8.07 Class Notes Fall 2011

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Table of Contents
(*LIVE LINKS*)

1 Fields and their Depiction 10
1.1 Learning objectives 10
1.2 Maxwell's electromagnetism as a fundamental advance 10
1.3 Why this course is different 11
1.3.1 The profound parts of E&M first 11
1.3.2 The easy E&M and the hard E&M 12
1.3.3 Energy and momentum in fields 13
1.3.4 Animations and visualizations 13
1.4 Reference texts 14
1.5 Representations of vector fields 14
1.5.1 The vector field representation of a vector field 14
1.5.2 The field line representation of a vector field 15
1.5.3 Line integral convolution representations 16
2 Conservations Laws for Scalar and Vector Fields 19
2.1 Learning objectives 19
2.2 Conservation laws for scalar quantites in integral and differential form 19
2.2.1 Density and creation rate of a conserved scalar quantity 19
2.2.2 The flux and flux density of a conserved scalar quantity 19
2.2.3 Gauss's Theorem and the differential form 21
2.3 Conservation laws for vector quantities in integral and differential form 21
3 The Dirac delta function and complete othorgonal sets of fucntions 22
3.1 Basic definition 22
3.2 Useful relations 24
3.3 Complete sets of orthorgonal functions on a finite interval 24
3.4 Representation of a delta function in terms of a complete set of functions 25
4 Conservation of energy and momentum in electromagnetism 26
4.1 Learning objectives 26
4.2 Maxwell's Equations 26
4.3 Conservation of charge 26
4.4 Conservation of energy 26
4.5 Conservation of momentum and angular momentum 27
4.5.1 The Maxwell stress tensor in statics 28
4.5.2 Calculating $\vec{\mathbf{T}}\cdot\hat{\mathbf{n}}\,da$ 30
5 The Helmholtz Theorem 36
5.1 Learning Objectives 36

5.2 The Helmholtz Theorem 36
5.2.1 Construction of the vector function F 37
5.2.2 The inverse of the Helmholtz Theorem 37
5.2.3 The Helmholtz Theorem in two dimensions 37
5.3 Examples of incompressible fluid flows 38
5.3.1 Irrotational flows 38
5.3.2 Flows with rotation 40
6 The Solution to the Easy E&M 41
6.1 Learning Objectives 41
6.2 The Easy Electromagnetism 41
6.2.1 The Solution to Maxwell's Equations 41
6.2.2 The free space-time dependent Green's function 43
6.2.3 The solution for $(\phi, \mathbf{A})$ given $(\rho, \mathbf{J})$ for all space and time 45
6.3 What does the observer see at time *t*? 45
6.3.1 The collapsing information gathering sphere 46
6.3.2 The backward light cone 47
7 E and B far from localized static sources 48
7.1 Learning objectives 48
7.2 A systematic expansion in powers of *d/r* 48
7.3 The magnetic dipole and electric dipole terms 50
7.4 Properties of a static dipole 51
7.5 The electric quadrupole term 52
8 Sources varying slowly in time 53
8.1 Learning objectives 53
8.2 E and B fields far from localized sources varying slowly in time 53
8.3 Electric dipole radiation 56
9 Examples of electric dipole radiation 58
9.1 Learning objectives 58
9.2 Dipole moment vector p varying in magnitude but not direction 58
9.3 The near, intermediate, and far zones 60
9.4 Examples of electric dipole radiation in the near, intermediate, and far zones 60
9.4.1 Dipole moment varying sinusoidally with total reversal, in the near zone... 62
9.4.2 Dipole moment with total reversal, in the intermediate zone 63
9.4.3 Dipole moment varying sinusoidally with total reversal, in the far zone 64
9.4.4 Dipole moment increasing 50% over time *T*, in the near zone 66
9.4.5 Dipole moment increasing over time *T*, in the intermediate zone 67
9.4.6 Dipole moment decreasing by 33% over time *T*, in the near zone 67
9.4.7 Dipole moment decreasing over time *T*, in the intermediate zone 68
9.5 Conservation of energy 69
10 The General Form of Radiation E and B Fields 72
10.1 Learning Objectives 72
10.2 General expressions for radiation E and B fields 72
10.3 Radiation of energy and momentum in the general case 73
11 Another Example of Electric Dipole Radiation 74
11.1 Moving non-relativistic point charges 74
11.2 Energy and momentum flux 77

12 Magnetic Dipole and Electric Quadrupole Radiation 78
12.1 Learning Objectives 78
12.2 Magnetic dipole radiation 78
12.3 Electric quadrupole radiation 81
12.4 An Example Of Electric Quadrupole Radiation 83

7 E and B far from localized static sources

7.1 Learning objectives

We first investigate the form of the electric and magnetic fields far from localized sources, assuming that these sources do not vary in time. "Localized" means that our sources vanish outside of a sphere of radius d. "Far from" mean we are at radii such that $r >> d$. Our major goal here is to show that everything "looks" the same if you get far enough away, and to introduce the idea of moments, that is, dipole moments, quadrupole moments, and so on.

7.2 A systematic expansion in powers of *d/r*

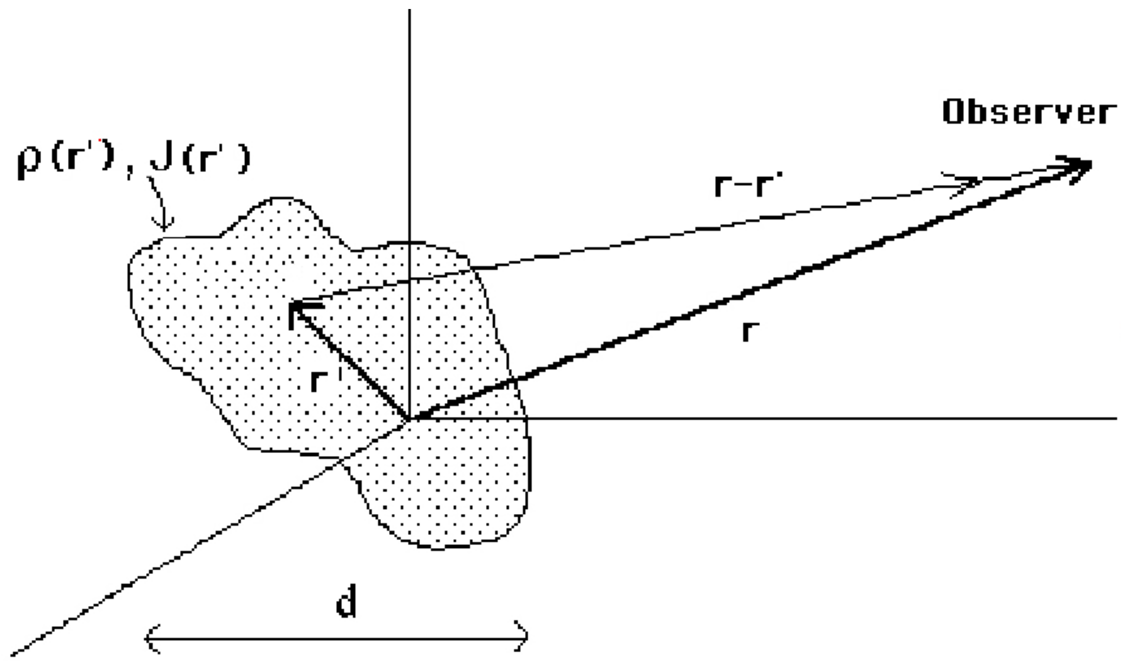


Figure 7-1: An observer far from a localized distribution of static sources.

In Figure 7-1, we show a distribution of time independent charges and currents which vanish outside a distance *d* from the orgin. I want to look at (6.2.13) and (6.2.15) when there is no time dependence and under the assumption that I am far away from the sources compared to *d*. For no time independence, I have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3x' \tag{7.2.1}$$

and

$$\phi(\mathbf{r}) = \frac{1}{4\pi\,\varepsilon_o}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3x' \tag{7.2.2}$$

Let's look at the $|\mathbf{r}-\mathbf{r}'|$ term in (7.2.2), assuming that the angle between $\mathbf{r}$ and $\mathbf{r}'$ is θ'. I have

$$|\mathbf{r}-\mathbf{r}'|^2 = (\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}') = r^2 + (r')^2 - 2rr'\cos\theta' \tag{7.2.3}$$

I define

$$\eta = \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta' = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right) \tag{7.2.4}$$

and I assume that $r >> r'$, since I am taking the distance *r* of the observer large compared to *d*, and the r' value in the integral in (7.2.3) can always be assumed to be less that *d*, since the sources vanish outside of *d*. Then it is clear that η is a small quantitiy, and

$$|\mathbf{r}-\mathbf{r}'| = r(1+\eta)^{1/2} \qquad \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}(1+\eta)^{-1/2} \tag{7.2.5}$$

I can expand $(1+\eta)^{-1/2}$ in a Taylor series, since I always have $\eta << 1$, giving

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}(1+\eta)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\eta + \frac{3}{8}\eta^2 - \frac{5}{16}\eta^3 + \ldots\right) \tag{7.2.6}$$

If I use the definition of η in (7.2.4) in equation (7.2.6), and gather terms in powers of $\left(\frac{r'}{r}\right)$, I find that

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\begin{pmatrix} 1 + \left(\frac{r'}{r}\right)\cos\theta' + \left(\frac{r'}{r}\right)^2 \dfrac{(3\cos^2\theta' - 1)}{2} \\ + \left(\frac{r'}{r}\right)^2 \dfrac{(5\cos^3\theta' - 3\cos\theta')}{2} + \ldots \end{pmatrix} \tag{7.2.7}$$

If I look back at the Legendre polynomials I found in Problem Set 1, I have

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta') \tag{7.2.8}$$

Putting this into (7.2.1) and (7.2.2), I have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int \mathbf{J}(\mathbf{r}')\left[\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta')\right] d^3x'$$
$$= \frac{\mu_o}{4\pi}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int \mathbf{J}(\mathbf{r}')(r')^l\, P_l(\cos\theta')\, d^3x' \tag{7.2.9}$$

and

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int \rho(\mathbf{r}')(r')^l\, P_l(\cos\theta')\, d^3x' \tag{7.2.10}$$

We have in (7.2.9) and (7.2.10) what we wanted to achieve, an expansion in powers of $\left(\frac{d}{r}\right)$, so that each successive term is smaller than the preceeding by a factor of $\frac{d}{r} << 1$ for a distant observer.

7.3 The magnetic dipole and electric dipole terms

The first term in the sum in (7.2.9) is

$$\frac{\mu_o}{4\pi}\frac{1}{r}\int \mathbf{J}(\mathbf{r}')P_0(\cos\theta')\, d^3x' = \frac{\mu_o}{4\pi}\frac{1}{r}\int \mathbf{J}(\mathbf{r}')\, d^3x' \tag{7.3.1}$$

In Problem Set 3 you will show that this term vanishes in the time independent case. The second term in (7.2.9) is

$$\frac{\mu_o}{4\pi}\frac{1}{r^2}\int \mathbf{J}(\mathbf{r}')(r')P_1(\cos\theta')\, d^3x' = \frac{\mu_o}{4\pi}\frac{1}{r^2}\int \mathbf{J}(\mathbf{r}')(r'\cos\theta')\, d^3x' \tag{7.3.2}$$

If $\hat{\mathbf{n}} = \mathbf{r}/r$ is a unit vector which points from the origin of our coordinate system to the observer at $\mathbf{r}$, then $r'\cos\theta' = \hat{\mathbf{n}}\cdot\mathbf{r}'$, and

$$\frac{\mu_o}{4\pi}\frac{1}{r^2}\int \mathbf{J}(\mathbf{r}')(r'\cos\theta')\, d^3x' = \frac{\mu_o}{4\pi}\frac{1}{r^2}\int \mathbf{J}(\mathbf{r}')(\mathbf{r}'\cdot\hat{\mathbf{n}})\, d^3x' \tag{7.3.3}$$

In Problem Set 3, you will show that this can be written as

$$\frac{\mu_o}{4\pi}\frac{1}{r^2}\int \mathbf{J}(\mathbf{r}')(\mathbf{r}'\cdot\hat{\mathbf{n}})\, d^3x' = \frac{\mu_o}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{n}}}{r^2} \tag{7.3.4}$$

where I have defined the "magnetic dipole moment" **m** by

$$\mathbf{m} = \frac{1}{2}\int \mathbf{r}'\times\mathbf{J}(\mathbf{r}')d^3x' \tag{7.3.5}$$

Note that once you are given $\mathbf{J}(\mathbf{r}')$, this moment is a *fixed constant vector*! If I take the curl of (7.3.4), I find that the magnetic field associated with the first non-vanishing term in (7.2.9) is

$$\mathbf{B}_{dipole}(\mathbf{r}) = \frac{\mu_o}{4\pi} \frac{\left[3\hat{\mathbf{n}}(\mathbf{m}\cdot\hat{\mathbf{n}}) - \mathbf{m}\right]}{r^3} \tag{7.3.6}$$

Now let us turn to the expansion of the electric potential in (7.2.10). The first non-vanishing term in the sum is

$$\frac{1}{4\pi\varepsilon_o}\frac{1}{r}\int \rho(\mathbf{r}') P_0(\cos\theta')\, d^3x' = \frac{1}{4\pi\varepsilon_o}\frac{1}{r}\int \rho(\mathbf{r}')\, d^3x' = \frac{1}{4\pi\varepsilon_o}\frac{Q_o}{r} \tag{7.3.7}$$

where Q_o is the total charge of the static distribution. This is just the potential of a point charge, and the associated electric field is that of a point charge. The next term is

$$\begin{aligned}
\frac{1}{4\pi\varepsilon_o}\frac{1}{r^2}\int \rho(\mathbf{r}')(r') P_1(\cos\theta')\, d^3x' &= \frac{1}{4\pi\varepsilon_o}\frac{1}{r^2}\int \rho(\mathbf{r}')(r'\cos\theta')\, d^3x' \\
&= \frac{1}{4\pi\varepsilon_o}\frac{1}{r^2}\int \rho(\mathbf{r}')\, \mathbf{r}'\cdot\hat{\mathbf{n}}\, d^3x' = \frac{1}{4\pi\varepsilon_o}\frac{\hat{\mathbf{n}}\cdot\mathbf{p}}{r^2}
\end{aligned} \tag{7.3.8}$$

where we have defined the electric dipole moment $\mathbf{p}$ as

$$\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')d^3x' \tag{7.3.9}$$

If we take the gradient of (7.3.8) to find the electric field corresponding to this term, we have

$$\mathbf{E}_{dipole}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \frac{\left[3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}}) - \mathbf{p}\right]}{r^3} \tag{7.3.10}$$

7.4 Properties of a static dipole

Both the magnetic and electric dipole fields have the same form. I discuss the relevant properties of an electric dipole oriented along the z-axis. If $\mathbf{p}$ is along the z-axis, then

$$\mathbf{E}(\mathbf{r}) = \frac{2p\cos\theta}{4\pi\,\varepsilon_o r^3}\hat{\mathbf{r}} + \frac{p\sin\theta}{4\pi\,\varepsilon_o r^3}\hat{\boldsymbol{\theta}} \tag{7.4.1}$$

In spherical polar coordinates, our differential equations (1.5.1) for the field lines becomes

$$\frac{dr(s)}{ds} = \frac{E_r}{E} \qquad \frac{rd\theta(s)}{ds} = \frac{E_\theta}{E} \tag{7.4.2}$$

or we can write the differential equation $r(\theta)$ for a given field line by dividing the first equation in (7.4.2) by the second to obtain

$$\frac{1}{r}\frac{dr(\theta)}{d\theta} = \frac{E_r}{E_\theta} = \frac{2\cos\theta}{\sin\theta} \tag{7.4.3}$$

If we gather the terms involving *r* and the terms involving θ in (7.4.3), we have

$$\frac{dr}{r} = \frac{2\cos\theta}{\sin\theta} d\theta = 2\frac{d}{d\theta}\ln\left(\sin\theta\right) \tag{7.4.4}$$

which can be integrated to give the equation for a dipole field line.

$$r = L\sin^2\theta \tag{7.4.5}$$

The parameter L characterizing a given field line is the equatorial crossing distance of that field line. Figure 7-2 shows a family of such field lines with equatorial crossing distances equally space.

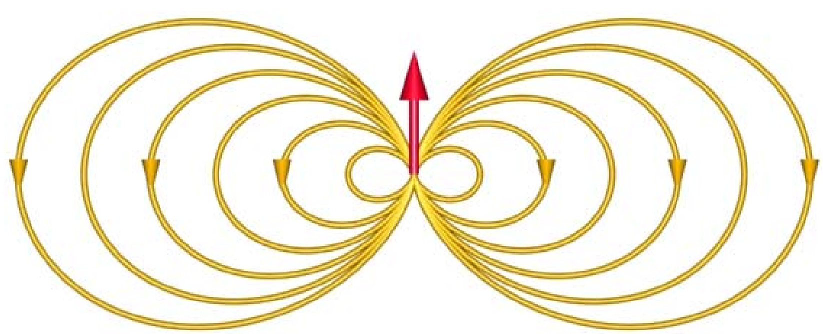


Figure 7-2: The field lines of a static dipole

7.5 The electric quadrupole term

If we go to the third term in the sum given in (7.2.10) for the electric potential, I can show that this term has the form

$$\frac{1}{4\pi\varepsilon_o}\frac{1}{2r^3}\sum_{j=1}^{3}\sum_{i=1}^{3}\hat{n}_i\hat{n}_j Q_{ij} = \frac{1}{4\pi\varepsilon_o}\frac{1}{2r^3}\hat{\mathbf{n}}\cdot\overleftrightarrow{\mathbf{Q}}\cdot\hat{\mathbf{n}} \tag{7.5.1}$$

where the electric quadrupole tensor $\overleftrightarrow{\mathbf{Q}}$ is given by

$$\left[\overleftrightarrow{\mathbf{Q}}\right]_{ij} = \int\left[3x_i'x_j' - \left(r'\right)^2\delta_{ij}\right]\rho(\mathbf{r})d^3x' \tag{7.5.2}$$

8 Sources varying slowly in time

8.1 Learning objectives

I now turn to the time case where our sources are localized and now are not static, but vary in time, but *slowly*, in the sense that any time dependence is slow compared to the speed of light travel time across the source, *d/c*. When I do this I will uncover the details what is arguably the most fundamental of all electromagnetic processes, the generation of electromagnetic waves. My major goal in this section is to show you how to systematically expand the solutions (6.2.13) and (6.2.15) in small parameters to get **E** and **B** for sources which vary slowly in time in the above sense. As I expect, I recover the static fields I have already seen above, but I also find much much more.

8.2 E and B fields far from localized sources varying slowly in time

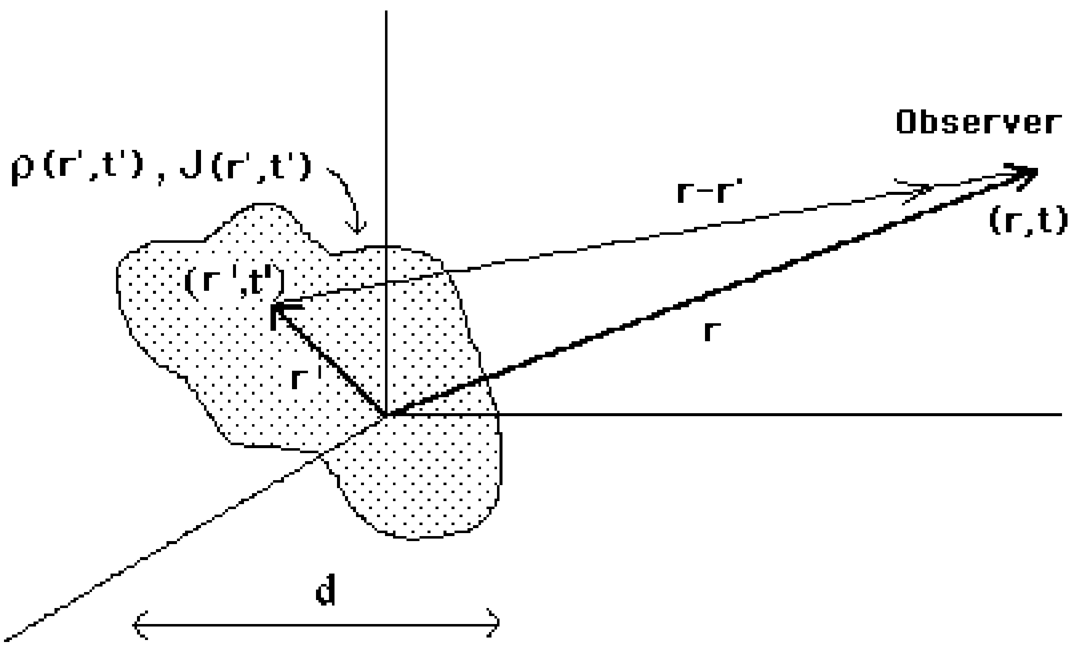


Figure 8-1: An observer far from a localized distribution of currents and charges

Suppose we have a localized distribution of charge and current near the origin, described by sources $\rho(\mathbf{r}',t')$ and $\mathbf{J}(\mathbf{r}',t')$ (Figure 8-1). The sources have a characteristic linear dimension *d*, such that the charge and current densities are zero for $r' > d$. Let the length of time for significant variation in the charge and current densities be *T*. We want to investigate the electromagnetic fields produced by these currents and charges as measured by an observer at $(\mathbf{r},t)$ located far from the sources, assuming that the sources vary slowly in time.

Again, as above, "far from" means that the observer's distance from the origin, *r*, is much greater than the maximum extent of the sources, *d*. "Slowly in time" means that the characteristic time *T* for significant variation is long compared to the speed of transit time across the sources, that is, $T >> d/c$. Under these two assumptions (*far from* in distance and *slowly* in time), we can develop straightforward expansions for $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ to various orders in the small quantities *d/r* and *d/cT*. Note that we have made no assumption as of yet about how the distance *r* compares to *cT*, and in fact we will be interested below in **three very different cases**: $r << cT$, $r \approx cT$, and $r >> cT$, the so-called near, intermediate, and far zones (also called the quasi-static, intermediate, and radiation zones). The solutions to our equations look quite different in these three regimes.

To obtain the electromagnetic fields at $(\mathbf{r},t)$, we first calculate the electromagnetic potentials. We have the exact solutions for $\phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$ at any observation point $(\mathbf{r},t)$

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\,\varepsilon_o}\int \rho(\mathbf{r}',t'_{ret})\frac{d^3x'}{|\mathbf{r}-\mathbf{r}'|}; \qquad \mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\int \mathbf{J}(\mathbf{r}',t'_{ret})\frac{d^3x'}{|\mathbf{r}-\mathbf{r}'|} \tag{8.2.1}$$

$$t'_{ret} = t - |\mathbf{r}-\mathbf{r}'|/c \tag{8.2.2}$$

For an observer far away ($r >> d$), I make the approximation that (see (7.2.3))

$$\begin{aligned} |\mathbf{r}-\mathbf{r}'| &= \sqrt{r^2 - 2\mathbf{r}\cdot\mathbf{r}' + r'^2} \approx r\sqrt{1-2\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}} \\ &\approx r\left(1-\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right) = r - \hat{\mathbf{n}}\cdot\mathbf{r}' + \ldots., \qquad \hat{\mathbf{n}} = \mathbf{r}/|\mathbf{r}| \end{aligned} \tag{8.2.3}$$

where I now go only to terms of first order in $\left(\frac{r'}{r}\right)$, as opposed to keeping all orders, as I did above in (7.2.7). I also have

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} + \frac{\hat{\mathbf{n}}\cdot\mathbf{r}'}{r^2} + \ldots. \tag{8.2.4}$$

Using (8.2.3) in (8.2.2) gives

$$t'_{ret} \cong t - r/c + \hat{\mathbf{n}}\cdot\mathbf{r}'/c \tag{8.2.5}$$

Expanding the exact solutions given in (8.2.1) is complicated for time varying sources because of the finite propagation time from field point to observation point. As we saw before, events which are recorded at the observation point at $(\mathbf{r},t)$ are due to time variations in the source at **r**' at a time $t'_{ret} \cong t - r/c + \hat{\mathbf{n}}\cdot\mathbf{r}'/c$, where t'_{ret} depends on **r**'. It is worth emphasizing this point.

Because of the finite propagation time from source to observer, at time *t* we are sampling what happened in parts of the source more distant from the observer at an earlier source time than parts of the source closer to the observer!

Thus to find $\phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$, we have to add up source variations which occur at different points in the source at *different* retarded times. Our basic assumption here is that the sources vary slowly enough in time that we can expand $\rho(\mathbf{r}',t'_{ret})$ as follows

$$\begin{aligned} \rho(\mathbf{r}',t'_{ret}) &= \rho(\mathbf{r}',t - r/c + \hat{\mathbf{n}}\cdot\mathbf{r}'/c + ...) \\ \rho(\mathbf{r}',t'_{ret}) &\cong \rho(\mathbf{r}',t - r/c) + \frac{\hat{\mathbf{n}}\cdot\mathbf{r}'}{c}\frac{\partial}{\partial t'}\rho(\mathbf{r}',t - r/c) + ... \end{aligned} \tag{8.2.6}$$

This is just a Taylor series expansion about $t - r/c$, where r/c is the propagation time from the center of the source region to the observer. Such an expansion will be good as long as the first term on the right hand side of equation (8.2.6) is much larger than the second, i.e.,

$$\frac{\hat{\mathbf{n}}\cdot\mathbf{r}'}{c}\frac{1}{\rho}\frac{\partial\rho}{\partial t'} << 1 \tag{8.2.7}$$

But $\frac{1}{\rho}\frac{\partial\rho}{\partial t'}$ is *1/T*, where *T* is a characteristic time for significant variation in ρ. Since |**r**'| is less than *d*, the maximum extent of our localized source region, for our approximation in (8.2.6) to be valid, we need

$$\frac{d}{c} << T \qquad \textbf{\textit{Electric Dipole Approximation}} \tag{8.2.8}$$

This approximation is known as the *electric dipole approximation*. For expansion (8.2.6) to hold, we must require (8.2.8) to hold, which says that the time required for light to propagate across our source must be small compared to characteristic times for significant variation in the source. Thus if we assume that

$$r >> d \quad \text{and} \quad \frac{d}{c} << T \tag{8.2.9}$$

then using (8.2.4) and (8.2.6), we can expand our exact solutions (8.2.1) to first order in the small quantities *d/r* and *d/cT* as

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\,\varepsilon_o}\int d^3x' \left[\rho(\mathbf{r}',t') + \frac{\hat{\mathbf{n}}\cdot\mathbf{r}'}{c}\frac{\partial}{\partial t'}\rho(\mathbf{r}',t') + ...\right]\left[\frac{1}{r}\left(1 + \frac{\hat{\mathbf{n}}\cdot\mathbf{r}'}{r} + ...\right)\right] \tag{8.2.10}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi\, r}\int d^3x' \mathbf{J}(\mathbf{r}',t') + \frac{\mu_o}{4\pi\, r}\int\left(\hat{\mathbf{n}}\cdot\mathbf{r}'\right)\left[\frac{\mathbf{J}(\mathbf{r}',t')}{r} + \frac{1}{c}\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}',t')\right]d^3x' + ... \tag{8.2.11}$$

where

$$t' = t - r/c \tag{8.2.12}$$

We could keep higher order terms in (8.2.10) and (8.2.11), but we will find that what information we need is contained in the orders we've kept. These terms will allow us to look at the properties of electric dipole radiation, magnetic dipole radiation, and electric quadrupole radiation. We define the electric monopole, electric dipole moment, magnetic dipole moment, and electric quadrupole moment of our sources by the equations

$$q(t') = \int \rho(\mathbf{r}',t')d^3x' = \mathrm{Q_o} \tag{8.2.13}$$

$$\mathbf{p}(t') = \int \mathbf{r}'\rho(\mathbf{r}',t')d^3x' \qquad \mathbf{m}(t') = \frac{1}{2}\int \mathbf{r}'\times\mathbf{J}(\mathbf{r}',t')d^3x' \tag{8.2.14}$$

$$\left[\vec{\mathbf{Q}}(t')\right]_{ij} = \int\left[3x_i'x_j' - \left(r'\right)^2\delta_{ij}\right]\rho(\mathbf{r},t')d^3x' \tag{8.2.15}$$

These are exactly analogous to our definitions in the static case, except now these vectors quantities vary in time. Note that since charge is conserved and our sources are localized (in particular there is no current flow to infinity), the total charge $q(t')$ is constant in time and equal to Q_O.

8.3 Electric dipole radiation

We gather all terms in **A** and ϕ which are proportional to Q_{O}, **p** or $d\mathbf{p}/dt$. From (8.2.10) we have

$$\phi(\mathbf{r},t) \cong \frac{1}{4\pi\,\varepsilon_o}\frac{Q_o}{r} + \frac{1}{4\pi\,\varepsilon_o}\frac{1}{r^2}\hat{\mathbf{n}}\cdot\int \mathbf{r}'\rho(\mathbf{r}',t')\,d^3x' + \frac{1}{4\pi\,\varepsilon_o}\frac{1}{c\,r}\hat{\mathbf{n}}\cdot\int \mathbf{r}'\frac{\partial}{\partial t'}\rho(\mathbf{r}',t')\,d^3x' \tag{8.3.1}$$

Using the definition of $\mathbf{p}(t)$ in (8.2.14), we have

$$\phi(\mathbf{r},t) \cong \frac{1}{4\pi\,\varepsilon_o}\frac{Q_o}{r} + \frac{1}{4\pi\,\varepsilon_o}\frac{1}{r^2}\hat{\mathbf{n}}\cdot\mathbf{p}(t') + \frac{1}{4\pi\,\varepsilon_o}\frac{1}{c\,r}\hat{\mathbf{n}}\cdot\frac{d\mathbf{p}(t')}{dt} \tag{8.3.2}$$
$$\text{where } t' = t - r/c$$

What about **A**? In Problem Set 3 you showed that

$$\int \mathbf{J}(\mathbf{r}',t')\,d^3x' = -\int \mathbf{r}'\left[\nabla'\cdot\mathbf{J}(\mathbf{r}',t')\right]d^3x' \tag{8.3.3}$$

and thus the first term in (8.2.11) becomes

$$\mathbf{A}(\mathbf{r},t) \cong -\frac{\mu_o}{4\pi}\frac{1}{r}\int d\tau'\;\mathbf{r}'(\nabla'\cdot\mathbf{J}(\mathbf{r}',t')) \tag{8.3.4}$$

Remember that charge conservation in differential form is

$$\nabla'\cdot\mathbf{J}(\mathbf{r}',t') + \frac{\partial}{\partial t'}\rho(\mathbf{r}',t') = 0 \tag{8.3.5}$$

so that the term in **A** proportional to $d\mathbf{p}/dt'$ is

$$\mathbf{A}(\mathbf{r},t) \cong +\frac{\mu_o}{4\pi}\frac{1}{r}\frac{d\mathbf{p}(t')}{dt'} \qquad t' = t - r/c \tag{8.3.6}$$

The other terms in (8.2.11) are proportional to electric quadrupole or magnetic dipole moments, as discussed later, so we for the moment ignore those terms and concentrate only on the terms that involve $\mathbf{p}(t')$ and its time derivatives. We define $\dot{\mathbf{p}}(t')$ to be $d\mathbf{p}(t')/dt'$, $\ddot{\mathbf{p}}(t')$ to be $d^2\mathbf{p}(t')/dt'^2$, and so on. Since $\mathbf{B} = \nabla\,\mathbf{x}\,\mathbf{A}$ we have using (6.1.1) that

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\left[\frac{\mu_o}{4\pi}\frac{1}{r}\dot{\mathbf{p}}(t')\right] = \frac{\mu_o}{4\pi}\frac{1}{r}\nabla\times\dot{\mathbf{p}}(t') - \frac{\mu_o}{4\pi}\dot{\mathbf{p}}(t')\times\nabla\left[\frac{1}{r}\right] \tag{8.3.7}$$

Obviously, $\nabla\frac{1}{r} = -\frac{1}{r^2}\hat{\mathbf{n}}$, and for functions of *t-r/c*,

$$\nabla g(t - r/c) = -\frac{\hat{\mathbf{n}}}{c}\;\frac{dg(t')}{dt'} \tag{8.3.8}$$

so that

$$\nabla\times\dot{\mathbf{p}}(t') = -\frac{\hat{\mathbf{n}}}{c}\times\ddot{\mathbf{p}}(t') \tag{8.3.9}$$

and thus

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\left[\frac{\dot{\mathbf{p}}}{r^2} + \frac{\ddot{\mathbf{p}}}{c\,r}\right]\times\hat{\mathbf{n}} \quad \text{evaluated at } t' = t - r/c \tag{8.3.10}$$

What about **E**? Well $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$ so from (8.3.2) and (8.3.6)

$$\mathbf{E}(\mathbf{r},t) = -\frac{1}{4\pi\,\varepsilon_o}\nabla\left[\frac{Q_o}{r} + \frac{1}{r^2}\hat{\mathbf{n}}\cdot\mathbf{p}(t') + \frac{1}{c\,r}\hat{\mathbf{n}}\cdot\dot{\mathbf{p}}(t')\right] - \frac{\mu_o}{4\pi}\frac{1}{r}\ddot{\mathbf{p}}(t') \tag{8.3.11}$$

We can derive expressions for $\nabla\left[\frac{\hat{\mathbf{n}}\cdot\mathbf{p}(t')}{r^2}\right]$ and $\nabla\left[\frac{\hat{\mathbf{n}}\cdot\dot{\mathbf{p}}(t')}{r}\right]$ as follows

$$\nabla\frac{\hat{\mathbf{n}}\cdot\mathbf{p}(t')}{r^2} = \frac{1}{r^3}\left[\mathbf{p} - 3(\mathbf{p}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}\right] - \frac{(\hat{\mathbf{n}}\cdot\dot{\mathbf{p}})}{cr^2}\hat{\mathbf{n}} \tag{8.3.12}$$

$$\nabla\frac{\hat{\mathbf{n}}\cdot\dot{\mathbf{p}}(t')}{r} = \frac{1}{r^2}\left[\dot{\mathbf{p}} - 2(\dot{\mathbf{p}}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}\right] - \frac{(\hat{\mathbf{n}}\cdot\ddot{\mathbf{p}})}{cr}\hat{\mathbf{n}} \tag{8.3.13}$$

Inserting these into equation (8.3.11) gives us

$$\mathbf{E}(\mathbf{r},t) = \underbrace{\frac{1}{4\pi\,\varepsilon_o}\frac{Q_o}{r^2}\hat{\mathbf{n}} + \frac{1}{r^3}\frac{\left[3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}}) - \mathbf{p}\right]}{4\pi\,\varepsilon_o}}_{\text{quasi-static}} + \underbrace{\frac{1}{c\,r^2}\frac{\left[3\hat{\mathbf{n}}(\dot{\mathbf{p}}\cdot\hat{\mathbf{n}}) - \dot{\mathbf{p}}\right]}{4\pi\,\varepsilon_o}}_{\text{induction}} + \underbrace{\frac{1}{rc^2}\frac{(\ddot{\mathbf{p}}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{4\pi\,\varepsilon_o}}_{\text{radiation}} \tag{8.3.14}$$

evaluated at $t' = t - r/c$. In the limit of no time variation, **E** is just a static monopole field and dipole field, and **B** is zero. If **p** is varying slowly in time, the static dipole field becomes quasi-static (slowly varying in time, but with the same mathematical form). In addition, there are terms proportional to $\dot{\mathbf{p}}/cr^2$ (the *induction* terms) and terms proportional to $\ddot{\mathbf{p}}/c^2r$ (the *radiation* terms).

9 Examples of electric dipole radiation

9.1 Learning objectives

We look at the solutions for electric dipole radiation for specific cases. We consider two different kinds of time behavior for the electric dipole moment, and define the near, intermediate, and far zones.

9.2 Dipole moment vector p varying in magnitude but not direction

We want to look at some specific cases so that we can understand what equations (8.3.10) and (8.3.14) mean. To do this we first assume that we have a dipole moment vector **p** that is *always in the same direction* but with a time varying magnitude $p(t)$, that is

$$\mathbf{p}(t) = \hat{\mathbf{z}}\, p(t) \tag{9.2.1}$$

If we insert (9.2.1) into the expressions for **E** and **B** above, we find that

$$\mathbf{E}(\mathbf{r},t) = \frac{p}{r^3}\frac{\left[3\hat{\mathbf{n}}(\hat{\mathbf{z}}\cdot\hat{\mathbf{n}})-\hat{\mathbf{z}}\right]}{4\pi\,\varepsilon_o} + \frac{\dot{p}}{c\,r^2}\frac{\left[3\hat{\mathbf{n}}(\hat{\mathbf{z}}\cdot\hat{\mathbf{n}})-\hat{\mathbf{z}}\right]}{4\pi\,\varepsilon_o} + \frac{\ddot{p}}{rc^2}\frac{(\hat{\mathbf{z}}\,\mathbf{x}\,\hat{\mathbf{n}})\,\mathbf{x}\,\hat{\mathbf{n}}}{4\pi\,\varepsilon_o} \tag{9.2.2}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\left[\frac{\dot{p}}{r^2}+\frac{\ddot{p}}{c\,r}\right]\hat{\mathbf{z}}\,\mathbf{x}\,\hat{\mathbf{n}} \tag{9.2.3}$$

Remember that $\hat{\mathbf{n}}$ is just $\hat{\mathbf{r}}$, pointing from the source located at the near the origin to the observer far from the origin. Let us specify the direction of $\hat{\mathbf{n}} = \mathbf{r}/r$ in spherical polar coordinates by the polar angle θ and the azimuth angle φ, as shown in Figure 9-1.

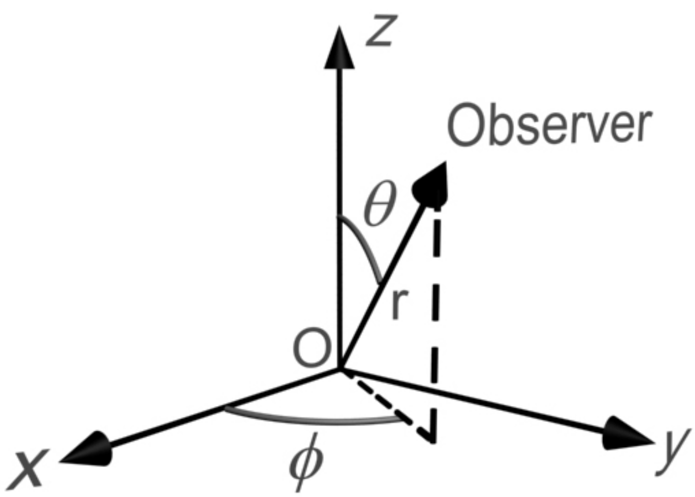


Figure 9-1: The vector to the observer

Thus we have

$$\hat{\mathbf{z}}\cdot\hat{\mathbf{n}} = \cos\theta \qquad \hat{\mathbf{z}}\,\mathbf{x}\,\hat{\mathbf{n}} = \hat{\boldsymbol{\varphi}}\sin\theta \qquad (\hat{\mathbf{z}}\,\mathbf{x}\,\hat{\mathbf{n}})\,\mathbf{x}\,\hat{\mathbf{n}} = \hat{\boldsymbol{\theta}}\sin\theta \tag{9.2.4}$$

So (9.2.2) and (9.2.3) become

$$\mathbf{E}(\mathbf{r},t) = \frac{2\cos\theta}{4\pi\,\varepsilon_o}\hat{\mathbf{r}}\left(\frac{p}{r^3}+\frac{\dot{p}}{cr^2}\right) + \frac{\sin\theta}{4\pi\,\varepsilon_o}\hat{\boldsymbol{\theta}}\left(\frac{p}{r^3}+\frac{\dot{p}}{cr^2}+\frac{\ddot{p}}{c^2 r}\right) \tag{9.2.5}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_o \sin\theta}{4\pi}\left(\frac{\dot{p}}{r^2} + \frac{\ddot{p}}{c\,r}\right)\hat{\boldsymbol{\varphi}} \tag{9.2.6}$$

9.3 The near, intermediate, and far zones

If I ask which terms are the dominate terms in equation (9.2.5), I must ask about the magnitude of r compared to cT, where T is a characteristic time scale for variations in the sources. To see this, we re-write (9.2.5) as follows in order of magnitude

$$E \approx \frac{1}{4\pi\,\varepsilon_o}\left(\frac{p}{r^3} + \frac{\dot{p}}{cr^2} + \frac{\ddot{p}}{c^2 r}\right) \approx \frac{1}{4\pi\,\varepsilon_o}\frac{p}{r^3}\left(1 + \frac{r\,\dot{p}}{cp} + \frac{r^2\ddot{p}}{c^2 p}\right) \tag{9.3.1}$$

I now set $\dot{p} \approx p/T$ and $\ddot{p} \approx p/T^2$, so that (9.3.1) becomes

$$E \approx \frac{1}{4\pi\,\varepsilon_o}\frac{p}{r^3}\left(1 + \frac{r}{cT} + \left[\frac{r}{cT}\right]^2\right) \tag{9.3.2}$$

There are three different possibilities:

(I) $r << cT$ **STATIC OR NEAR ZONE.** The dominant term in (9.3.2) is the first one, which represents the quasi-static dipole electric fields, varying in time, but in essence just a dipole field with E falling off as $1/r^3$.

(II) $r >> cT$ **RADIATION OR FAR ZONE**. The dominant term in (9.3.2) the last one, and it falls off as $1/r$, and these are the radiation fields. As we will see below, these terms carry energy to infinity, that is energy that is lost irreversibly and cannot be recovered.

(III) $r \approx cT$ **INDUCTION OR INTERMEDIATE ZONE**, all of the terms in (9.3.2) are of comparable importance.

9.4 Examples of electric dipole radiation in the near, intermediate, and far zones

In the sections below, I look at a examples of the time behavior of *p(t)*, each chosen to illustrate various features of (9.2.5) and (9.2.6). The examples can all be grouped into time behaviors of two general types. The first type of behavior is a sinusoidal time dependence, that is

$$p(t) = p_0 + p_1 \cos \omega t \tag{9.4.1}$$

The second type of time behavior is an electric dipole which has been constant at one value of the dipole moment, p_0, up to time $t = 0$, and then smoothly transitions to another value of the dipole moment, p_1, over a time T. In this type of behavior, I take the time dependence of $p(t)$ to be

$$p(t) = \begin{cases} p_o & \text{for } t < 0 \\ p_0 + p_1\left[6\left(\dfrac{t}{T}\right)^5 - 15\left(\dfrac{t}{T}\right)^4 + 10\left(\dfrac{t}{T}\right)^3\right] & \text{for } 0 < t < T \\ p_0 + p_1 & \text{for } t > T \end{cases} \tag{9.4.2}$$

The time dependence of this function and its first and second time derivatives is shown in Figure 9-2. In the case plotted the dipole moment increases by 25% from its initial value. For clarity, so that they are more easily seen, I have multiplied the first and second derivatives of the dipole moment as a function of time by a factor of 10 in Figure 9-2.

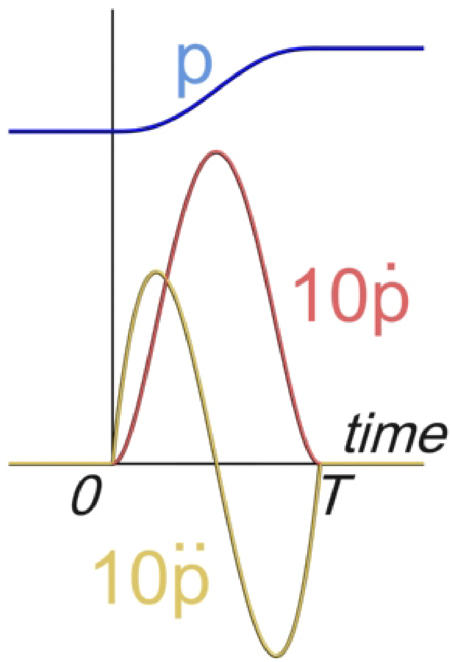


Figure 9-2: The time dependence of a dipole changing in time *T*

In each of the examples I show below, I will also show movies in class that animate the radiation time sequence at the three different scales for that example. The field lines and texture patterns move in these movies, with a velocity at each point in time given by

$$\mathbf{V}_{\text{field line}}(\mathbf{r},t) = c^2 \frac{\mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)}{E^2(\mathbf{r},t)} \tag{9.4.3}$$

I will justify the use of this velocity for a moving electric field line later on in the course. For the moment, the only thing you need to know is that this velocity is in the direction of the local value of $\mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)$ at every point in space and time, and that this vector represents the direction of electromagnetic energy flow, as we discuss at length soon. So motion in these movies represents direction of electromagnetic energy flow, and you should view these movies in this light.

9.4.1 Dipole moment varying sinusoidally with total reversal, in the near zone

In this case we set p_0 to zero in equation (9.4.1). Figure 9-3 shows the field line configuration at a time near the maximum value of the dipole moment, in the near zone. Successive figures show this pattern at different phases in the dipole cycle.

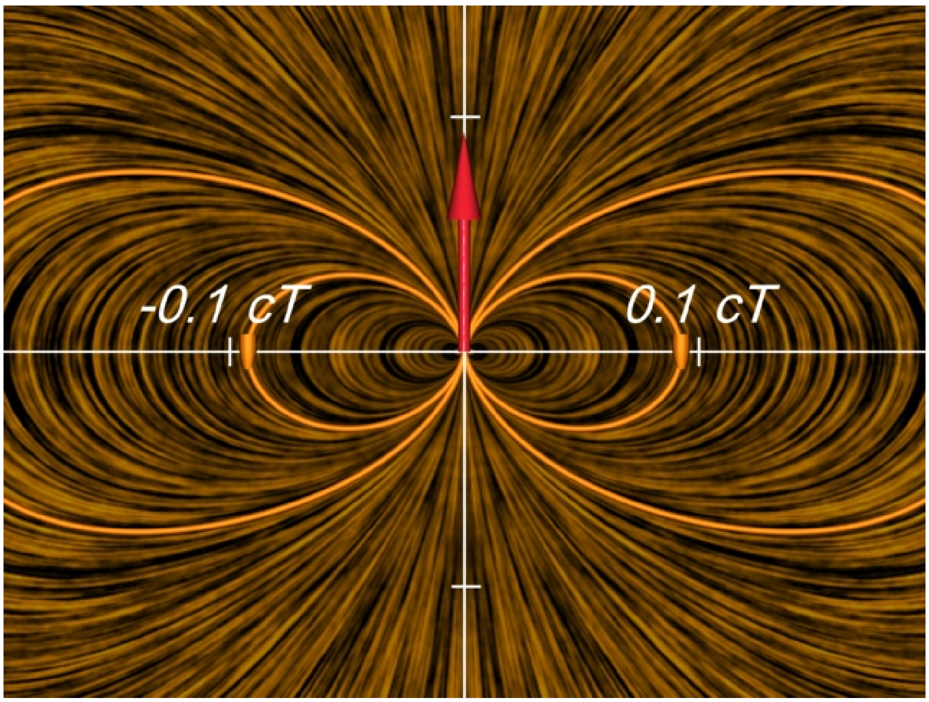


Figure 9-3: An oscillating dipole at the maximum of the dipole moment, in the near zone.

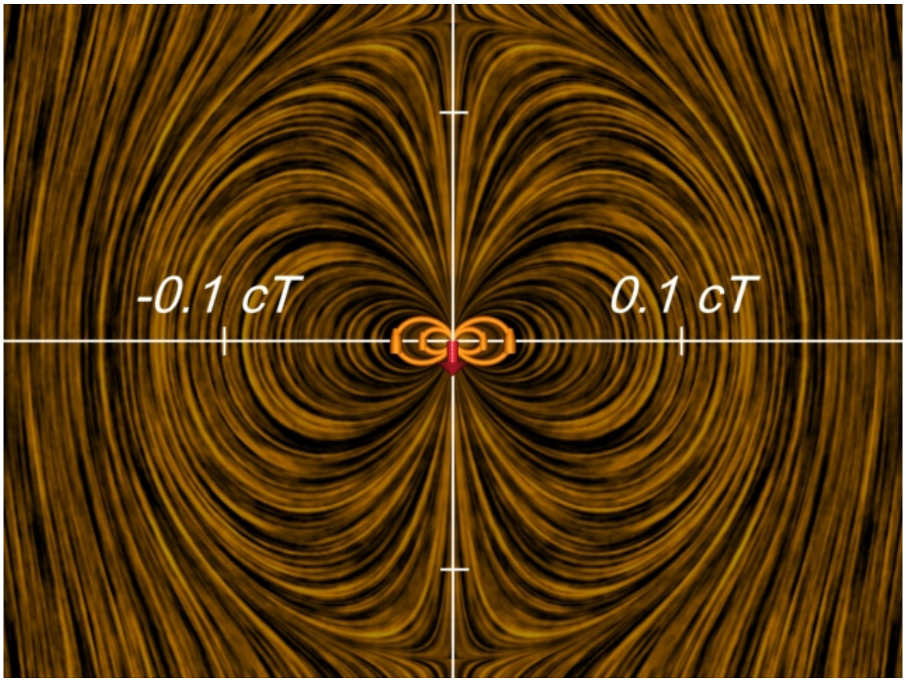


Figure 9-4: An oscillating dipole just after of the dipole moment has reversed, in the near zone.

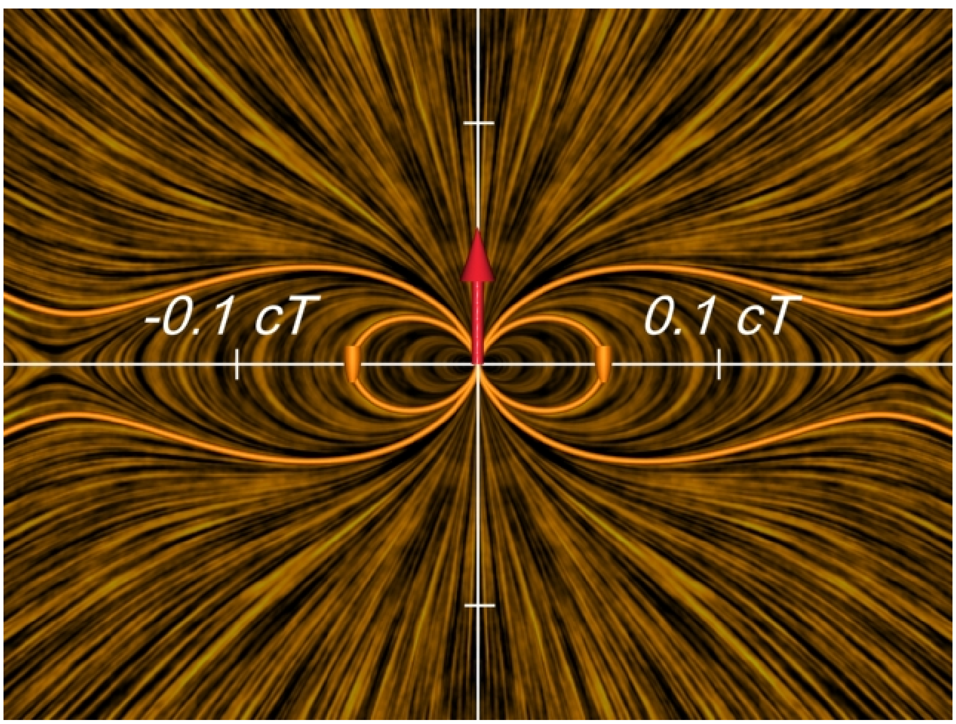


Figure 9-5: An oscillating dipole as the dipole moment approaches its maximum, in the near zone.

9.4.2 Dipole moment with total reversal, in the intermediate zone

Now we look at exactly the same thing as above, but we include distance further from the origin. The pattern changes qualitatively. The figures below show various phases.

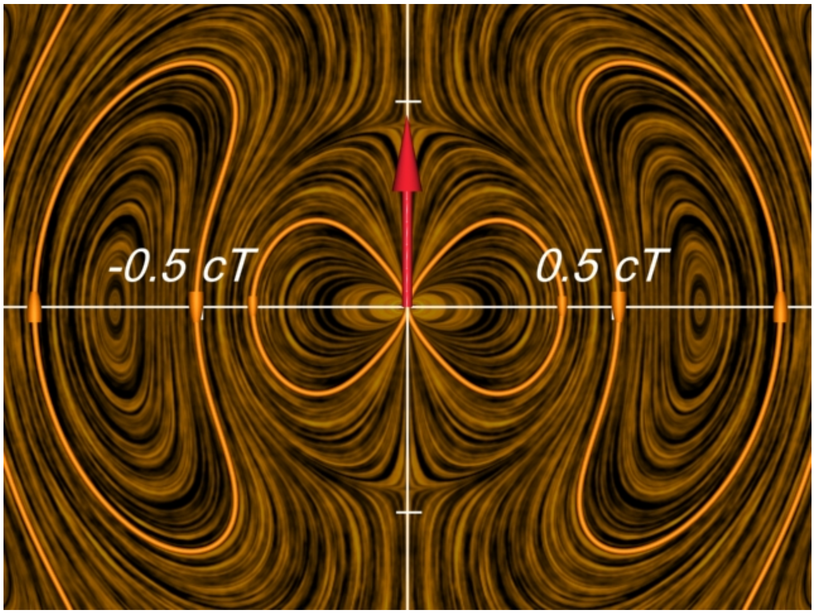


Figure 9-6: An oscillating dipole at the maximum of the dipole moment, in the intermediate zone

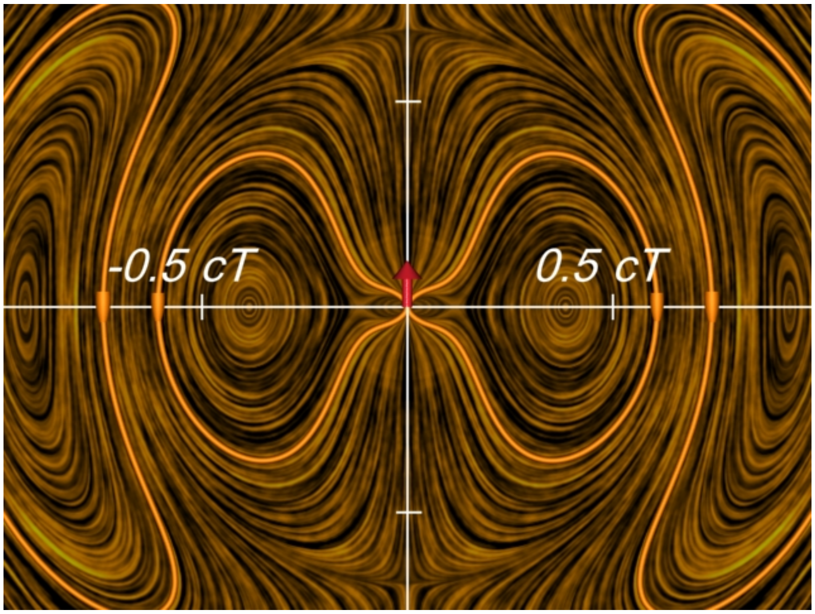


Figure 9-7: An oscillating dipole just before the dipole moment reverses, in the intermediate zone

9.4.3 Dipole moment varying sinusoidally with total reversal, in the far zone

Now we show the pattern even further out, and we see the characteristic electric dipole radiation pattern.

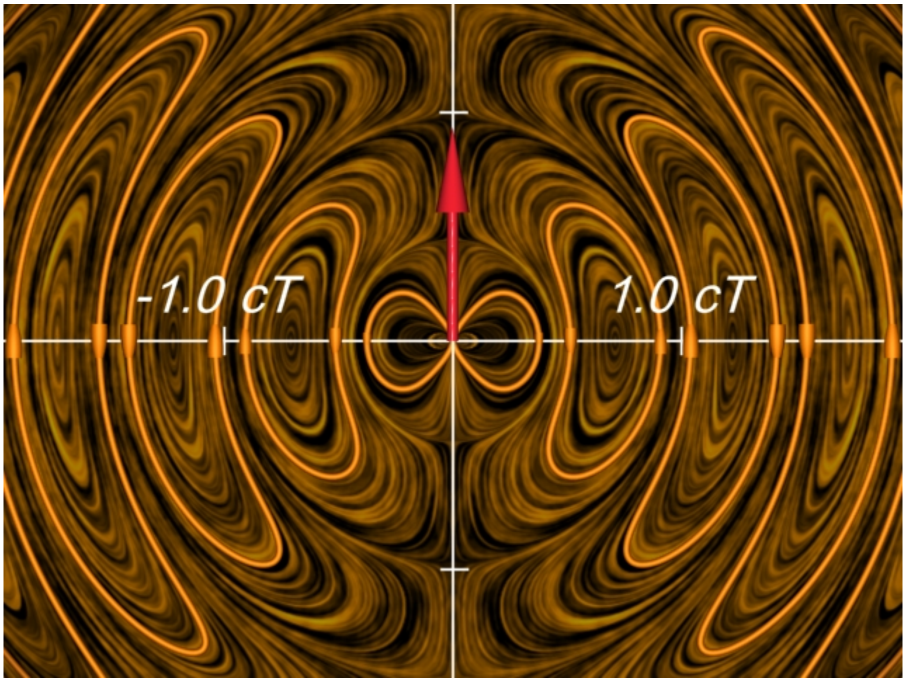


Figure 9-8: An oscillating dipole at the maximum of the dipole moment, in the far zone

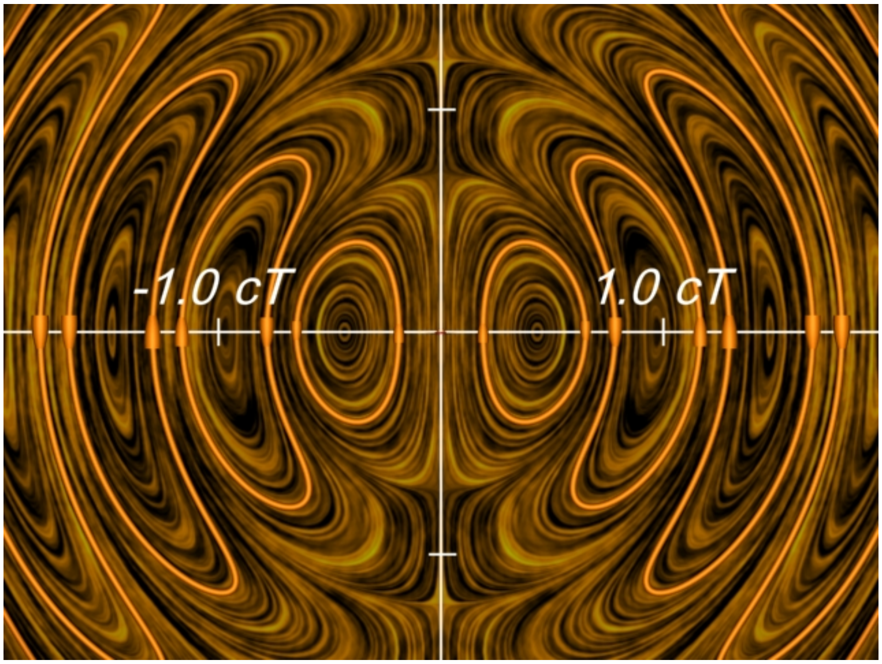


Figure 9-9: An oscillating dipole at the zero of the dipole moment, in the far zone

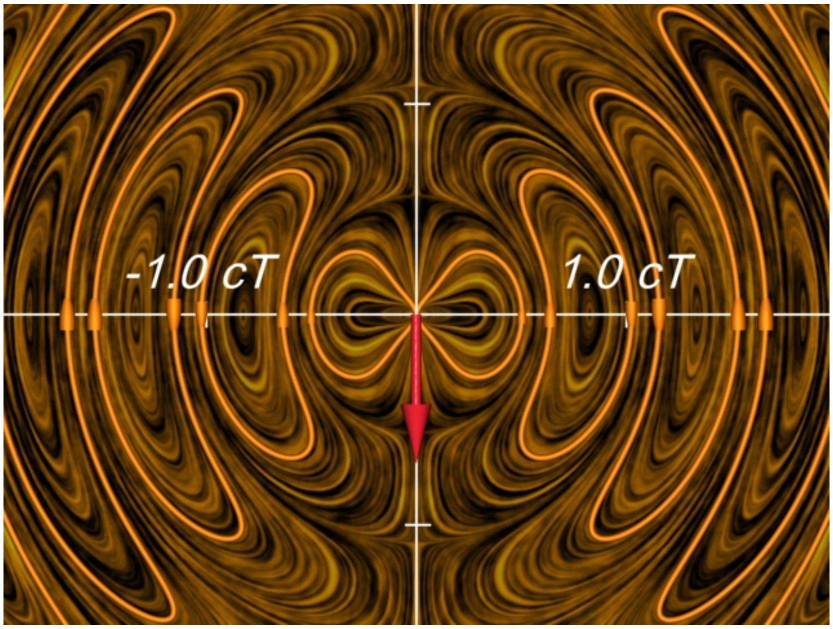


Figure 9-10: An oscillating dipole as the dipole moment approaches its downward maximum, in the far zone

9.4.4 Dipole moment increasing 50% over time *T*, in the near zone

Now I look at the behavior described by equation (9.4.2), in the case where the dipole magnitude increases by 50% over time *T* and then remains constant, first in the near zone.

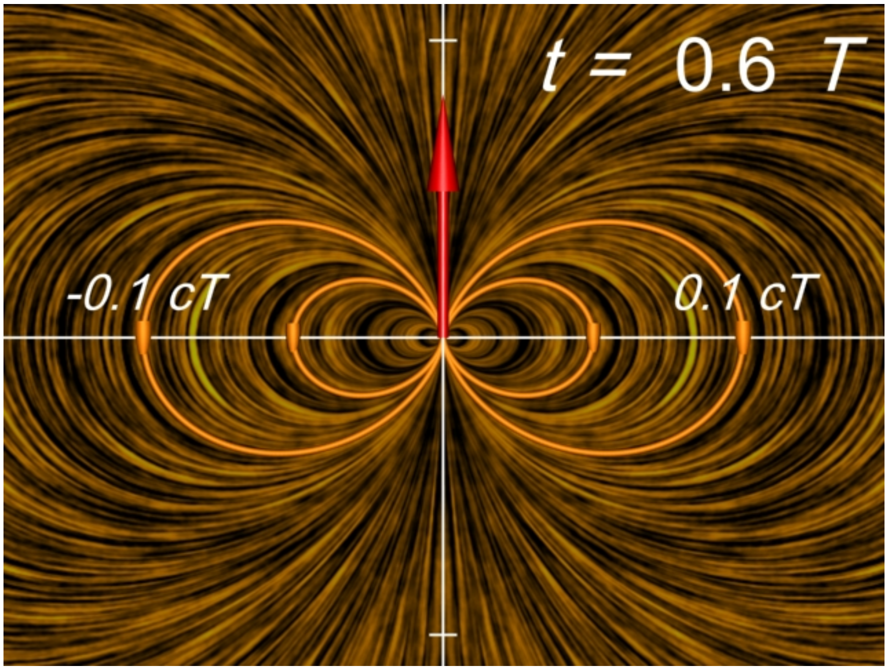


Figure 9-11: Dipole increasing over time *T* in the near zone

9.4.5 Dipole moment increasing over time *T*, in the intermediate zone

Now we look at the pattern in the intermediate zone.

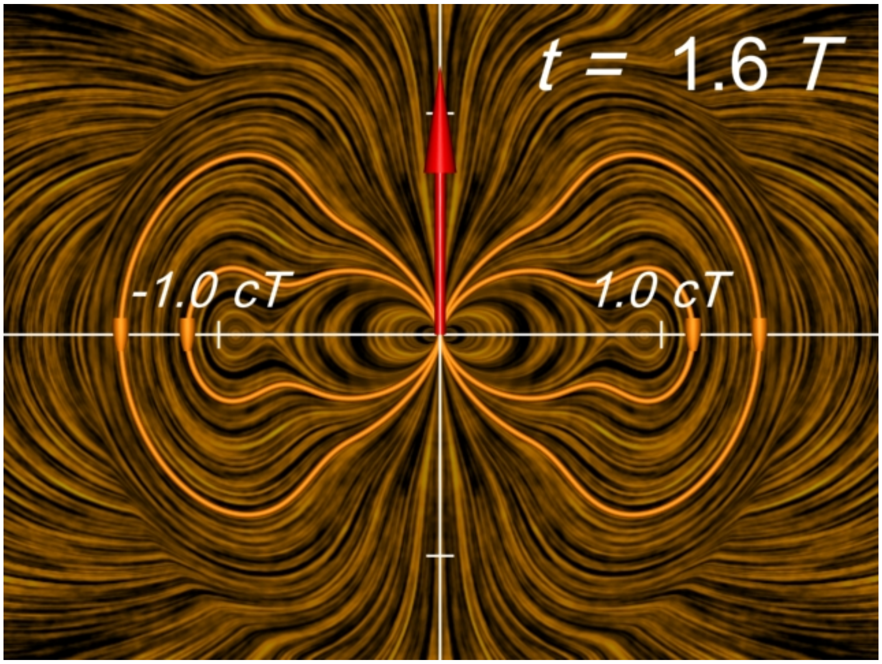


Figure 9-12: Dipole increasing over time *T* in the intermediate zone

9.4.6 Dipole moment decreasing by 33% over time *T*, in the near zone

Now we repeat the same sequence as above, except for the case that the dipole is decreasing in magnitude by 33% over time *T*.

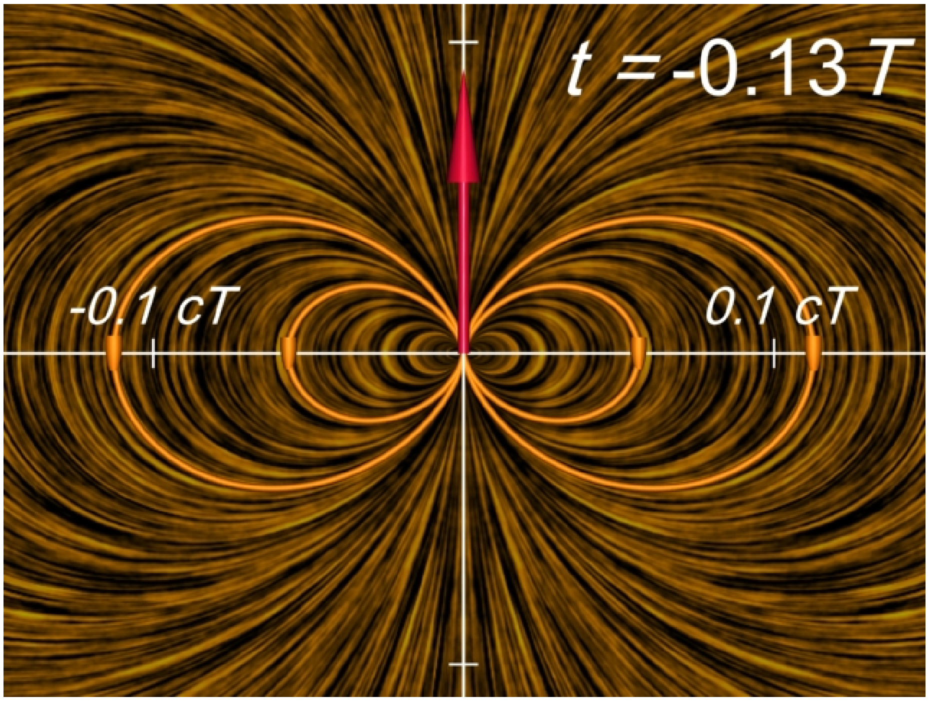


Figure 9-13: Dipole decreasing over time *T* in the near zone

9.4.7 Dipole moment decreasing over time *T*, in the intermediate zone

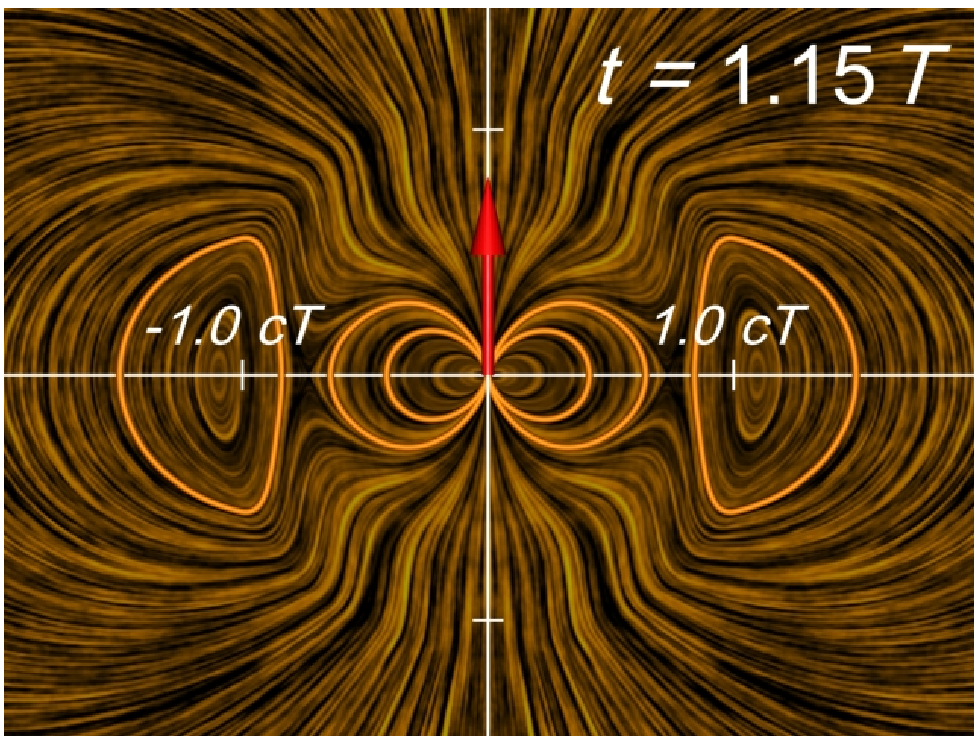


Figure 9-14: Dipole decreasing over time *T* in the intermediate zone.

9.5 Conservation of energy

Let us consider how the conservation of energy applies to our solutions above. First of all, our solutions above are only good for $r > d$, that is we are outside the source region containing charges and currents. Therefore in the region in which they apply, there are no sources or sinks of electromagnetic energy. We simply have energy flowing to fill up space with the local energy density of the electromagnetic field.

Second, we can calculate an expression for the flux of electromagnetic energy, that is rate at which total energy flows across a sphere of radius R_o per second. To get some feel for this, I consider the second type of time dependence for the dipole moment I considered above, when the dipole moment starts out at one value of the dipole moment, say p_1, and changes over a time T to another value of the dipole moment, say p_2. If we want to calculate the total amount of energy that has moved across a sphere of radius R_o in this process, we simply calculate the area integral over the surface of the sphere as follows:.

$$\text{Energy through R}_o = \int_{-\infty}^{\infty} dt \int \frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\cdot\hat{\mathbf{n}}\, da \tag{9.5.1}$$

If I refer to equations (9.2.5) and (9.2.6), I see that

$$\begin{aligned}\frac{\mathbf{E}\times\mathbf{B}}{\mu_o} &= \frac{\left(E_r\hat{\mathbf{r}}+E_\theta\hat{\boldsymbol{\theta}}\right)\times B_\varphi\hat{\boldsymbol{\varphi}}}{\mu_o} = \frac{\left(E_\theta B_\varphi\hat{\mathbf{r}}-E_r B_\varphi\hat{\boldsymbol{\theta}}\right)}{\mu_o} \\ &= \left[\hat{\mathbf{r}}\frac{\sin^2\theta}{(4\pi)^2\,\varepsilon_o}\left(\frac{\dot{p}}{r^2}+\frac{\ddot{p}}{c\,r}\right)\left(\frac{p}{r^3}+\frac{\dot{p}}{cr^2}+\frac{\ddot{p}}{c^2r}\right)-\hat{\boldsymbol{\theta}}\frac{2\sin\theta\cos\theta}{(4\pi)^2\,\varepsilon_o}\left(\frac{p}{r^3}+\frac{\dot{p}}{cr^2}\right)\left(\frac{\dot{p}}{r^2}+\frac{\ddot{p}}{c\,r}\right)\right]\end{aligned} \tag{9.5.2}$$

Since we are considering a spherical surface of radius R_o, $\hat{\mathbf{n}}=\hat{\mathbf{r}}$, and we have

$$\text{Energy through R}_o = \int_{-\infty}^{\infty} dt \int \frac{\mathbf{E}\times\mathbf{B}}{\mu_o}\cdot\hat{\mathbf{r}}\; r^2 d\Omega = \int_{-\infty}^{\infty} dt \int \frac{(\mathbf{E}\times\mathbf{B})_r}{\mu_o} r^2 d\Omega \tag{9.5.3}$$

With a little work (which you will do on a problem on Problem Set 4) this can be shown to be

$$\text{Energy through R}_o = \frac{1}{12}\frac{\left(p_2^2-p_1^2\right)}{4\pi\,\varepsilon_o R_o^3} + \int_{-\infty}^{\infty}\frac{\ddot{p}^2}{6\pi\,\varepsilon_o c^3}dt \tag{9.5.4}$$

Note that the second term on the right side of this equation is independent of R_o. This term represents the energy radiated away to infinity, and this is an irreversible process. We can easily see that the instantaneous rate at which energy is radiated away is

$$\text{Power in radiation (joules per sec)} = \frac{\ddot{p}^2}{6\pi\,\varepsilon_o c^3} \tag{9.5.5}$$

and this is known as Larmor's formula. What does the first term in (9.5.4) represent? Let's calculate the total amount of energy in an electrostatic dipole outside of a sphere of radius R_o. Using (7.4.1), you will show the following. The electrostatic energy of an electric dipole in the volume external to a sphere of radius R_o is given by

$$\text{Electrostatic energy of dipole outside } \mathrm{R}_o = \frac{1}{12}\frac{p^2}{4\pi\,\varepsilon_o R_o^3} \tag{9.5.6}$$

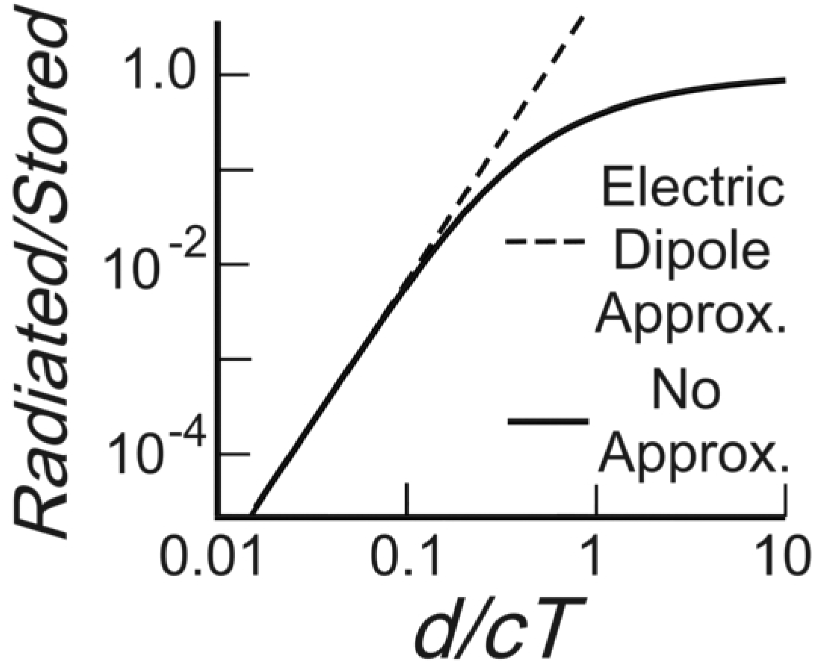
and thus we see that the first term in (9.5.4) represents the electrostatic energy needed to change the field from a dipole moment of p_1 to a dipole moment of p_2. Note that this can be positive or negative, depending on the relative sizes of these dipole moments, so that energy flows either inward or outward, depending on whether these quasi-electrostatic fields are being destroyed are created.

We can also compare the energy radiated away to the energy stored in or taken out of the electrostatic field, by looking at the ratio second to the first term in (9.5.4). If we take R_o to be the size of our source region, *d*, then we find that

$$\frac{\text{Radiated energy}}{\text{Stored energy}} \approx \frac{\ddot{p}^2 T}{6\pi\,\varepsilon_o c^3} \Big/ \frac{1}{12}\frac{\left(p_2^2 - p_1^2\right)}{4\pi\,\varepsilon_o d^3} \approx \frac{8d^3\left(p^2/T^4\right)T}{c^3\left(p_2^2 - p_1^2\right)} \approx \left[\frac{d}{cT}\right]^3 \tag{9.5.7}$$

Since our entire derivation above assumes the electric dipole approximation, that is $d/cT \ll 1$, we see that the radiated energy is always a small fraction of the energy that is stored or taken out of the electrostatic energy. Thus the irreversible energy loss due to radiation is small compared to the reversible energy storage in the electrostatic field.

Our course our whole expansion scheme rested on assuming $d < cT$ so that although you might thank that Eq. (9.5.7) implies that as *d* become larger than *cT* the radiated energy would exceed the stored energy by an arbitrarily large amount, in fact Eq. (9.5.7) is not valid when *d* becomes comparable to *cT*, so we can conclude nothing about the ratio of radiated to stored energy in such a case. A much more difficult and much more complicated calculation that does not make the electric dipole approximating in fact shows that when *d* becomes comparable to *cT* or greater, the radiated and stored energy are in fact exactly the same. I do not do that calculation here but I show a graph of the result of that calculation in Figure 9-15. For small values of *d/cT* we recover the electric dipole approximation result given in (9.5.7), but as *d/cT* approaches or exceeds one the radiated energy just equals the stored energy.



9-15: Energy radiated to stored energy as a function of *d/cT*

10 The General Form of Radiation E and B Fields

10.1 Learning Objectives

We stop and consider the general form of **E** and **B** radiation fields

10.2 General expressions for radiation E and B fields

I look at the terms in (8.3.10) and (8.3.14) which are a pure radiation field, that is the terms which carry energy off to infinity. These terms are the *1/r* terms, and are given for electric dipole radiation by

$$\begin{aligned} \mathbf{B}_{el\,dip}(\mathbf{r},t) &= \frac{\mu_o}{4\pi}\left[\frac{\ddot{\mathbf{p}}\times\hat{\mathbf{n}}}{c\,r}\right] \\ \mathbf{E}_{el\,dip}(\mathbf{r},t) &= \frac{1}{4\pi\,\varepsilon_o}\frac{1}{rc^2}(\ddot{\mathbf{p}}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}} = c\,\mathbf{B}_{el\,dip}\times\hat{\mathbf{n}} \end{aligned} \tag{10.2.1}$$

Before proceeding, I pause for a moment and consider the generalization of the form of the expressions in equation (10.2.1). As a general rule, whenever I deal only with radiation fields (terms falling off as *1/r*) of any kind (electric dipole, magnetic dipole, electric quadrupole, etc.), if we ignore everything except terms falling off as *1/r* , we will always have

$$\mathbf{B}_{rad} = +\frac{1}{c}\dot{\mathbf{A}}_{rad}\ \mathbf{x}\ \hat{\mathbf{n}} \quad \text{and} \quad \mathbf{E}_{rad} = (\dot{\mathbf{A}}_{rad}\ \mathbf{x}\ \hat{\mathbf{n}})\ \mathbf{x}\ \hat{\mathbf{n}} = c\,\mathbf{B}_{rad}\ \mathbf{x}\ \hat{\mathbf{n}} \tag{10.2.2}$$

These equations follow from assuming that the radiation part of the vector potential $\mathbf{A}(\mathbf{r},t)$ in (8.2.1) is *1/r* times some function of $t'_{ret} = t - r/c$, and dropping everything but *1/r* terms after taking derivatives. With this approach, it is clear that $\mathbf{B} = \nabla\times\mathbf{A}$ leads to the expression for $\mathbf{B}_{rad}$ in equation (10.2.2). What about the expression for $\mathbf{E}_{rad}$ in (10.2.2)? We appeal to the fact that outside the sources, where $\mathbf{J} = 0$, we have $\nabla\times\mathbf{B} = \mu_o\,\varepsilon_o\frac{\partial}{\partial t}\mathbf{E} = \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}$. Using the expression in (10.2.2) for the radiation $\mathbf{B}_{rad}$ field, and again dropping non-radiation terms, $\nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}$ tells us that we must have

$$\frac{1}{c^2}\left[\ddot{\mathbf{A}}_{rad}\ \mathbf{x}\ \hat{\mathbf{n}}\right]\ \mathbf{x}\ \hat{\mathbf{n}} = \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}_{rad} \tag{10.2.3}$$

Integrating (10.2.3) with respect to time, we recover the expression for the radiation $\mathbf{E}_{rad}$ field in (10.2.2) . We see that these equations are in particular appropriate for electric dipole radiation by inserting equation (18) into (23) and comparing to (22). Note that

$\mathbf{E}_{rad}$, $\mathbf{B}_{rad}$ and $\hat{\mathbf{n}}$ are all mutually perpendicular, and $cB_{rad} = E_{rad}$. This later equality means that the energy density in the magnetic field is equal to the energy density in the electric field.

10.3 Radiation of energy and momentum in the general case

What is the energy flux radiated into unit solid angle for the general radiation fields given in (10.2.2). From (10.2.2), we have for the radiation fields that

$$\frac{1}{\mu_o}(\mathbf{E}_{rad} \times \mathbf{B}_{rad}) = \frac{1}{\mu_o}\left(c\left(\mathbf{B}_{rad} \times \hat{\mathbf{n}}\right) \times \mathbf{B}_{rad}\right) = \hat{\mathbf{n}}\frac{c}{\mu_o}B_{rad}^2 - \frac{c}{\mu_o}\left(\mathbf{B}_{rad} \cdot \hat{\mathbf{n}}\right)\mathbf{B}_{rad} \tag{10.3.1}$$

Since $\mathbf{B}_{rad}$ and $\hat{\mathbf{n}}$ are perpendicular for the radiation fields, we have in general that

$$\frac{1}{\mu_o}(\mathbf{E}_{rad} \times \mathbf{B}_{rad}) = \hat{\mathbf{n}}\frac{c}{\mu_o}B_{rad}^2 \tag{10.3.2}$$

Now, suppose we take a very large sphere of radius r centered at the origin, and consider a surface element $\hat{\mathbf{n}}\,da$ on that sphere at a point (r,θ,ϕ), with

$$\hat{\mathbf{n}}\,da = r^2 d\Omega\,\hat{\mathbf{r}} = r^2 \sin\theta\, d\theta\, d\phi\,\hat{\mathbf{r}} \tag{10.3.3}$$

In the context of the energy conservation law that we have developed, we know that the quantity $\frac{(\mathbf{E}\,\mathbf{x}\,\mathbf{B})}{\mu_o}\cdot\hat{\mathbf{n}}\,da\,dt$ represents the amount of electromagnetic energy dW in joules flowing through $\hat{\mathbf{n}}\,da$ in a time dt :

$$dW = \frac{1}{\mu_o}(\mathbf{E}\,\mathbf{x}\,\mathbf{B})\cdot\hat{\mathbf{n}}\,da\,dt = \frac{c}{\mu_o}(\mathbf{E}\,\mathbf{x}\,\mathbf{B})\cdot(r^2 d\Omega\,\hat{\mathbf{r}})\,dt \tag{10.3.4}$$

or

$$\frac{dW}{d\Omega\;dt} = \frac{c}{\mu_o}(\mathbf{E} \times \mathbf{B})\cdot(r^2\,\hat{\mathbf{r}}) \tag{10.3.5}$$

Equation (10.3.5) is a general expression, good for any r and for any fields (quasi-static, induction, or radiation fields). However, it is clear that if we consider only the energy per second radiated to infinity, we need only include terms in **E** and **B** which fall off as *1/r*, since terms which fall off faster than this in the expression (10.3.5) for *dW* will vanish as *r* goes to infinity, and therefore carry no energy to infinity. Thus the electromagnetic energy radiated to infinity per unit time per unit solid angle is given by

$$\frac{dW_{rad}}{d\Omega\, dt} = \frac{r^2(\mathbf{E}_{rad}\,\mathbf{x}\,\mathbf{B}_{rad})\cdot\hat{\mathbf{n}}}{\mu_o} = \frac{c\, r^2 B_{rad}^2}{\mu_o} = \frac{r^2\left|\dot{\mathbf{A}}_{rad}\,\mathbf{x}\,\hat{\mathbf{n}}\right|^2}{\mu_o\, c} \tag{10.3.6}$$

where we have used equation (10.2.2) and (10.3.2) to obtain the various forms in (10.3.6)

In the context of the momentum conservation law that we have developed, we know that the quantity $-\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}\, da\, dt$ represents the amount of electromagnetic momentum $d\mathbf{P}_{rad}$ flowing through $\hat{\mathbf{n}}\, da$ in a time *dt*. As above, the momentum radiated per unit time per unit solid angle is thus

$$\frac{d\mathbf{P}_{rad}}{dt\, d\Omega} = -r^2\, \overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}} \tag{10.3.7}$$

Again, if we are only interested in the momentum radiated to infinity, we see that we need only consider terms in $\overleftrightarrow{\mathbf{T}}$ which fall off as $1/r^2$, since terms which fall off faster than this will vanish as *r* goes to infinity, and therefore carry no momentum to infinity. Since $\overleftrightarrow{\mathbf{T}}$ involves the square of the fields, we need only keep the radiation terms in calculating the momentum radiated to infinity.

11 Another Example of Electric Dipole Radiation

11.1 Moving non-relativistic point charges

I first point out one general expression for the amount of power radiated in electric dipole radiation. Using equation (10.2.1) and equation (10.3.6), I have for electric dipole radiation that

$$\frac{dW_{el\,dip}}{d\Omega\, dt} = \frac{c\, r^2}{\mu_o}\left[\frac{\mu_o}{4\pi}\frac{\ddot{\mathbf{p}}\,\mathbf{x}\,\hat{\mathbf{n}}}{c\, r}\right]^2 = \frac{\mu_o\, \ddot{p}^{\,2}}{(4\pi)^2 c}\sin^2\theta \tag{11.1.1}$$

where θ is the angle between $\ddot{\mathbf{p}}$ and $\hat{\mathbf{n}}$. If I integrate this expression over solid angle, taking $\ddot{\mathbf{p}}$ to lie along the z-axis for convenience, I obtain the expression for the total energy per second radiated in electric dipole radiation,

$$\frac{dW_{el\,dip}}{dt} = \frac{\mu_o}{4\pi}\frac{2\left|\ddot{\mathbf{p}}\right|^2}{3c} = \frac{1}{4\pi\,\varepsilon_o}\frac{2\left|\ddot{\mathbf{p}}\right|^2}{3c^3} \quad \text{or} \quad \frac{dW_{el\,dip}}{dt} = \frac{1}{4\pi\,\varepsilon_o}\frac{2}{3}\frac{q^2a^2}{c^3} \tag{11.1.2}$$

In the last form in equation (11.1.2), we have given an expression appropriate for the specific case where the radiation is due to a single point charge of charge *q* which is at $\mathbf{R}(t)$ at time t. It is clear that in such a situation, $\rho(\mathbf{r}',t') = q\,\delta^3(\mathbf{r}' - \mathbf{R}(t'))$, and that therefore, using $\mathbf{p}(t') = \int \mathbf{r}'\rho(\mathbf{r}',t')d^3x'$, that $\mathbf{p}(t) = q\ \mathbf{R}(t)$ and $\ddot{\mathbf{p}}(t) = q\ \ddot{\mathbf{R}}(t) = q\,\mathbf{a}(t)$, where $\mathbf{a}$(t) is the acceleration of the particle at time t.

Thus for a single particle, the instantaneous rate at which it radiates electric dipole energy is proportional the square of its charge and the square of its instantaneous acceleration. Note that if the particle is moving a speed V, it will travel across a region of length $d = VT$ in time T. Thus if T is the time it takes for the speed V to increase significantly, our requirement that d/cT be small compared to one for our expansion to be valid becomes for particle motion the requirement that $d/cT = VT/cT = V/c$ be small compared to one, i.e. that the particle speed be non-relativistic. Indeed, the radiation patterns for relativistic particles look nothing like the simple dipole and quadrupole radiation patterns we will derive here.

We now look at a specific example of electric dipole radiation, by looking at the fields of two point charges. We emphasize, however, that the methods we develop here can be applied in much more general situations than just individual point charge motions. All we need to do to apply them is to compute the overall moments of the charge and current distributions, as in equations (8.2.13) through (8.2.15).

Consider the following time varying source functions. We have two point charges, one at rest at the origin, with charge $-q_o$, and one moving up and down on the z-axis, with charge $+q_o$, and with its position described by

$$\mathbf{r}'(t') = \hat{\mathbf{z}}\ R_o \cos\omega_o t' \qquad (11.1.3)$$

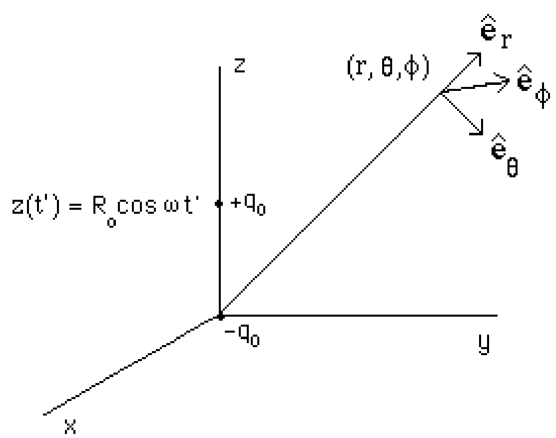


Figure 11-1: An electric dipole formed by moving point charges

We want to find the electric and magnetic fields appropriate to this source, including quasi-static, induction, and radiation fields. The charge density is given by the charge $\pm\, q_o$ times three dimensional delta functions at the positions of the charges. The charge density is thus

$$\rho(\mathbf{r}',t') = q_o\,\delta(x')\,\delta(y')\,\delta(z' - R_o\cos\omega_o t') - q_o\,\delta(x')\,\delta(y')\,\delta(z') \tag{11.1.4}$$

This charge distribution has no net charge. The electric dipole moment is (cf equation (8.2.14))

$$\begin{aligned}&\mathbf{p}(t') = \int \mathbf{r}'\rho(\mathbf{r}',t')d^3x' = \\ &q_o\int \mathbf{r}'\left[\delta(x')\,\delta(y')\,\delta(z' - R_o\cos\omega_o t') - \delta(x')\,\delta(y')\,\delta(z')\right]d^3x'\end{aligned} \tag{11.1.5}$$

or

$$\mathbf{p}(t') = q_o R_o \cos\omega_o t'\ \hat{\mathbf{z}} = p_o\cos\omega_o t'\ \hat{\mathbf{z}} \tag{11.1.6}$$

with $p_o = q_o R_o\, p_o$. From (8.3.10) and (8.3.14), with this expression for **p**(t), and defining $k = \omega_o / c$, we have for **E** and **B** the expressions

$$\begin{aligned}\mathbf{B}(\mathbf{r},t) &= -\hat{\mathbf{e}}_\phi \frac{\mu_o}{4\pi}\frac{p_o c\,k^2\sin\theta}{r}\left[\cos\omega_o(t - r/c) + \frac{\sin\omega_o(t - r/c)}{kr}\right] \\ \mathbf{E}(\mathbf{r},t) &= \hat{\mathbf{e}}_r \frac{1}{4\pi\ \varepsilon_o}\frac{2p_o\cos\theta}{r^3}\left[\cos\omega_o(t - r/c) - kr\sin\omega_o(t - r/c)\right] \\ &\quad + \hat{\mathbf{e}}_\theta \frac{1}{4\pi\ \varepsilon_o}\frac{p_o\sin\theta}{r^3}\left[(1 - k^2r^2)\cos\omega_o(t - r/c) - kr\sin\omega_o(t - r/c)\right]\end{aligned} \tag{11.1.7}$$

Terms in equation (11.1.7) like $\cos\omega_o(t - r/c)$ represent traveling waves moving away from origin with a frequency ω_o and a wave length $\lambda = 2\pi / k$, with period $T = 2\pi / \omega_o$. Note that our conditions in equation (8.2.9) are now $r >> d$ and $\lambda >> d$ (this requirement on λ is equivalent to the requirement that the maximum speed of the moving charge be small compared to the speed of light, as we saw above). Note however that we have made *no* requirement on r as compared to λ, only that both be much greater than d.

We now restrict ourselves to the radiation terms in equation (11.1.7), that is the terms that go to zero as $1/r$ as $r \to \infty$. In the limit that $kr >> 1$, the dominant terms in these equations are the radiation terms,

$$\begin{aligned}\mathbf{B}(\mathbf{r},t) &= -\hat{\mathbf{e}}_\phi \frac{\mu_o}{4\pi}\frac{c\ p_o\,k^2\sin\theta}{r}\cos\omega_o(t - r/c) \\ \mathbf{E}(\mathbf{r},t) &= -\hat{\mathbf{e}}_\theta \frac{\mu_o}{4\pi}\frac{c^2\ p_o\,k^2\sin\theta}{r}\cos\omega_o(t - r/c)\end{aligned} \tag{11.1.8}$$

where to get the form for the electric field in (11.1.8), we have used the fact that

$$c^2 = \frac{1}{\mu_o \, \varepsilon_o}$$

11.2 Energy and momentum flux

The energy radiated into a solid angle $d\Omega$ is just the Poynting flux into that solid angle, that is

$$\frac{dW_{rad}}{dt} = \left(\frac{\mathbf{E}_{rad} \ \mathbf{x} \, \mathbf{B}_{rad}}{\mu_o} \right) \cdot \hat{\mathbf{n}} \, r^2 d\Omega \tag{11.2.1}$$

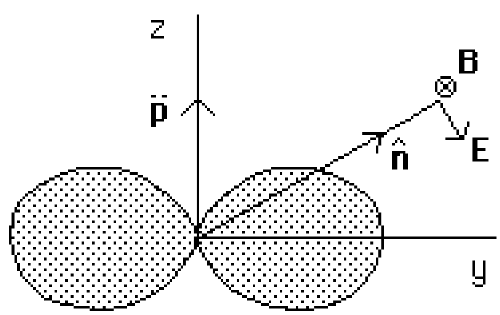
or

$$\frac{dW_{rad}}{d\Omega \, dt} = \frac{r^2 (\mathbf{E}_{rad} \ \mathbf{x} \, \mathbf{B}_{rad}) \cdot \hat{\mathbf{n}}}{\mu_o} \tag{11.2.2}$$

Using the radiation fields in (11.1.8) in (11.2.2), we have

$$\frac{dW_{rad}}{d\Omega \, dt} = + \frac{1}{(4\pi)^2 \varepsilon_o} c \, p_o{}^2 k^4 \sin^2\theta \ \cos^2 \omega_o (t - r/c) \tag{11.2.3}$$

The radiation electric field **E** for electric dipole radiation is polarized in the plane of $\hat{\mathbf{n}}$ and **p**, and the radiation magnetic field **B** is out of that plane.



If we average over one period T and integrate over all solid angles, we find the total energy flux per second (in ergs/sec) is

$$\left\langle \frac{dW_{rad}}{dt} \right\rangle = \frac{c \ p_o{}^2 k^4}{12\pi \ \varepsilon_o} = \frac{1}{4\pi \ \varepsilon_o} \frac{2}{3} \frac{q_o{}^2 < a^2 >}{c^3} \tag{11.2.4}$$

where $<a^2>$ is equal to the value of the square of the acceleration averaged over one period (the average square of the acceleration is just one-half of the square of the peak acceleration). Compare equation (11.2.4) to equation (11.1.2) for the instantaneous rate at which energy is radiated.

So equation (11.2.4) gives the rate at which energy is radiated away. What about momentum? We use equation (10.3.7) We need to compute $-\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$. The *j-th* component of $-\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}}$ is given by

$$(-\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}})_j = -T_{ji}n_i = -T_{jr} \quad \text{since } \hat{\mathbf{n}} = \hat{\mathbf{r}} \tag{11.2.5}$$

Since $B_r = 0$, we have

$$-\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}} = -\left[\varepsilon_o \mathbf{E}E_r - \frac{1}{2}\hat{\mathbf{e}}_r(\varepsilon_o E^2 + B^2/\mu_o)\right] \tag{11.2.6}$$

But $\mathbf{E}\,E_r$ is proportional to $1/r^3$, so we can drop this first term, and only keep radiation fields in the $\varepsilon_o E^2 + B^2/\mu_o$ term, so that equation(10.3.7) becomes

$$\frac{d\mathbf{P}_{rad}}{dt\,d\Omega} = +\frac{1}{(4\pi)^2\varepsilon_o}\hat{\mathbf{e}}_r\, k^4 p_o^{\,2}\sin^2\theta\,\cos^2\omega_o(t - r/c) \tag{11.2.7}$$

This vector is radial and in magnitude is just $1/c$ times the energy per unit time passing through $\hat{\mathbf{n}}\,da$ (cf. equation (11.2.3)). A photon has energy $\hbar\omega$ and momentum $\hbar\omega/c$. We can time average $\frac{d\mathbf{P}_{rad}}{dt\,d\Omega}$ over one cycle, but if we integrate $\frac{d\mathbf{P}_{rad}}{dt\,d\Omega}$ over $d\Omega$ we get a net of zero (to do this must *first* express $\frac{d\mathbf{P}_{rad}}{dt\,d\Omega}$ in Cartesian components, and *then* integrate over $d\Omega$).

12 Magnetic Dipole and Electric Quadrupole Radiation

12.1 Learning Objectives

We now consider the properties of magnetic dipole radiation and electric quadrupole radiation.

12.2 Magnetic dipole radiation

In looking at electric dipole radiation, we have just scratched the surface of the radiation produced by time varying sources. Electric dipole radiation is the dominate mode of radiation, but if it vanishes there are other modes we now review. We only want to consider two other characteristic forms of radiation, which for $d/cT << 1$ turn out to be important only if the electric dipole moment vanishes. Consider the higher order terms in equation (8.2.11) for $\mathbf{A}(\mathbf{r},t)$:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{4\pi\, r^2}\int \left(\hat{\mathbf{n}}\cdot\mathbf{r}'\right)\mathbf{J}(\mathbf{r}',t')d^3x' + \frac{\mu_o}{4\pi\, r\, c}\int \left(\hat{\mathbf{n}}\cdot\mathbf{r}'\right)\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}',t')\, d^3x' \qquad (12.2.1)$$

By using your results on Problem 3-1(c) of Problem Set 3, this can be written as

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_o}{8\pi\, r^2}\int \left[\left(\mathbf{r}'\,\mathbf{x}\,\mathbf{J}\right)\mathbf{x}\,\hat{\mathbf{n}}\right]d^3x' + \frac{\mu_o}{8\pi\, r\, c}\int \left[\left(\mathbf{r}'\,\mathbf{x}\,\frac{\partial}{\partial t'}\mathbf{J}\right)\mathbf{x}\,\hat{\mathbf{n}}\right]\; d^3x'$$
$$+\frac{\mu_o}{8\pi\, r^2}\int \left[\mathbf{r}'(\hat{\mathbf{n}}\cdot\mathbf{J}) + \mathbf{J}\,(\hat{\mathbf{n}}\cdot\mathbf{r}')\right]\, d^3x' + \frac{\mu_o}{8\pi\, r\, c}\int \left[\mathbf{r}'\;(\hat{\mathbf{n}}\cdot\frac{\partial}{\partial t'}\mathbf{J}) + \frac{\partial}{\partial t'}\mathbf{J}\;(\hat{\mathbf{n}}\cdot\mathbf{r}')\right] d^3x'$$
$$(12.2.2)$$

Let us first treat the first two terms in (12.2.2), which will give us magnetic dipole radiation.. With the definition $\mathbf{m}(t') = \frac{1}{2}\int \mathbf{r}'\mathbf{x}\mathbf{J}(\mathbf{r}',t')d^3x'$ from (8.2.14), we have for the magnetic dipole part of **A**:

$$\mathbf{A}_{mag\ dip}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\left[\frac{\mathbf{m}(t')}{r^2} + \frac{\dot{\mathbf{m}}(t')}{cr}\right]\mathbf{x}\,\hat{\mathbf{n}} \qquad (12.2.3)$$

The first term here is just static magnetic dipole vector potential. To get the full **B**, we must compute $\nabla\times\mathbf{A}$. This is messy, and we can avoid the work by noting that outside the source, we have from Maxwell's equations in vacuum that

$$\nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} \qquad \text{and} \qquad \nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \qquad (12.2.4)$$

In particular, the first equation on the left in (12.2.4) must be true for the electric dipole **B** and **E** given in (8.3.10) and (8.3.14) above. Thus

$$\nabla\times\frac{\mu_o}{4\pi}\left\{\left[\frac{\ddot{\mathbf{p}}}{c\,r} + \frac{\dot{\mathbf{p}}}{r^2}\right]\times\hat{\mathbf{n}}\right\} = \frac{1}{c^2}\frac{\partial}{\partial t}\left\{\frac{\left[3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}})-\mathbf{p}\right]}{4\pi\,\varepsilon_o r^3} + \frac{\left[3\hat{\mathbf{n}}(\dot{\mathbf{p}}\cdot\hat{\mathbf{n}})-\dot{\mathbf{p}}\right]}{4\pi\,\varepsilon_o c\, r^2} + \frac{(\ddot{\mathbf{p}}\,\mathbf{x}\,\hat{\mathbf{n}})\,\mathbf{x}\,\hat{\mathbf{n}}}{4\pi\,\varepsilon_o rc^2}\right\} \quad (12.2.5)$$

Since $\frac{\partial}{\partial t} = \frac{\partial}{\partial t'}$, we can integrate both sides of equation (12.2.5) with respect to t' to obtain

$$\nabla\,\mathbf{x}\left\{\frac{\mu_o}{4\pi}\left[\frac{\dot{\mathbf{p}}}{cr} + \frac{\mathbf{p}}{r^2}\right]\mathbf{x}\,\hat{\mathbf{n}}\right\} = \frac{1}{c^2}\left\{\frac{\left[3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}})-\mathbf{p}\right]}{4\pi\,\varepsilon_o\, r^3} + \frac{\left[3\hat{\mathbf{n}}(\dot{\mathbf{p}}\cdot\hat{\mathbf{n}})-\dot{\mathbf{p}}\right]}{4\pi\,\varepsilon_o c\, r^2} + \frac{(\ddot{\mathbf{p}}\,\mathbf{x}\,\hat{\mathbf{n}})\,\mathbf{x}\,\hat{\mathbf{n}}}{4\pi\,\varepsilon_o r\, c^2}\right\} \qquad (12.2.6)$$

If we just let **p** go to **m** this equation tells us what the curl of **A**(t') is in equation (12.2.3). Thus, we have the expression for **B** for terms proportional to **m** and its time derivatives:

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\left\{\frac{1}{r^3}\left[3\hat{\mathbf{n}}(\mathbf{m}\cdot\hat{\mathbf{n}}) - \mathbf{m}\right] + \frac{1}{c\,r^2}\left[3\hat{\mathbf{n}}(\dot{\mathbf{m}}\cdot\hat{\mathbf{n}}) - \dot{\mathbf{m}}\right] + \frac{1}{rc^2}(\ddot{\mathbf{m}}\,\mathbf{x}\,\hat{\mathbf{n}})\,\mathbf{x}\,\hat{\mathbf{n}}\right\} \tag{12.2.7}$$

quasi-static induction radiation

Again, we see that we have the quasi-static magnetic dipole term, plus induction and radiation terms.

What about **E** for the terms involving **m**? Well, we could go back and pull it out of $\mathbf{E}(\mathbf{r},t) = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$, but this is not necessary since we can again use Maxwell's equations. We know (12.2.5) is true, and it is just as true if we replace **p**(t') in that equation with **m**(t'). That is, we must have (using $c^2 = \frac{1}{\mu_o \varepsilon_o}$)

$$\nabla\,\mathbf{x}\,\frac{\mu_o}{4\pi}\left\{\left[\frac{\ddot{\mathbf{m}}}{c\,r} + \frac{\dot{\mathbf{m}}}{r^2}\right]\mathbf{x}\,\hat{\mathbf{n}}\right\} = \frac{\partial}{\partial t}\mu_o\left\{\frac{\left[3\hat{\mathbf{n}}(\mathbf{m}\cdot\hat{\mathbf{n}}) - \mathbf{m}\right]}{4\pi\,r^3} + \frac{\left[3\hat{\mathbf{n}}(\dot{\mathbf{m}}\cdot\hat{\mathbf{n}}) - \dot{\mathbf{m}}\right]}{4\pi\,c\,r^2} + \frac{(\ddot{\mathbf{m}}\,\mathbf{x}\,\hat{\mathbf{n}})\,\mathbf{x}\,\hat{\mathbf{n}}}{4\pi\,rc^2}\right\}$$

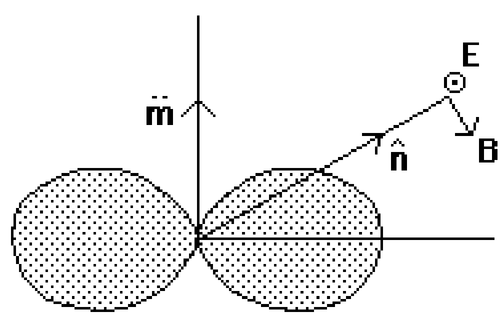
But the term in brackets on the right hand side of this equation is just the **B** field which involves magnetic dipole terms, so that

$$\nabla\,\mathbf{x}\left\{\frac{\mu_o}{4\pi}\left[\frac{\ddot{\mathbf{m}}}{c\,r} + \frac{\dot{\mathbf{m}}}{r^2}\right]\mathbf{x}\,\hat{\mathbf{n}}\right\} = \frac{\partial}{\partial t}\mathbf{B} \tag{12.2.8}$$

But we know from Maxwell's equations that the **E** field must satisfy $\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$. Comparing this equation with equation (12.2.8), we see that the **E** field for terms involving **m** and its time derivatives must be given by

$$\mathbf{E}(\mathbf{r},t) = -\frac{\mu_o}{4\pi}\left[\frac{\ddot{\mathbf{m}}}{c\,r} + \frac{\dot{\mathbf{m}}}{r^2}\right]\mathbf{x}\,\hat{\mathbf{n}} \tag{12.2.9}$$

If we abstract from equations (12.2.7) and (12.2.9) the radiation terms, we have the expressions for magnetic dipole radiation:



$$\mathbf{B}_{mag\ dip}(\mathbf{r},t) = \frac{\mu_o}{4\pi\,rc^2}(\ddot{\mathbf{m}}\,\mathbf{x}\,\hat{\mathbf{n}})\,\mathbf{x}\,\hat{\mathbf{n}}$$

$$\mathbf{E}_{mag\ dip}(\mathbf{r},t) = -\frac{\mu_o}{4\pi\,rc}\ddot{\mathbf{m}}\,\mathbf{x}\,\hat{\mathbf{n}}$$

(12.2.10)

For magnetic dipole radiation, **B** is in the plane of $\hat{\mathbf{n}}$ and $\ddot{\mathbf{m}}$ and **E** is perpendicular to that plane, *just the opposite of the situation for electric dipole radiation*. The angular distribution of the power radiated per unit solid angle is the same as for electric dipole radiation. That is, it goes as $\sin^2\theta$, where θ is the angle between $\hat{\mathbf{n}}$ and $\ddot{\mathbf{m}}$.

The total energy per second radiated is given by a form similar to equation (11.1.2), with $\ddot{\mathbf{p}}$ replaced by $\ddot{\mathbf{m}}/c$, that is

$$\frac{dW_{mag\ dip}}{dt} = \frac{\mu_o}{4\pi}\frac{2\left|\ddot{\mathbf{m}}\right|^2}{3c^3} \quad . \tag{12.2.11}$$

It is important to note that for particle motion, **J** in equation (8.2.14) defining **m** is a charge density times a velocity V of a particle, and simple dimensional analysis leads to the conclusion that:

For non-relativistic particle motion, we always have that the ratio of the power radiated into magnetic dipole radiation to that radiated into electric dipole radiation is $(V/c)^2$, unless $\left|\mathbf{p}\right|$ happens to be zero.

12.3 Electric quadrupole radiation

We now turn to electric quadrupole radiation, which we can obtain from the 3rd and 4th terms in equation (12.2.2). This is complicated mathematically, and let us start out by stating what the important points are.

The energy radiated into electric quadrupole will be down by a factor $(d/\lambda)^2$ compared to that radiated into electric dipole radiation. Thus, unless the electric dipole moment is zero, electric quadrupole radiation is an unimportant addition to the radiated energy for $d << \lambda$. Also, the frequency that emerges if we use the oscillating charge example of Section V above is *twice* the frequency ω_o with which the charge oscillates (see Section VIII below).

The following is an identity for the current density **J**

$$\int\left[\mathbf{r}'(\hat{\mathbf{n}}\cdot\mathbf{J}) + \mathbf{J}(\hat{\mathbf{n}}\cdot\mathbf{r}')\right]d^3x' = +\int\mathbf{r}'(\hat{\mathbf{n}}\cdot\mathbf{r}')\frac{\partial}{\partial t'}\rho(\mathbf{r}',t')\,d^3x' \tag{12.3.1}$$

Using this equation, we can write the electric quadrupole part of **A** in (12.2.2) as

$$\mathbf{A}_{el\ quad}(\mathbf{r},t) = \frac{\mu_o}{8\pi r^2}\int\left[\mathbf{r}'(\hat{\mathbf{n}}\cdot\mathbf{r}')\frac{\partial\rho(\mathbf{r}',t')}{\partial t'}\right]d^3x' + \frac{\mu_o}{8\pi c\, r}\int\left[\mathbf{r}'(\hat{\mathbf{n}}\cdot\mathbf{r}')\frac{\partial^2\rho(\mathbf{r}',t')}{\partial t'^2}\right]d^3x' \tag{12.3.2}$$

From now on, we drop all but the radiation terms in our expressions, since if we keep the full expansion, things get really messy. Thus we drop the first term in (12.3.2) So

$$\mathbf{A}_{el\ quad}(\mathbf{r},t) = \frac{\mu_o}{8\pi\, c\, r}\frac{d^2}{dt'^2}\int\left[\mathbf{r}'(\hat{\mathbf{n}}\cdot\mathbf{r}')\rho(\mathbf{r}',t')\right]d^3x' \tag{12.3.3}$$

For the moment, we define the 2nd rank tensor $\overleftrightarrow{\mathbf{H}}(t')$ by the equation

$$\overleftrightarrow{\mathbf{H}}(t') = \frac{d^2}{dt'^2}\int\left[\mathbf{r}'\,\mathbf{r}'\rho(\mathbf{r}',t')\right]d^3x' = \int\left[\mathbf{r}'\,\mathbf{r}'\ddot{\rho}(\mathbf{r}',t')\right]d^3x' \tag{12.3.4}$$

Then

$$\mathbf{A}_{el\ quad}(\mathbf{r},t) = \frac{\mu_o}{8\pi\, c\, r}\,\hat{\mathbf{n}}\cdot\overleftrightarrow{\mathbf{H}}(t') \tag{12.3.5}$$

and thus

$$\mathbf{B}_{el\ quad}(\mathbf{r},t) = \nabla\times\mathbf{A}_{el\ quad} = \nabla\times\frac{\mu_o\,\hat{\mathbf{n}}\cdot\overleftrightarrow{\mathbf{H}}(t')}{8\pi\, c\, r} \tag{12.3.6}$$

$$\left(\nabla\times\mathbf{A}_{el\ quad}\right)_j = \varepsilon_{jkl}\,\frac{\mu_o}{8\pi\, c}\frac{\partial}{\partial x_k}\frac{n_i H_{il}}{r} \tag{12.3.7}$$

$$\left(\nabla\times\mathbf{A}_{el\ quad}\right)_j = \varepsilon_{jkl}\,\frac{\mu_o}{8\pi\, c}\frac{n_i}{r}\frac{\partial}{\partial x_k}H_{il} + \varepsilon_{jkl}\,\frac{\mu_o}{8\pi\, c}H_{il}\frac{\partial}{\partial x_k}\frac{x_i}{r^2} \tag{12.3.8}$$

The second term on the right side of equation (12.3.8) is proportional to $1/r^2$, and since we are keeping only radiation terms, we drop it. Using our prescription for taking gradients of functions of $t' = t - r/c$, we have

$$B_j = -\varepsilon_{jkl}\,\frac{\mu_o}{8\pi\, c^2}\frac{n_k n_i}{r}\dot{H}_{il} \quad \text{or} \quad \mathbf{B}_{el\ quad} = -\hat{\mathbf{n}}\times\frac{\mu_o}{8\pi\, c^2}\frac{\left[\hat{\mathbf{n}}\cdot\dot{\overleftrightarrow{\mathbf{H}}}(t')\right]}{r} \tag{12.3.9}$$

Using the definition of $\overleftrightarrow{\mathbf{H}}(t')$,

$$\mathbf{B}_{el\ quad}(\mathbf{r},t) = -\left[\frac{\mu_o}{8\pi\, c^2}\frac{\hat{\mathbf{n}}}{r}\right]\times\left[\hat{\mathbf{n}}\cdot\int\left[\mathbf{r}'\,\mathbf{r}'\,\dddot{\rho}(\mathbf{r}',t')\right]d\tau'\right] \tag{12.3.10}$$

which can be written as

$$\mathbf{B}_{el\ quad}(\mathbf{r},t) = -\left[\frac{\mu_o}{8\pi\, c^2}\frac{\hat{\mathbf{n}}}{r}\right]\times\frac{\partial^3}{\partial t'^3}\left[\hat{\mathbf{n}}\cdot\int\left[\mathbf{r}'\,\mathbf{r}' - \frac{1}{3}\overleftrightarrow{\mathbf{I}}(r')^2\right]\rho(\mathbf{r}',t')d\tau'\right] \tag{12.3.11}$$

where we have added a term involving the identity tensor. The term that we have added is proportional to $\hat{\mathbf{n}}\times\hat{\mathbf{n}}$ and is therefore zero. Using (8.2.15) for the definition of the quadrupole moment, we have

$$\mathbf{B}_{el\ quad}(\mathbf{r},t) = -\frac{\mu_o}{24\pi c^2}\frac{1}{r}\hat{\mathbf{n}}\times\left[\hat{\mathbf{n}}\cdot\dddot{\overleftrightarrow{\mathbf{Q}}}\right] \tag{12.3.12}$$

For quadrupole radiation, then, using equation (10.3.6) and (12.3.12), the energy flux into unit solid angle is given by

$$\frac{dW_{el\ quad}}{d\Omega\, dt} = \frac{\mu_o}{(24)^2\pi^2 c^3}\left|\hat{\mathbf{n}}\times\left[\hat{\mathbf{n}}\cdot\dddot{\overleftrightarrow{\mathbf{Q}}}\right]\right|^2 \tag{12.3.13}$$

A tedious integration of equation (12.3.13) over all solid angles gives the total radiated power

$$\frac{dW_{el\ quad}}{dt} = \frac{\mu_o}{720\pi c^3}\sum_{i=1}^{3}\sum_{j=1}^{3}\left|\dddot{Q}_{ij}\right|^2 \tag{12.3.14}$$

12.4 An Example Of Electric Quadrupole Radiation

We take the same problem as for the electric dipole example above, except now we compute the quadrupole radiation. What is $\overleftrightarrow{\mathbf{Q}}$? Well, using the definition in equation (8.2.15) and the charge density in equation **Error! Reference source not found.**, we have $Q_{xy} = Q_{yz} = Q_{xz} = 0$. Moreover,

$$\begin{aligned}
Q_{xx}(t') &= q_o\int \delta(x')\delta(y')\delta(z'-R_o\cos\omega_o t')\left[3x'^2-(x'^2+y'^2+z'^2)\right]d^3x' \\
Q_{xx}(t') &= -q_oR_o{}^2\cos^2\omega_o t' = Q_{yy}(t') \\
Q_{zz}(t') &= q_o\int \delta(x')\delta(y')\delta(z'-R_o\cos\omega_o t')\left[3z'^2-(x'^2+y'^2+z'^2)\right]d^3x' \\
Q_{zz}(t') &= +2q_oR_o{}^2\cos^2\omega_o t'
\end{aligned}$$

If we use the trig identity $\cos^2\omega_o t' = \tfrac{1}{2}(1+\cos 2\omega_o t')$ we can write $\overleftrightarrow{\mathbf{Q}}$ as

$$\overleftrightarrow{\mathbf{Q}}(t') = q_oR_o{}^2(1+\cos 2\omega_o t')\begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1\end{pmatrix} \tag{12.4.1}$$

Note that we now have a ***time variation at a frequency of*** $2\omega_o$. This is because the quadrupole goes as the square of the position of the charge. Higher moments, which go as higher powers of the position of the charge, will for that reason exhibit time variations at higher multiples of ω_o. Note also that the trace of $\overleftrightarrow{\mathbf{Q}}$ is 0, as it must be. Taking the appropriate derivatives, we have

$$\dddot{\overleftrightarrow{\mathbf{Q}}}(t') = 8q_o R_o{}^2 \omega_o{}^3 \sin 2\omega_o t' \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12.4.2}$$

Let's look at the angular distribution of this radiation. From (12.3.12), we see that we first need to compute $\hat{\mathbf{n}} \cdot \dddot{\overleftrightarrow{\mathbf{Q}}}$. First of all, $\hat{\mathbf{n}}$ is the unit vector in the radial direction, and in Cartesian coordinates that vector is $\hat{\mathbf{n}} = \hat{\mathbf{e}}_r = \sin\theta\cos\phi\,\hat{\mathbf{e}}_x + \sin\theta\sin\phi\,\hat{\mathbf{e}}_y + \cos\theta\,\hat{\mathbf{e}}_z$. Now,

$$\left[\hat{\mathbf{e}}_r \cdot \dddot{\overleftrightarrow{\mathbf{Q}}}\right]_i = \sum_{j=1}^{3} e_{rj}\dddot{Q}_{ji} = e_{ri}\dddot{Q}_{ii} \tag{12.4.3}$$

since $\dddot{\overleftrightarrow{\mathbf{Q}}}$ is diagonal, so that

$$\begin{gathered} \hat{\mathbf{e}}_r \cdot \dddot{\overleftrightarrow{\mathbf{Q}}}(t') = C(t')\left[-\tfrac{1}{2}\sin\theta\cos\phi\,\hat{\mathbf{e}}_x - \tfrac{1}{2}\sin\theta\sin\phi\,\hat{\mathbf{e}}_y + \cos\theta\,\hat{\mathbf{e}}_z\right] \\ \text{where } \mathrm{C(t')} = 8q_o R_o{}^2 \omega_o{}^3 \sin 2\omega_o t' \end{gathered} \tag{12.4.4}$$

We want to express this vector in spherical components, using the standard relationships:

$$\begin{aligned} \hat{\mathbf{e}}_x &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_r + \cos\theta\cos\phi\,\hat{\mathbf{e}}_\theta - \sin\phi\,\hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_y &= \sin\theta\sin\phi\,\hat{\mathbf{e}}_r + \cos\theta\sin\phi\,\hat{\mathbf{e}}_\theta + \cos\phi\,\hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_z &= \cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\;\hat{\mathbf{e}}_\theta \end{aligned} \tag{12.4.5}$$

This gives

$$\hat{\mathbf{e}}_r \cdot \dddot{\overleftrightarrow{\mathbf{Q}}}(t') = C(t')\begin{bmatrix} +\hat{\mathbf{e}}_r(\cos^2\theta - \frac{1}{2}\sin^2\theta\cos^2\phi - \frac{1}{2}\sin^2\theta\sin^2\phi) \\ +\hat{\mathbf{e}}_\theta(-\sin\theta\cos\theta - \frac{1}{2}\sin\theta\cos\theta\cos^2\phi - \frac{1}{2}\sin\theta\cos\theta\sin^2\phi) \\ +\hat{\mathbf{e}}_\phi(\frac{1}{2}\sin\theta\cos\phi\sin\phi - \frac{1}{2}\sin\theta\cos\phi\sin\phi) \end{bmatrix} \tag{12.4.6}$$

or

$$\hat{\mathbf{e}}_r \cdot \dddot{\overleftrightarrow{\mathbf{Q}}}(t') = C(t')\left[\hat{\mathbf{e}}_r(\cos^2\theta - \tfrac{1}{2}\sin^2\theta) - \hat{\mathbf{e}}_\theta\,\tfrac{3}{2}\sin\theta\cos\theta\right] \tag{12.4.7}$$

and finally we have

$$\begin{aligned} \hat{\mathbf{n}}\ \mathbf{x}\left[\hat{\mathbf{n}} \cdot \dddot{\overleftrightarrow{\mathbf{Q}}}\right] &= \hat{\mathbf{e}}_r\ \mathbf{x}\left[\hat{\mathbf{e}}_r \cdot \dddot{\overleftrightarrow{\mathbf{Q}}}\right] = -\tfrac{3}{2}\sin\theta\cos\theta\, C\,\hat{\mathbf{e}}_r\ \mathbf{x}\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{n}}\ \mathbf{x}\left[\hat{\mathbf{n}} \cdot \dddot{\overleftrightarrow{\mathbf{Q}}}\right] &= -\tfrac{3}{2}\sin\theta\cos\theta\, C\,\hat{\mathbf{e}}_\phi \end{aligned} \tag{12.4.8}$$

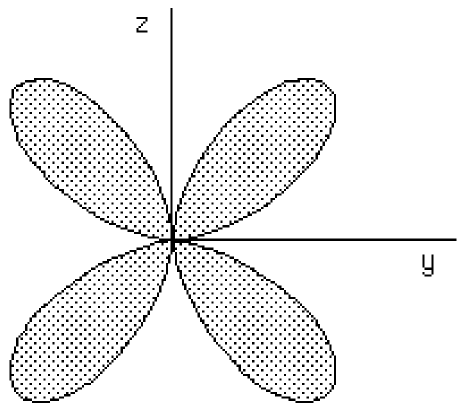
From (12.4.8), we see that **B** is in the $\hat{\mathbf{e}}_\phi$ direction, and thus **E** is in the $\hat{\mathbf{e}}_\theta$ direction, for this radiation field. From (12.3.13), we have that

$$\begin{aligned}\frac{dW_{el\ quad}}{d\Omega\, dt} &= \frac{\mu_o}{256\pi^2 c^3} C^2(t')\sin^2\theta\cos^2\theta \\ &= \frac{\mu_o c^3}{4\pi^2} k^6 q_o{}^2 R_o{}^4 \sin^2 2\omega_o t' \sin^2\theta\cos^2\theta\end{aligned} \tag{12.4.9}$$

Time averaging over one period gives the average power radiated per solid angle

$$\left\langle \frac{dW_{el\ quad}}{d\Omega\, dt} \right\rangle = \frac{\mu_o c^3}{8\pi^2} k^6 q_o{}^2 R_o{}^4 \sin^2\theta\cos^2\theta \tag{12.4.10}$$

The angular distribution of this radiation is shown in the sketch.



Note that if we compare this to the amount of power radiated into electric dipole radiation by this same system (equation (11.2.3)), we see that the ratio of quadrupole to dipole radiated power goes as $k^2 R_o{}^2 \approx R_o{}^2 / \lambda^2 << 1$, by assumption. So the power radiated into quadrupole radiation is unimportant under this assumption, unless the electric dipole moment is identically zero.

> Note: in this example, the quadrupole radiation is emitted at an angular frequency of $2\omega_o$, and *not* ω_o, the frequency at which electric dipole radiation is emitted. In general, the oscillation of a charge as in this example here will result in radiation of ω_o, $2\omega_o$, $3\omega_o$ ...., that is all harmonics of ω_o. If the electric dipole approximation is not satisfied, the radiation emitted will emerge at higher and higher multiples of the fundamental. Sychrotron radiation of relativistic particles is a good example, where the radiation emitted extends up to $\gamma^3\omega_o$.

We go to no higher orders. To properly treat the expansion to all orders in d/λ, we need to introduce vector spherical harmonics (e.g., see Jackson, *Classical Electrodynamics*, Chapter 16).