DISTRIBUTIONS AND THE DIRAC DELTA FUNCTION

Technically, δ(x) is not a function at all, since its value is not finite at x = 0; in the mathematical literature it is known as a generalized function, or distribution.

But what is a distribution?

It is, if you like, the limit of a sequence of functions, such as rectangles $R_n(x)$, of height $n$ and width $1/n$, or isosceles triangles $T_n(x)$, of height $n$ and base $2/n$.

For any value of $\sigma$, the Gaussian integrates to one:

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)} \implies \int_{-\infty}^{\infty} g_\sigma(x) \, dx = 1.$$
Can We Define \( \delta(x) \equiv \lim_{\sigma \to 0} g_\sigma(x) \)?

Unfortunately, we **can’t**!

Why not?

Because even though

\[
\lim_{\sigma \to 0} \int_{-\infty}^{\infty} g_\sigma(x) \, dx = 1 ,
\]

it nonetheless turns out that

\[
\int_{-\infty}^{\infty} \left[ \lim_{\sigma \to 0} g_\sigma(x) \right] \, dx = 0 .
\]

It is crucial that we integrate first, and then take the limit!

---

You Can’t Believe Everything You Read

Griffiths goes on to write:

If \( f(x) \) is some “ordinary” function (that is, not another delta function—in fact, just to be on the safe side, let’s say that \( f(x) \) is continuous), then the product \( f(x)\delta(x) \) is zero everywhere except at \( x = 0 \). It follows that

\[
f(x)\delta(x) = f(0)\delta(x) .
\] (1.88)

(This is the most important fact about the delta function, so make sure you understand why it is true: since the product is zero anyway [anywhere?] except at \( x = 0 \), we may as well replace \( f(x) \) by the value it assumes at the origin.) In particular

\[
\int_{-\infty}^{\infty} f(x)\delta(x) \, dx = f(0) \int_{-\infty}^{\infty} \delta(x) \, dx = f(0) .
\] (1.89)

Under an integral, then, the delta function “picks out” the value of \( f(x) \) at \( x = 0 \).

---

Discussion of Griffiths’ Statement

Griffiths’ conclusion, that

\[
\int_{-\infty}^{\infty} f(x)\delta(x) \, dx = f(0) \int_{-\infty}^{\infty} \delta(x) \, dx = f(0)
\] (1.89)

is correct, but it does not follow from the fact that \( f(x)\delta(x) = 0 \) except at \( x = 0 \). We will see shortly that \( f(x)\delta'(x) \), where \( \delta'(x) \) is the derivative of the delta function, is also zero except at \( x = 0 \), but

\[
\int_{-\infty}^{\infty} f(x)\delta'(x) \, dx = -f'(0) \int_{-\infty}^{\infty} \delta(x) \, dx = -f'(0) .
\]

---

Griffiths Comes Very Close to the Target

Although \( \delta \) itself is not a legitimate function, integrals over \( \delta \) are perfectly acceptable. In fact, it’s best to think of the delta function as something that is always intended for use under an integral sign. In particular, two expressions involving delta functions (say, \( D_1(x) \) and \( D_2(x) \)) are considered equal if

\[
\int_{-\infty}^{\infty} f(x)D_1(x) \, dx = \int_{-\infty}^{\infty} f(x)D_2(x) \, dx ,
\]

for all (“ordinary”) functions \( f(x) \).

**Issue:** how can a mathematical definition depend on someone’s “intentions”??
**Continuation**

A continuation of this discussion can be found in Problem 8 of Problem Set 2.

**References**


**Dedication from Lighthill’s Book**

TO

PAUL DIRAC
who saw that it must be true

LAURENT SCHWARTZ
who proved it,

AND

GEORGE TEMPLE
who showed how simple it could be made