “induced” electric field right where the time varying magnetic field was non-zero. What has changed in our circuit above to make the electric field zero in the wires of the (resistanceless) inductor zero, even though there is a time changing magnetic flux through it? This is a very suble point and a source of endless confusion, so let’s look at it very carefully.

Your intuition that there should be an electric field in the wires of an inductor is based on doing problems like that shown in Figure 25-3 below. We have a loop of wire of radius *a* and total resistance *R* immersed in an external magnetic field which is out of the page and increasing with time as shown. In considering this circuit, unlike in our “one-loop” circuit above, we neglect the the magnetic field due to the currents in the wire itself, assuming that the external field is much bigger than the self-field. The conclusions we arrive at here can be applied to the self-inductance case as well.

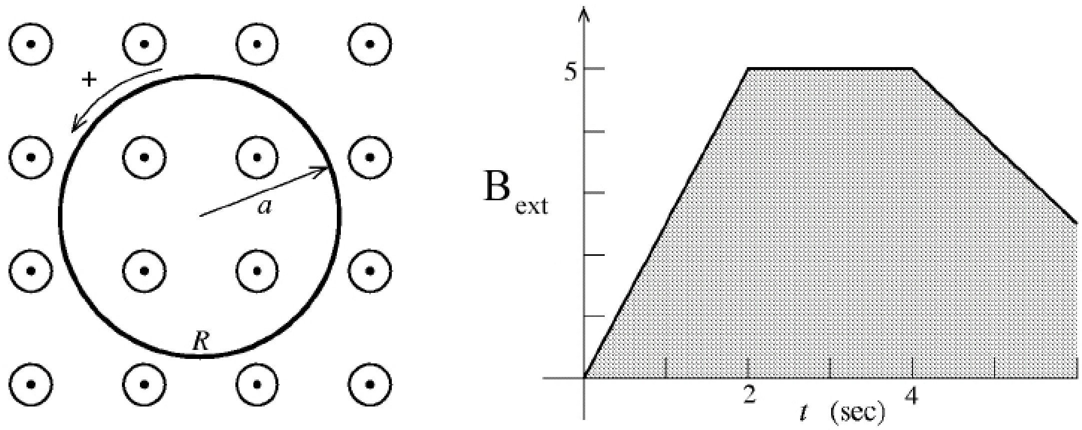

Figure 25-3: A loop of wire sitting in a time-changing external magnetic field

The changing extermal magnetic field will give rise to an "induced" electric field in the loop of the wire, with

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\left(B_{ext}\pi a^2\right) \tag{25.8.1}$$

This "induced" electric field is azimuthal and uniformly distributed around the loop as long as the resistance in the loop is uniform, and in the loop itself we easily have from (25.8.1) that the electric field right at the loop is given by

$$\oint \mathbf{E} \cdot d\mathbf{l} = 2\pi a E_\phi = -\frac{d}{dt}\left(B_{ext}\pi a^2\right) \quad \Rightarrow \mathbf{E}\big|_{r=a} = -\hat{\boldsymbol{\phi}}\frac{a}{2}\frac{dB_{ext}}{dt} \tag{25.8.2}$$

Thus if the resistance is distribed uniformly around the wire loop, we get a uniform induced electric field in the loop, circulating clockwise for the external magnetic field increasing in time (see Figure 25-4). This electric field causes a current to flow, and the current will circulate clockwise in the same sense as the electric field. The total current in the loop will be the total "potential drop" around the loop divided by its resistance *R*, or

$$I = \frac{\pi a^2}{R}\frac{dB_{ext}}{dt} \tag{25.8.3}$$

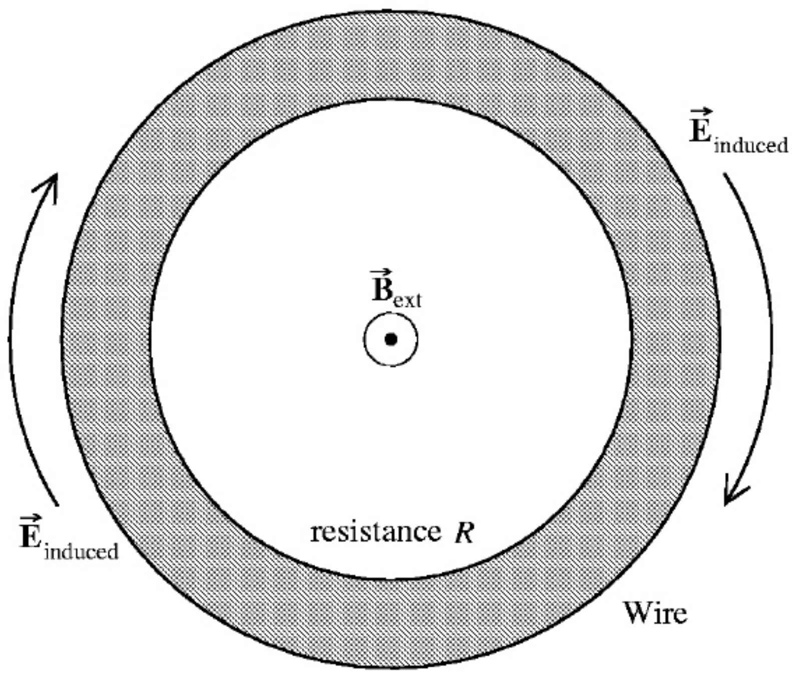

Figure 25-4: A loop of wire with resistance *R* in an external field out of the page

But what happens if we don't distribute the resistance uniformly around the wire loop? For example, let us make the left half of our loop out of wire with resistance R_1 and the right half of the loop out of wire with resistance R_2, with $R_1 + R_2 = R$, so that we have the same total resistance as before (see Figure 25-5). Let us futher assume that $R_1 < R_2$. How is the electric field distributed around the loop now?

First of all, the electromotive force around the loop (see (25.8.1)) is the same, as is the resistance, so that the current *I* has to be the same as in (25.8.3). Moreover it is the same on both sides of the loop by charge conservation. But the electric field in the left half of the loop $\mathbf{E}_1$ must now be different from the electric field in the right half of the loop $\mathbf{E}_2$. This is so because the line integral of the electric field on the left side is $\pi a E_1$, and from Ohm's Law in macroscopic form, this must be equal to IR_1. Similarly, $\pi a E_2 = IR_2$. Thus

$$\frac{E_1}{E_2} = \frac{R_1}{R_2} \Rightarrow E_1 < E_2 \text{ since } R_1 < R_2 \tag{25.8.4}$$

This makes sense. We get the same current on both sides, even though the resistances are different, and we do this by adjusting the electric field on the side with the smaller resistance to ***be*** smaller. Because the resistance is also smaller, we produce the same current as on the opposing side, with this smaller electric field.

But what happened to our uniform electric field. Well there are two ways to produce electric fields—one from time changing currents and their associated time changing magnet fields, and the other from electric charges. Nature accomplishes the reduction of $\mathbf{E}_1$ compared to $\mathbf{E}_2$ by charging at the junctions separating the two wire segments (see Figure 25-5), positive on top and negative on bottom.

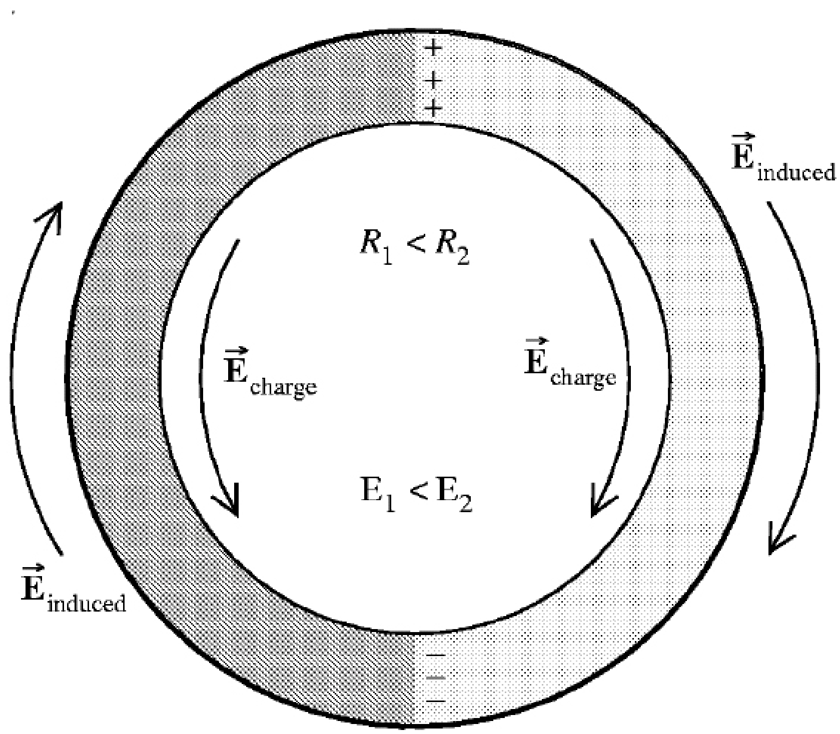

Figure 25-5: The electric field in the case of unequal resistances in the loop

The total electric field is the sum of the “induced” electric field and the electric field associated with the charges, as shown in the Figure above. It is clear that the addition of these two contributions to the electric field will reduce the total electric field on the left (side 1) and enhance it on the right (side 2). The field $\mathbf{E}_1$ will always be clockwise, but it can be made arbitrarily small by making $R_1 << R_2$.

Thus we see that we can make a non-uniform electric field in an inductor by using non-uniform resistance, even though our intuition tells us (correctly) that the “induced” electric field should be uniform at a given radius. All that Faraday’s Law tells us is that the line integral of the electric around a closed loop is equal to the negative of the time rate of change of the magnet flux through the enclosed surface. It does not tell us at what locations the electric field is non-zero around the loop, and it may be non-zero (or zero!) in unexpected places. The field in the wire making up the “one-loop” inductor we considered above is zero (or least very small) for exactly the kinds of reasons we have been discussing here.