26 Interactions with matter I

26.1 Learning Objectives

We begin our discussion of the interactions of electromagnetic fields with matter. First we look at insulators, that is dielectrics and magnetic materials.

26.2 The Displacement Current

Before we begin actually discussing matter, we first consider the origins of the displacement current term introduced by Maxwell. Before Maxwell, Ampere's Law was well known, which has the differential form

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} \tag{26.2.1}$$

Maxwell noted that this equation must be missing a term because if we take the divergence of both sides of (26.2.1), since the divergence of any curl is zero, we have

$$\nabla \cdot \mathbf{J} = 0 \tag{26.2.2}$$

But this cannot be because charge conservation in differential form is

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{26.2.3}$$

Maxwell suggested that (26.2.1) should instead be

$$\nabla \times \mathbf{B} = \mu_o \left(\mathbf{J} + \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} \right) \tag{26.2.4}$$

With this term and $\nabla \cdot \mathbf{E} = \rho / \varepsilon_o$, we no longer have a problem with the divergence of (26.2.4) being zero on both sides. The term $\varepsilon_o \frac{\partial}{\partial t}\mathbf{E}$ in equation (26.2.4) has units of current density and is called the displacement current, ***even though it has nothing to do with displacement and nothing to do with current.*** The reason it is called that is because Maxwell knew about the polarization current, which we discuss below in (26.3.4), and he could ascribe this current to the real motion of charges producing a real current. He hypothesized that the aether was made up of positive and negative changes and that an electric field would produce a similar "displacement current" by moving those charges around. There is no such thing, and the concept is totally wrong. Nonetheless we continue to call $\varepsilon_o \frac{\partial}{\partial t}\mathbf{E}$ the displacement current.

26.3 The average dipole moment per unit volume

We begin our consideration of the interaction of material media with electromagnetic fields by realizing that for the most part matter is neutral, and that the interaction with fields is not through any net charge but through the effects of electric and magnetic dipoles. We are motivated therefore to define the two vectors **P** and **M** as the electric dipole moment per unit volume and the magnetic dipole moment per unit volume. That is if we have a small volume ΔV of material located at **r** at time *t*, with *N* electric dipoles $\{\mathbf{p}_i\}_{i=1}^{N}$ and *N* magnetic dipoles $\{\mathbf{m}_i\}_{i=1}^{N}$, then we define **P** and **M** at (**r**,t) as

$$\mathbf{P}(\mathbf{r},t) = \frac{1}{\Delta V}\sum_{i=1}^{N}\mathbf{p}_i \qquad\qquad \mathbf{M}(\mathbf{r},t) = \frac{1}{\Delta V}\sum_{i=1}^{N}\mathbf{m}_i \tag{26.3.1}$$

Here we assume the averaging volume ΔV is large enough to contain many particles, but small enough that any macroscopic variation in the material medium is on scales much larger than $\Delta V^{1/3}$. Since the dimensions of electric dipole moment are charge time length, **P** has units of charge per area. Similary, since the dimensions of magnetic dipole moment are current time area, **M** has units of current per length.

A variation of $\mathbf{P}(\mathbf{r},t)$ with space can be an accompanied by a "polarization" charge density. The easiest way to see this is to think of a long cylinder of uniformly polarized material with cross sectional area *A* and length *l* (see the left image of Figure 24-0-1). The total dipole moment of this cylinder is lAP, and we expect therefore from far away it should look like a charge +*Q* on the top end accompanied by a charge –*Q* on the bottom end such that the dipole moment of this arrangement, Ql, is equal to lAP. This means that there must be a dipole surface charge density on the ends of the cylinder of $\sigma_{pol} = Q / A = P$.

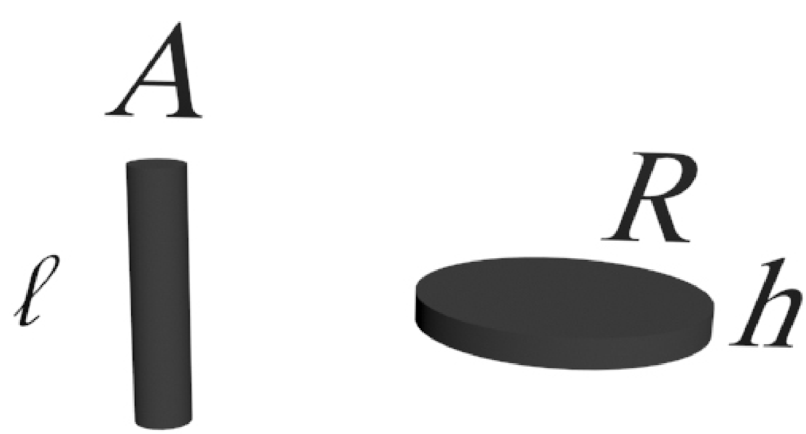

Figure 24-0-1: A uniform polarized "needle" and "disk"

Similary, a variation of $\mathbf{M}(\mathbf{r},t)$ with space can be accompanied by a "magnetization" current. . The easiest way to see this is to think of a disk of uniformly magnetized material with cross sectional radius *R* and height *l* (see the right image of Figure 24-0-1). The total magnetic dipole moment of this cylinder is $h\pi R^2 M$, and we expect therefore from far away it should look like a disk with a current *I* running around its circumference such that the dipole moment $I\pi R^2$ of this arrangement is equal to $h\pi R^2 M$. This means that there must be a current per unit height of the disk given by $\kappa_{mag} = I/h = M$.

When we go over to a non-uniform **P** amd **M**, then we expect there to be corresponding polarization charge densities and current densities given by

$$\rho_{pol} = -\nabla\cdot P \qquad\qquad \mathbf{J}_{mag} = \nabla\times\mathbf{M} \tag{26.3.2}$$

At the boundary of a dielectric or magnetic material, we will have polarization surface charges and magnetization surface currents given by

$$\sigma_{pol} = \hat{\mathbf{n}}\cdot\mathbf{P} \qquad\qquad \boldsymbol{\kappa}_{mag} = \mathbf{M}\times\hat{\mathbf{n}} \tag{26.3.3}$$

where $\hat{\mathbf{n}}$ is the normal pointing out of the material medium.

The polarization charge density in (26.3.2) will be accompanied by a polarization current to conserve polarization charge, that is there is a $\mathbf{J}_{pol}$ such that

$$\frac{\partial\rho_{pol}}{\partial t} + \nabla\cdot\mathbf{J}_{pol} = 0 \qquad\qquad \Rightarrow \mathbf{J}_{pol} = \frac{\partial\mathbf{P}}{\partial t} \tag{26.3.4}$$

Since $\nabla\cdot\mathbf{J}_{mag} = \nabla\cdot\left(\nabla\times\mathbf{M}\right) = 0$, there is no corresponding "magnetization" charge density, nor would we expect one.

26.4 Unformly polarized spheres

26.4.1 A dielectric sphere

Suppose we have a uniformly polarized dielectric sphere of radius *R*, with the direction of polarization in the z-direction. The polarization surface charge on the surface of the sphere will be

$$\sigma_{pol} = \hat{\mathbf{n}}\cdot\mathbf{P} = \hat{\mathbf{r}}\cdot P\hat{\mathbf{z}} = P\cos\theta \tag{26.4.1}$$

We have seen that a surface charge on a sphere with this angular dependence will produce a sphere which has an electric field given by

$$\mathbf{E} = \begin{cases} -E_o\hat{\mathbf{z}} & r < R \\ \left[\begin{array}{l} \dfrac{2p_o\cos\theta}{4\pi\varepsilon_o r^3}\hat{\mathbf{r}} \\ +\dfrac{p_o\sin\theta}{4\pi\varepsilon_o r^3}\hat{\boldsymbol{\theta}} \end{array}\right] & r > R \end{cases} \tag{26.4.2}$$

where

$$E_o = \frac{P}{3\varepsilon_o} \qquad\qquad p_o = \frac{4\pi R^3}{3}P \tag{26.4.3}$$

26.4.2 A magnetized sphere

Suppose we have a uniformly polarized magnetic sphere of radius *R*, with the direction of polarization in the z-direction. The polarization surface current on the surface of the sphere will be

$$\kappa_{mag} = \mathbf{M}\times\hat{\mathbf{n}} = M\,\hat{\mathbf{z}}\times\hat{\mathbf{r}} = M\sin\theta\,\hat{\boldsymbol{\phi}} \tag{26.4.4}$$

We have seen that a surface current on a sphere with this angular dependence will produce a sphere which has a magnetic field given by

$$\mathbf{B} = \begin{cases} B_o\hat{\mathbf{z}} & r < R \\ \left[\begin{array}{l} \dfrac{\mu_o}{4\pi}\dfrac{2m_o\cos\theta}{r^3}\hat{\mathbf{r}} \\ +\dfrac{\mu_o}{4\pi}\dfrac{m_o\sin\theta}{r^3}\hat{\boldsymbol{\theta}} \end{array}\right] & r > R \end{cases} \tag{26.4.5}$$

where

$$B_o = \frac{2}{3}\mu_o M \qquad\qquad m_o = \frac{4\pi R^3}{3}M \tag{26.4.6}$$

26.5 Maxwell's Equations in the presence of matter

So the presence of spatial and temporal variations in **P** and **M** mean that we have to modify Maxwell's equations to include the assoicated currents and charge densities, since **E** and **B** are produced by all currents and charges, regardless of from which they arise. That is if ρ_{free} and $\mathbf{J}_{free}$ are the "free" charge density and current density (that is, the ones we control) we have

$$\rho = \rho_{free} + \rho_{pol} = \rho_{free} - \nabla\cdot\mathbf{P} \tag{26.5.1}$$

$$\mathbf{J} = \mathbf{J}_{free} + \mathbf{J}_{mag} + \mathbf{J}_{mag} = \mathbf{J}_{free} + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \tag{26.5.2}$$

Maxwell's equations with these additional terms now become become (Faraday's Law and the divergence of B is zero are unchanged)

$$\nabla \cdot \mathbf{E} = \frac{\rho_{free}}{\varepsilon_o} - \frac{\nabla \cdot \mathbf{P}}{\varepsilon_o} \tag{26.5.3}$$

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J}_{free} + \mu_o \nabla \times \mathbf{M} + \mu_o \frac{\partial \mathbf{P}}{\partial t} + \mu_o \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} \tag{26.5.4}$$

or

$$\nabla \cdot \left(\varepsilon_o \mathbf{E} + \mathbf{P} \right) = \rho_{free} \tag{26.5.5}$$

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_o} - \mathbf{M} \right) = \mathbf{J}_{free} + \frac{\partial}{\partial t} \left(\mathbf{P} + \varepsilon_o \mathbf{E} \right) \tag{26.5.6}$$

We define

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P} \tag{26.5.7}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_o} - \mathbf{M} \qquad \text{or} \qquad \mathbf{B} = \mu_o \left(\mathbf{H} + \mathbf{M} \right) \tag{26.5.8}$$

So then our equations are

$$\nabla \cdot \mathbf{D} = \rho_{free} \tag{26.5.9}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_{free} + \frac{\partial}{\partial t} \mathbf{D} \tag{26.5.10}$$

26.6 Linear media

We assume that we have media where the only polarization arises *because of external applied fields.* Note that this is *very* different from the situation we discussed in Section 26.4 above, where somehow the dipoles were aligned on their own (e.g. as in a permanent magnet) and we then looked to see what fields this polarization produced. Not only are we going to suppose that the polarization arises because of external fields, but we are going to assume there is a linear relationship between the external field and the resultant polarization, that is we assume that

$$\mathbf{P} = \varepsilon_o \chi_e \mathbf{E} \tag{26.6.1}$$

$$\mathbf{M} = \chi_m H \tag{26.6.2}$$

You might argue that if we are going to make **P** proportional to **E** we should correnspondingly make **M** proportional to **B**, not **H**, but this is the custom and we do not deviate from it. The constant of proportionality χ_e is called the electric susceptibility and the constant of proportionality χ_m is called the magnetic susceptibility. These are dimensionless quantities that vary from one substance to another. If we go back to (26.5.7) and (26.5.8), we see that for linear media, we have

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P} = \varepsilon_o \mathbf{E} + \varepsilon_o \chi_e \mathbf{E} = \varepsilon_o \left(1 + \chi_e\right) = \varepsilon \mathbf{E} \tag{26.6.3}$$

$$\mathbf{B} = \mu_o \left(\mathbf{H} + \mathbf{M}\right) = \mu_o \left(1 + \chi_m\right) \mathbf{H} = \mu \mathbf{H} \tag{26.6.4}$$

where

$$\mu = \mu_o \left(1 + \chi_m\right) \tag{26.6.5}$$

$$\varepsilon = \varepsilon_o \left(1 + \chi_e\right) \tag{26.6.6}$$

We call ε the permittivity of the material, and μ the permeability of the material. We also call $\left(1+\chi_e\right)$ the dielectric constant of the material, defined as

$$K_e = 1 + \chi_e \tag{26.6.7}$$

In the case of linear materials, we can therefore write Maxwell's equations as

$$\nabla \cdot \varepsilon \mathbf{E} = \rho_{free} \tag{26.6.8}$$

$$\nabla \times \mathbf{B} = \mu \mathbf{J}_{free} + \mu\varepsilon \frac{\partial}{\partial t} \mathbf{E} \tag{26.6.9}$$

26.7 Boundary conditions of E, B, H, and D and spheres in uniform fields

26.7.1 Boundary conditions

If you look at our boundary conditions on E and B in Section Sections 19.6 and 20.7 above, it is clear that with material media, the boundary conditions on D, H, B, and E are

$$\mathbf{E}_{2t} = \mathbf{E}_{1t} \tag{26.7.1}$$

$$B_{2n} = B_{1n} \tag{26.7.2}$$

$$\mathbf{H}_{2t} - \mathbf{H}_{1t} = \boldsymbol{\kappa}_{free} \times \hat{\mathbf{n}} \tag{26.7.3}$$

$$D_{2n} - D_{1n} = \hat{\mathbf{n}} \cdot \left(\mathbf{D}_2 - \mathbf{D}_1 \right) = \sigma_{free} / \varepsilon_o \tag{26.7.4}$$

26.7.2 A linear dielectric sphere in a uniform field

Suppose we have sphere made out of linear dielectric material with dielectric constant K_e sitting in a constant field in the z-direction, $\mathbf{E} = E_o \hat{\mathbf{z}}$. We are going to guess our solution is of the form

$$\mathbf{E} = \begin{cases} E_1 \hat{\mathbf{z}} & r < R \\ \begin{bmatrix} E_o \hat{\mathbf{z}} + \dfrac{2 p_1 \cos\theta}{4\pi\varepsilon_o r^3} \hat{\mathbf{r}} \\ + \dfrac{p_1 \sin\theta}{4\pi\varepsilon_o r^3} \hat{\boldsymbol{\theta}} \end{bmatrix} & r > R \end{cases} \tag{26.7.5}$$

where the unknowns here are E_1 and p_1, and we will determine these constants using the boundary conditions in the problem. At the poles we have from (26.7.4) that

$$\varepsilon_o K_e E_1 = \varepsilon_o \left(E_o + \frac{2 p_1}{4\pi\varepsilon_o R^3} \right) \tag{26.7.6}$$

and at the equator we have from (26.7.1) that

$$E_1 = E_o - \frac{p_1}{4\pi\varepsilon_o R^3} \tag{26.7.7}$$

Solving these equations gives

$$p_1 = 4\pi\varepsilon_o R^3 E_o \frac{\left(K_e - 1\right)}{\left(K_e + 2\right)} \tag{26.7.8}$$

$$E_1 = E_o \frac{3}{\left(2 + K_e\right)} \tag{26.7.9}$$

In the limit that $K_e = 1$, we recover the field we expect. In the limit that $K_e = \infty$, we have zero field inside the sphere. This limit is the same as if the sphere were made out of conducting material. We can find the polarization surface charge now that we have solved the problem by computing

$$\sigma_{pol} = \varepsilon_o \left(E_{2n} - E_{1n} \right) = \left(\varepsilon_o E_o \cos\theta + \frac{2 p_1 \cos\theta}{4\pi\varepsilon_o r^3} \right) - \varepsilon_o E_1 \cos\theta \tag{26.7.10}$$

$$\sigma_{pol} = \varepsilon_o \cos\theta \left(E_o - E_1 + \frac{2p_1}{4\pi\varepsilon_o R^3} \right) = \varepsilon_o E_o \cos\theta \left(1 - \frac{3}{(2+K_e)} + \frac{2(K_e - 1)}{(K_e + 2)} \right)$$
$$\sigma_{pol} = \varepsilon_o E_o \cos\theta \frac{3(K_e - 1)}{(2+K_e)} \tag{26.7.11}$$

Again we get the corrent limits when $K_e = 1$ and when $K_e = \infty$. When $K_e = 1$ the polarization charge vanishes, as we expect, and when $K_e = \infty$ the polarization charge is $\sigma_{pol} = 3\varepsilon_o E_o \cos\theta$, which is what we need to make the electric field just outside the sphere at the pole to drop from $3E_o$ to zero just inside the pole. The field in this case ($K_e = \infty$) is

$$\mathbf{E} = E_o \begin{cases} 0 \quad r < R \\ \begin{bmatrix} \hat{\mathbf{z}} + 2\cos\theta \left(\frac{R}{r} \right)^3 \hat{\mathbf{r}} \\ + \sin\theta \left(\frac{R}{r} \right)^3 \hat{\boldsymbol{\theta}} \end{bmatrix} \quad r > R \end{cases} \tag{26.7.12}$$

We see that the presence of the linear dielectric decreased the electric field inside the sphere below what it would have otherwise be with out the dielectric, as we expect. In the corresponding magnetic material case (a sphere with permeability μ sitting in a constant magnetic field $\mathbf{B} = B_o \hat{\mathbf{z}}$), the magnetic field inside the sphere will be enhanced instead of diminished, as we expect.

26.8 Conservation of energy in linear media

We can carry out the following formal manipulations, assuming μ and ε are not functions of space, and using (26.5.10)

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})$$
$$= -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \left(\mathbf{J}_{free} + \frac{\partial}{\partial t} \mathbf{D} \right) = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial}{\partial t} \mathbf{D} - \mathbf{E} \cdot \mathbf{J}_{free} \tag{26.8.1}$$

Or

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \mathbf{E} \cdot \mathbf{D} + \frac{1}{2} \mathbf{B} \cdot \mathbf{H} \right) + \nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{E} \cdot \mathbf{J}_{free} \tag{26.8.2}$$

This suggests that we take the energy density in linear media to be

$$u = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{B}\cdot\mathbf{H} \tag{26.8.3}$$

and the energy flux density to be

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} \tag{26.8.4}$$

We could write down similar formal equations for the conservation of momentum, but this is a complicated subject and we refer the reader to e.g. Jackson's treatment.

26.9 Propagation speed of electromagnetic waves

Suppose we have no free currents or charges and are looking at the propagation of waves in the media. The taking the curl of (26.6.9), we have

$$\nabla\times\left(\nabla\times\mathbf{B}\right) = \mu\varepsilon\frac{\partial}{\partial t}\nabla\times\mathbf{E} = -\mu\varepsilon\frac{\partial^2\mathbf{B}}{\partial t^2} = -\nabla^2\mathbf{B} \tag{26.9.1}$$

where we have used Faraday's Law and the fact that $\nabla\cdot\mathbf{B} = 0$ in (26.9.1). We thus have

$$\nabla^2\mathbf{B} - \mu\varepsilon\frac{\partial^2\mathbf{B}}{\partial t^2} = 0 \tag{26.9.2}$$

which means that a plane wave will no longer propagate at speed speed of *c*, but rather at the speed

$$\frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\left(1+\chi_e\right)\left(1+\chi_m\right)}} \tag{26.9.3}$$

We define the index of refraction *n* of a medium by

$$n = \frac{c}{\omega/k} = \sqrt{\left(1+\chi_e\right)\left(1+\chi_m\right)} \tag{26.9.4}$$

27 Why is the speed of light not *c* in a dielectric?

27.1 Learning Objectives

We want to discuss the propagation of light when there is matter present. We look at the case where we only have a dielectric present, which leads to electromagnetic plane waves propagating at speeds different from the speed of light, as we saw in (26.9.3). We want to see what this means and how it is possible. To this end, we will make a "model" of a dielectric medium where we deduce a physical mechanism producing the dipole moment per unit volume **P**, and thus offer up a physical basis for the origin of (26.6.1).

27.2 Review of plane wave generated by an oscillating sheet of current

We know that a current sheet oscillating in the *y*-direction at the origin, $\mathbf{J}(x,t)=\hat{\mathbf{y}}\,\kappa(t)\,\delta(x)$, will generate electromagnetic plane waves propagating in the $+x$ and $-x$ direction, with **E** and **B** fields given by

$$\mathbf{E}(x,t)=-\hat{\mathbf{y}}\frac{c}{2}\mu_o\,\kappa\left(t-\frac{|x|}{c}\right) \tag{27.2.1}$$

$$\mathbf{B}(x,t)=-\hat{\mathbf{z}}\frac{1}{2}\mu_o\,\kappa\left(t-\frac{|x|}{c}\right)sign(x) \tag{27.2.2}$$

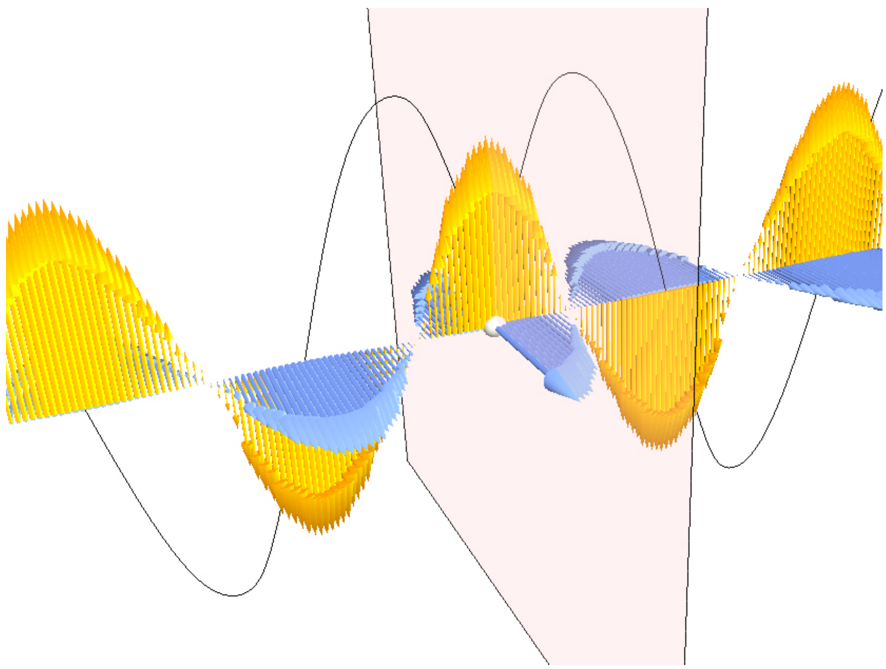

Figure 27-1: The E and B fields of an oscillating current sheet at *x = 0*

We will use these expressions below to calculate the electromagnetic wave generated by a thin dielectric sheet oscillating up and down.

27.3 The slow-down of electromagnetic waves traversing a thin dielectric sheet

27.3.1 The "polarization" current density

Suppose we add matter consisting of many point dipoles. For example, we could have a number of massive ions which we regard as fixed in space, with number density *n*, and with each of the ions having a movable electron attached to it by a spring with spring constant *k*. Suppose the vector separation between the electron and the ion for the i[th] ion/electron pair is $\Delta\mathbf{r}_e$, and the electron sees an electric field **E** and magnetic field **B** at its location. Then the non-relativistic equation of motion for the electron is

$$m_e \frac{d}{dt}\mathbf{u}_e = -e\left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}\right) - k\Delta\mathbf{r}_e \tag{27.3.1}$$

We will assume that that $cB \le E$ and that the speed of the electron is much less than c, in which case we can neglect the $\mathbf{u}_e \times \mathbf{B}$ compared to the **E** term in (27.3.1), so that we have

$$m_e \frac{d}{dt}\mathbf{u}_e = -e\mathbf{E} - k\Delta\mathbf{r}_e \tag{27.3.2}$$

or

$$\frac{d}{dt}\mathbf{u}_e = -\frac{e}{m_e}\mathbf{E} - \omega_o^2\Delta\mathbf{r}_i \qquad \omega_o^2 = \frac{k}{m_e} \tag{27.3.3}$$

If we now assume that the electric field varies as $\cos(\omega t)$ and that $\omega \ll \omega_o = \sqrt{k/m_e}$, we can neglect the $\frac{d}{dt}\mathbf{u}_e$ term in (27.3.3) compared to the $\omega_o^2\Delta\mathbf{r}_i$ term, giving us

$$0 = -\frac{e}{m_e}\mathbf{E} - \omega_o^2\Delta\mathbf{r}_e \qquad \Rightarrow \quad \Delta\mathbf{r}_e(t) = -\frac{e}{\omega_o^2 m_e}\mathbf{E}(t) \tag{27.3.4}$$

What (27.3.4) tells us is that if the electron sees an electric field varying with time at its position, and if the frequency of the electric field variation is low compared to the natural oscillation frequency of the electron, ω_o, then the position of the electron will vary with the electric field **E**, and its velocity will vary as

$$\mathrm{u}_e(\mathrm{t}) = \frac{d}{dt}\Delta\mathbf{r}_e(t) = -\frac{e}{\omega_o^2 m_e}\frac{d}{dt}\mathbf{E}(t) \tag{27.3.5}$$

This means that a time varying electric field in a dielectric will induce a current due to the motion of the “bound” charges, and this is called the “polarization current”. We can compute this current using (27.3.5) and the definition of current density to get

$$\mathbf{J}_{polarization}(t) = -ne\mathrm{u}_e(t) = +\frac{ne^2}{\omega_o^2 m_e}\frac{d}{dt}\mathbf{E}(t) \tag{27.3.6}$$

It is clear that if we have an electric field that varies in space and time, then (27.3.6) will generalize to

$$\mathbf{J}_{polarization}(\mathbf{r},t) = -ne\mathrm{u}_e(t) = +\frac{ne^2}{\omega_o^2 m_e}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) \tag{27.3.7}$$

We define

$$\chi_e = \frac{n e^2}{\varepsilon_o \omega_o^2 m_e} \tag{27.3.8}$$

and with this definition (27.3.7) becomes

$$\mathbf{J}_{polarization}(\mathbf{r},t) = \varepsilon_o \chi_e \frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) \tag{27.3.9}$$

Thus we have come up with a physical model which explains why we would have a relation like (26.6.1).

27.3.2 Change in the speed in the dielectric due to the polarization current

We can easily see that the polarization current given in (27.3.9) will change the speed of electromagnetic waves in the dielectric. We saw above that if we only have a dielectric, and $\mu = \mu_o$, the velocity of propagation of a plane electromagnetic wave is ***NOT*** the speed of light *c*, but

$$\frac{\omega}{k} = \frac{1}{\sqrt{\mu_o \varepsilon_o (1+\chi_e)}} = \frac{c}{\sqrt{(1+\chi_e)}} = \frac{c}{\sqrt{K_e}} \tag{27.3.10}$$

where we have defined the dielectric constant K_e such that

$$K_e = 1 + \chi_e \tag{27.3.11}$$

So the addition of our bound charges and their associated polarization currents in the presence of time changing electric fields leads to a slowing of the propagation speed of electromagnet waves. How can this be?

That is, how does the presence of a dielectric slow down an electromagnetic wave propagating through it? What happens is that the time-varying electric field of the incoming wave drives an oscillating current in the dielectric. These oscillating current sheets, of necessity, must generate electromagnetic waves. The new waves are out of phase with the incident wave, and as a result of the interference between these two waves, the phase of the combination differs from the phase of the incident light. We observe this change in phase as a change in speed. Let us make this qualitative description quantitative[8].

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[8] This treatment is suggested by a similar approach found in Chapter 31 of Feynman, Leighton, and Sands, *The Feynman Lectures on Physics, Vol 1*, Addison-Wesley, 1963.

27.3.3 The effects of a thin dielectric slab

Suppose we have totally empty space except for a thin dielectric slab lying in the *y-z* plane at *x = 0*. A plane electromagnetic wave whose electric field is polarized in the *y*-direction propagates in the +*x*-direction and encounters the thin dielectric slab of width *D* with dielectric constant K_e . We assume that this is very close to one, that so that the speed of light in the slab is only slightly less than *c*. If there we no dielectric present, the wave would propagate through the slab in a time *D/c*. We now show that the additional time Δt it takes to get through the slab because the thin dielectric slab "slows" down the light can be written as $\Delta t = D\chi_e / 2c$, which corresponds to the slow-down we expect because the speed of the wave in the slab is no longer *c* but $c/\sqrt{1+\chi_e} \approx c - c\chi_e/2$, so that the time to "get through" the slab is no longer D/c but $\dfrac{D}{c(1-\chi_e/2)} \approx \dfrac{D}{c} + \dfrac{D\chi_e}{2c}$.

The electric field of our incident plane wave is given by

$$\delta\mathbf{E}_K = \hat{\mathbf{y}}\,\delta E_o \cos\omega\left(t - \frac{x}{c}\right) \tag{27.3.12}$$

as shown in the top wave form in Figure 27-2. We put at the origin a dielectric sheet with dielectric constant $K_e = 1 + \chi_e$ and width *D*, which we assume is small compared to a wavelength of our incoming wave, that is $\lambda = 2\pi c/\omega >> D$. Because of the presence of the bound charges in the slab, which will oscillate up and down under the influence of the electric field of the incident wave, we will see a polarization current density given by (see (27.3.9) and (27.3.12))

$$\mathbf{J}_{bound} = \varepsilon_o \chi_e \frac{\partial}{\partial t}\mathbf{E} = -\varepsilon_o \chi_e \delta E_o \omega \sin(\omega t)\,\hat{\mathbf{y}} \tag{27.3.13}$$

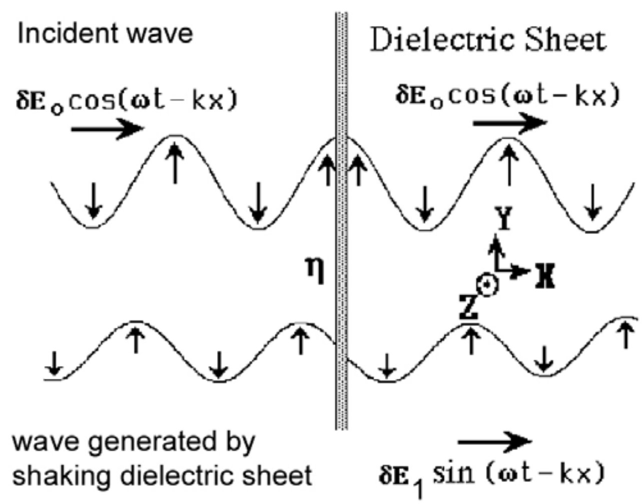

Figure 27-2: An electromagnetic wave encoungers a thin dielectric sheet

Thus the electric field of our incoming wave sets up an oscillating sheet of current with current per unit length in the *y*-direction of

$$\kappa(t) = -\varepsilon_o \chi_e \delta E_o D\omega \sin\left(\omega t\right) \qquad (27.3.14)$$

Using equation (27.2.1) above, for $x > 0$, we see that the electric field of the wave *generated by this current sheet* is given by

$$\delta\mathbf{E}_1 = -\hat{\mathbf{y}}\frac{c}{2}\mu_o\,\kappa(t-\frac{x}{c}) = \hat{\mathbf{y}}\left[\frac{\omega D\chi_e}{2c}\right]\delta E_o \sin\omega\left(t-\frac{x}{c}\right) \quad for\ x\ >\ 0 \qquad (27.3.15)$$

This wave is shown by the lower wave form in Figure 27-2 as a function of *x* for a given instant of time $t = 0$. The total electric field for $x > 0$ is thus

$$\delta\mathbf{E}_{total} = \delta\mathbf{E}_0 + \delta\mathbf{E}_1 = \hat{\mathbf{y}}\delta E_o\left\{\cos\omega\left(t-\frac{x}{c}\right) + \frac{\omega D\chi_e}{2c}\sin\omega\left(t-\frac{x}{c}\right)\right\} \quad for\ x\ >\ 0$$

(27.3.16)

To see that (27.3.16) implies that the wave is delayed in going through the dielectric, we assume that

$$\frac{\omega D\chi_e}{2c} = \frac{\pi D\chi_e}{\lambda} << 1 \qquad (27.3.17)$$

where λ is the wavelength of the wave. We now use the fact that if $\beta << 1$ then

$$\cos\left(\psi-\beta\right) = \cos\psi\cos\beta + \sin\psi\sin\beta \approx \cos\psi + \beta\sin\psi \qquad (27.3.18)$$

And comparing (27.3.18) to (27.3.16) assuming (27.3.17) allows us to write (27.3.16) as

$$\delta\mathbf{E}_{total}(x,t) = \hat{\mathbf{y}}\delta E_o\cos\omega\left(t-\Delta t-\frac{x}{c}\right) \quad for\ x\ >\ 0 \qquad (27.3.19)$$

where $\Delta t = D\chi_e / 2c$. Equation (27.3.19) shows that the peaks in amplitude for x > 0, are delayed by a time Δt from the time we expect if there were no dielectric, and this delay in the peak corresponds the a longer time for the wave to "get through" the dielectric sheet, as we noted above.

Note that we have gotten this delay by adding up two waves in (27.3.16), ***both of which are traveling at the speed of light.***

28 Interactions with matter II

28.1 Learning Objectives

We turn now from insulators to conductors. We first look at the "microscopic" relation between **J** and **E** in conductors, or Ohm's Law, and then move on to consider the consequences of this relationship.

28.2 The microscopic form of Ohm's Law

In constrast to our treatment of the relationship between **J** and **E** for dielectrics in Section 27.3.1 above, we consider here a different model for currents in a material. Again, we suppose we have a number of massive ions which we regard as fixed in space, with number density *n*, with an equal number of electrons, but now with the electrons free to move and not bound to the ions. However, the electrons do "collide" with the ions, and this results in a frictional drag. Suppose the vector displacement of an electron from its rest position is $\Delta\mathbf{r}_e$, and that the electron sees an electric field **E** and magnetic field **B** at its location, with $\frac{d}{dt}\Delta\mathbf{r}_e = \mathbf{u}_e$. Then the non-relativistic equation of motion for the electron is

$$m_e \frac{d}{dt}\mathbf{u}_e = -e\left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}\right) - m_e \gamma_c \mathbf{u}_e \tag{28.2.1}$$

where γ_c is a collision frequency and the $-m_e\gamma_c\mathbf{u}_e$ term represents the frictional drag. Again, we assume that that $cB \le E$ and that the speed of the electron is much less than c, in which case we can neglect the $\mathbf{u}_e \times \mathbf{B}$ compared to the **E** term in (28.2.1), so that we have

$$m_e \frac{d}{dt}\mathbf{u}_e = -e\mathbf{E} - m_e \gamma_c \mathbf{u}_e \tag{28.2.2}$$

or

$$\frac{d}{dt}\mathbf{u}_e = -\frac{e}{m_e}\mathbf{E} - \gamma_c \mathbf{u}_e \tag{28.2.3}$$

If we now assume that the electric field varies as $\cos(\omega t)$ and that $\omega \ll \gamma_c$, we can neglect the $\frac{d}{dt}\mathbf{u}_e$ term in (28.2.3) compared to the $-\gamma_c\mathbf{u}_e$ term, giving us

$$0 = -\frac{e}{m_e}\mathbf{E} - \gamma_c \mathbf{u}_e \quad \Rightarrow \quad \mathbf{u}_e\left(t\right) = -\frac{e}{\gamma_c m_e}\mathbf{E}\left(t\right) \tag{28.2.4}$$

What (28.2.4) tells us is that if the electron sees an electric field varying with time at its position, and if the frequency of the electric field variation is low compared to the collision frequency, then the velocity of the electron will vary with the electric field **E** in the manner given in (28.2.4). This means that a time varying electric field in a conductor will induce a current due to the motion of the "free" electrons. We can compute this current using (27.3.5) and the definition of current density to get

$$\mathbf{J}(t) = -ne\mathrm{u}_e(t) = +\frac{n\,e^2}{\gamma_e m_e}\mathbf{E}(t) \tag{28.2.5}$$

It is clear that if we have an electric field that varies in space and time, then (28.2.5) will generalize to

$$\mathbf{J}(\mathbf{r},t) = -ne\mathrm{u}_e(t) = +\frac{n\,e^2}{\gamma_e m_e}\mathbf{E}(\mathbf{r},t) \tag{28.2.6}$$

We define the conductivity σ_c to be

$$\sigma_c = \frac{n\,e^2}{\gamma_e m_e} \tag{28.2.7}$$

and with this definition (28.2.6) becomes

$$\mathbf{J}(\mathbf{r},t) = \sigma_c\,\mathbf{E}(\mathbf{r},t) \tag{28.2.8}$$

28.3 Reflection of an electromagnetic wave by a conducting sheet

28.3.1 The conceptual basis

How does a very good conductor reflect an electromagnetic wave falling on it? In words, what happens is the following. The time-varying electric field of the incoming wave drives an oscillating current on the surface of the conductor, following Ohm's Law. That oscillating current sheet, of necessity, must generate waves propagating in both directions from the sheet. One of these waves is the reflected wave. The other wave cancels out the incoming wave inside the conductor. Let us make this qualitative description quantitative.

Suppose we have an infinite plane wave propagating to the right, generated by currents far to the left and not shown. Suppose that the electric and magnetic fields of this wave are given by

$$\mathbf{E}_{incident}(x,t) = \hat{\mathbf{y}}\,\delta E_o \cos\omega\left(t-\frac{x}{c}\right) \qquad \mathbf{B}_{incident}(x,t) = \hat{\mathbf{z}}\,\delta B_o \cos\omega\left(t-\frac{x}{c}\right) \tag{28.3.1}$$

as shown in the top wave form in the Figure 28-1. We put at the origin (*x = 0*) a conducting sheet with width *D*, which we assume is small compared to a wavelength of our incoming wave. This conducting sheet will *reflect* our incoming wave. How?

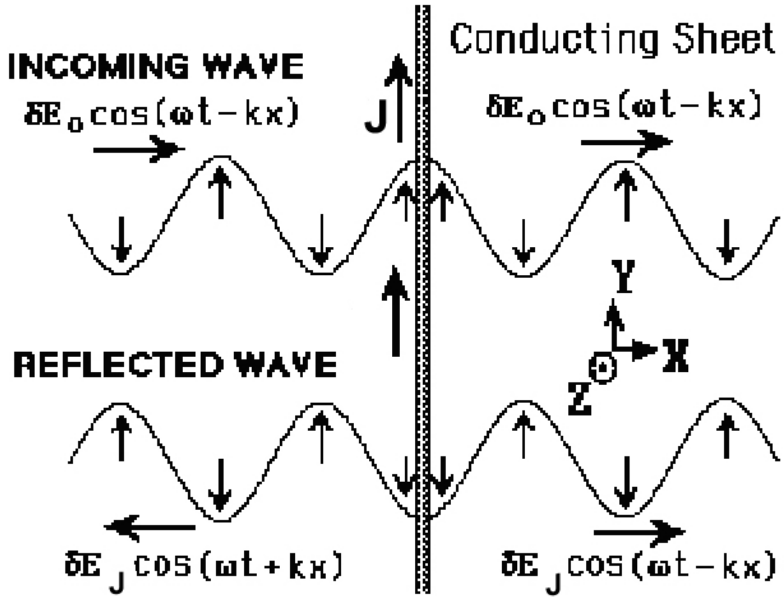

Figure 28-1: An incoming electromagnetic wave reflected by a conducing sheet.

The electric field of the incoming wave will cause a current $\mathbf{J} = \sigma_c \mathbf{E}$ to flow in the sheet, where σ_c is the conductivity. Moreover, the direction of $\mathbf{J}$ *will be in the same direction as the electric field of the incoming wave*, as shown in Figure 28-1. Thus our incoming wave sets up an oscillating sheet of current with current per unit length $\boldsymbol{\kappa} = D\mathbf{J}$. As in our discussion in Section 27.3.3, this current sheet *will generate electromagnetic waves*, moving both to the right and to the left (see Figure 28-1, lower wave form) away from the oscillating sheet of charge. What is the amplitude of these waves?

Using equation (27.2.1) above, for $x > 0$ the electric field of the wave generated by the current **J**, which we denote by $\mathbf{E}_\mathbf{J}(x,t)$ will be

$$\mathbf{E}_\mathbf{J}(x,t) = -\frac{c\mu_o}{2} D\mathbf{J}\left(t - \frac{|x|}{c}\right) \tag{28.3.2}$$

and this represents a wave propagating both to the left and to the right at the speed of light. The sign of this electric field at *x = 0*; it is *down* when the sheet of current is *up*, and vice-versa. Thus, for *x > 0*, the electric field $\mathbf{E}_\mathbf{J}(x,t)$ generated by the current in the sheet will always be *opposite* the direction of the electric field of the incoming wave, and *it will tend to cancel out the incoming wave for x > 0*. For a very good conductor, in fact, we show below that DJ will be equal to $2\delta E_o / c\mu_o$, so that for $x > 0$ we will have $\mathbf{E}_\mathbf{J}(x,t) = -\hat{\mathbf{y}}\delta E_o \cos(\omega t - kx)$. That is, for a very good conductor, the electric field of the wave generated by the current *will exactly cancel* the electric field of the incoming wave *for x > 0!*

And that's what a very good conductor does. It supports exactly the amount of current per unit length needed to cancel out the incoming wave for *x > 0* ($2\delta E_o / c\mu_o$, or equivalently $2\delta B_o / \mu_o$). For *x < 0, this same current* generates a "reflected" wave propagating back in the direction from which the original incoming wave came, *with the same amplitude as the original incoming wave*. This is how a very good conductor *totally* reflects electromagnetic waves. Below we show that our current density will in fact approach the value needed to accomplish this in the limit that the conductivity σ_C approaches infinity. We also have you obtain this result in the more standard manner in Problem Set 10.

28.3.2 The quantitative result in the limit of infinite conductivity

We show here that a perfect conductor will perfectly reflect an incident wave. To approach the limit of a perfect conductor, we first consider the finite resistivity case, and then let the conductivity go to infinity. As we pointed out above, the electric field of the incoming wave will, by Ohm's Law, cause a current $\mathbf{J} = \sigma_c \mathbf{E}$ to flow in the sheet, where σ_c is the conductivity. Since the sheet is assumed thin compared to a wavelength, we can assume that the entire sheet sees essentially the same electric field, so that $\mathbf{J}$ will be uniform across the thickness of the sheet, and outside of the sheet we will see fields appropriate to a equivalent surface current $\boldsymbol{\kappa} = D\mathbf{J}$. This current sheet will generate electromagnetic waves, moving both to the right and to the left, away from the oscillating current sheet. The total electric field, $\mathbf{E}_{total}(x,t)$, will be the sum of the incident electric field and the electric field generated by the current sheet. Using equations (28.3.1) and (28.3.2) above, we thus have for the total electric field the following expression:

$$\mathbf{E}_{total}(x,t) = \mathbf{E}_{incident}(x,t) + \mathbf{E}_{\mathbf{J}}(x,t) = \hat{\mathbf{y}}\delta E_o \cos\omega\left(t - \frac{x}{c}\right) - \frac{c\mu_o}{2} D\mathbf{J}\left(t - \frac{|x|}{c}\right) \qquad (28.3.3)$$

We also have a relation between the current density $\mathbf{J}$ and $\mathbf{E}_{total}$ from Ohm's Law, which is

$$\mathbf{J}(t) = \sigma_c \mathbf{E}_{total}(0,t) \qquad (28.3.4)$$

Where $\mathbf{E}_{total}(0,t)$ is the total electric field at the position of the conducting sheet (which remember is very thin compared to a wavelength of the wave). Note that is appropriate to use the ***total*** electric field in Ohm's Law--the currents arise from the total electric field, irrespective of the origin of that field. So we have

$$\boldsymbol{\kappa}(t) = D\mathbf{J}(t) = D\sigma_c \mathbf{E}_{total}(0,t) \qquad (28.3.5)$$

If we look at (28.3.3) at $x = 0$, we have

$$\mathbf{E}_{total}(0,t) = \hat{\mathbf{y}}\,\delta E_o \cos\omega(t) - \frac{c\mu_o}{2} D\mathbf{J}(t) \tag{28.3.6}$$

or using (28.3.5)

$$\mathbf{E}_{total}(0,t) = \hat{\mathbf{y}}\,\delta E_o \cos\omega(t) - \frac{c\mu_o}{2} D\sigma_c \mathbf{E}_{total}(0,t) \tag{28.3.7}$$

We can now solve equation (28.3.7) for $\mathbf{E}_{total}(0,t)$, with the result that

$$\mathbf{E}_{total}(0,t) = \left[\frac{1}{1+\frac{c\mu_o D\sigma_c}{2}}\right]\hat{\mathbf{y}}\,\delta E_o \cos\omega(t) \tag{28.3.8}$$

and therefore, using equation (28.3.8) and (28.3.5)

$$\boldsymbol{\kappa}(t) = D\sigma_c \mathbf{E}_{total}(0,t) = \left[\frac{D\sigma_c}{1+\frac{c\mu_o D\sigma_c}{2}}\right]\hat{\mathbf{y}}\,\delta E_o \cos\omega(t) \tag{28.3.9}$$

If we take the limit that σ_c approaches infinity (no resistance, that is, a perfect conductor), then we can easily see using equation (28.3.8) that $\mathbf{E}_{total}(0,t)$ goes to zero, and that using equation (28.3.9)

$$\boldsymbol{\kappa}(t) = \hat{\mathbf{y}}\,\frac{2\delta E_o}{c\mu_o}\cos\omega(t) = \hat{\mathbf{y}}\,\frac{2\delta B_o}{\mu_o}\cos\omega(t) \tag{28.3.10}$$

In this same limit equation (28.3.3) becomes

$$\mathbf{E}_{total}(x,t) = \hat{\mathbf{y}}\,\delta E_o \left[\cos\omega\left(t - \frac{x}{c}\right) - \cos\omega\left(t - \frac{|x|}{c}\right)\right] \tag{28.3.11}$$

or

$$\mathbf{E}_{total}(x,t) = \begin{cases} 0 \quad \text{for } x > 0 \\ \hat{\mathbf{y}}\,\delta E_o \left[\cos\omega\left(t - \frac{x}{c}\right) - \cos\omega\left(t + \frac{x}{c}\right)\right] \quad \text{for } x < 0 \end{cases} \tag{28.3.12}$$

Again in the same limit of infinite conductivity, our total magnetic fields become

$$\mathbf{B}_{total}(x,t) = \begin{cases} 0 \quad \text{for } x > 0 \\ \hat{\mathbf{z}}\delta B_o \left[\cos\omega\left(t - \frac{x}{c}\right) + \cos\omega\left(t + \frac{x}{c}\right) \right] \quad \text{for } x < 0 \end{cases} \tag{28.3.13}$$

Thus, we see that we get ***no electromagnetic wave for x > 0***, and standing electromagnetic waves for $x < 0$. Note that right at $x = 0$, the total electric field vanishes. The current per unit length $\boldsymbol{\kappa}(t) = \hat{\mathbf{y}}\,\frac{2\delta B_o}{\mu_o}\cos\omega(t)$ is just the current per length we need to bring the magnetic field down from its value at $x < 0$ to zero for $x > 0$

You may be perturbed by the fact that in the limit of a perfect conductor, the electric field vanishes at $x = 0$, since it is the electric field at $x = 0$ that is driving the current there! But this is ok in this limit, since the conductivity is going to infinity in the same limit. In the limit of very small resistance, the electric field required to drive any current you want can go to zero, because the product of the infinite conductivity and the zero electric field can assume any value you need. That is, even a very small value of the electric field can generate a perfectly finite value of the current.

28.4 Radiation pressure on a perfectly conducting sheet

In the process of the reflection, there is a force per unit area exerted on the perfect conductor. This is just the $\mathbf{J}\times\mathbf{B}$ force due to the current flowing in the presence of the magnetic field of the incoming wave. If we calculate the total force $d\mathbf{F}$ acting on a cylindrical volume with area dA and length D of the conductor, using (28.3.10) and (28.3.13) we find that it is in the $+x$ direction, as follows

$$\begin{aligned} d\mathbf{F}(t) &= dA\,D\mathbf{J}\times\mathbf{B}_{incident}(0,t) \\ &= dA\,\boldsymbol{\kappa}(t)\times\mathbf{B}_{incident}(0,t) \\ &= \hat{\mathbf{x}}\,dA\left[\frac{2\delta B_o^2}{\mu_o}\right]\cos^2(\omega t) \\ &= \hat{\mathbf{x}}dA\left(\frac{2\delta E_o\delta B_o}{c\mu_o}\right)\cos^2(\omega t) \end{aligned} \tag{28.4.1}$$

so that the force per unit area, $d\mathbf{F}/dA$, or the radiation pressure, is just twice the Poynting flux divided by the speed of light c. Note that in (28.4.1) we use on the magnetic field due to the incident wave. Including the magnetic field due to the generated wave is unnecessary, since that wave reverses sign across $x = 0$ and thus generates net force on the sheet. The factor of two is appropriate for total reflection, as is the case here. For total absorption, the factor of two becomes unity. We can obtain this same expression by integrating the Maxwell stress tensor over surface of the same cylindrical volume.

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