November 19, 2009

Massachusetts Institute of Technology

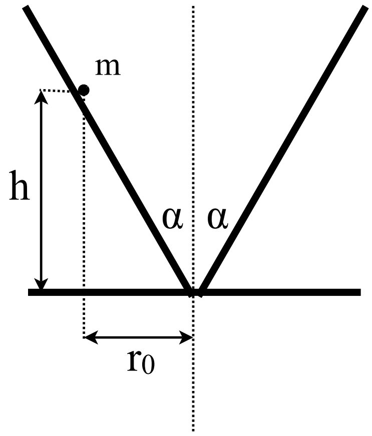
Department of Physics - Physics 8.09 - Fall 2009

Exam II

General comments

- This is a ‘closed-book’ exam. No books, notes or other reference material may be used except the attached equation sheet.
- The exam has three problems.
- Budget your time carefully!
- Good luck!

Problem 1 (20 points)



Consider a point mass m inside a cone with an opening angle 2α as shown above. The point mass experiences the impact of a uniform gravitational field. The height at which circular motion occurs is given by h.

(a) (4 points) Determine the Lagrange function.

(b) (2 points) Determine the radial equation of motion in terms of only r, g, m, α as well as potentially conserved quantities.

(c) (4 points) Determine the radius r_0 for circular motion in terms of g, m, α as well as potentially conserved quantities.

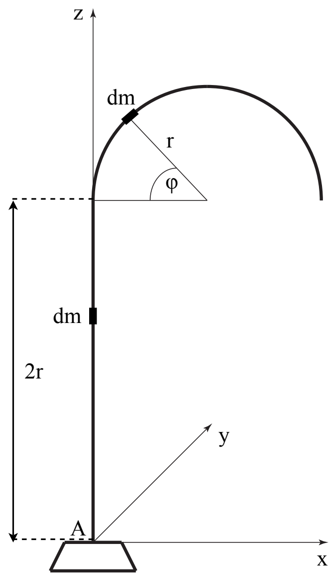
(d) (2 points) Determine the angular velocity ω_0 for circular motion in terms of g, h and α.

(e) (4 points) Consider now a small perturbation of circular motion r', i.e. $r = r_0 + r'$, giving rise to a small change of the radius r_0. Formulate the radial equation of motion in terms of r'.

(f) (2 points) Determine now the angular velocity for these oscillations around r_0 in terms of g, h and α.

(g) (2 points) Formulate a condition for closed orbital motion on both the angular velocity for circular motion and the angular velocity of small perturbations around r_0.
Hint: The solution does not require an elaborate calculation. Use your results from part d) and f).

Problem 2 (20 points)



A pole as shown above is bent at the end and is rotating around its straight section with a non-uniform angular velocity ω. The z-axis of the body system (x, y, z) coincides with the z_I axis of the inertial system (x_I, y_I, z_I). The straight section has a length of $2r$. The bent section has a radius of r. The bent pole has a uniform mass density per unit length of λ. The total mass of this arrangement is given by m.

(a) (2 points) Determine the mass of the straight section and bent section separately as a function of m.

(b) (4 points) Determine the moment of inertia tensor of the bent pole arrangement as shown above.

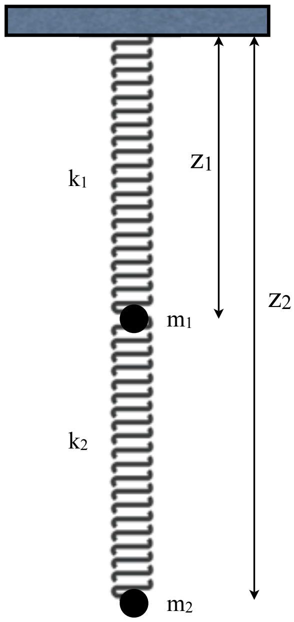
(c) (4 points) Determine the principal moments of the moment of inertia tensor. Hint: To simplify the calculation you can use for this step an approximation of $\pi \approx 3$!

(d) (2 points) Determine the angular momentum in the body system.

(e) (4 points) Determine the torque in the body system.

(f) (4 points) Determine the torque in the inertial system.

Problem 3 (20 points)



Consider the above spring arrangement in series. Mass m_1 is connected to spring 1 and mass m_2 is connected to spring 2. The mass of each spring can be ignored. Both masses are subject to a uniform gravitational field. The unstretched length of each spring is denoted by l_1 and l_2, respectively.

(a) (4 points) Determine the Lagrange function of this coupled spring system using the z location of each mass as generalized coordinates.

(b) (4 points) Determine the equations of motion.

(c) (4 points) Determine the equilibrium locations z_{01} and z_{02} for each mass.

(d) (4 points) Formulate now the equations of motion in terms of η_1 and η_2 where $\eta_1 = z_1 - z_{01}$ and $\eta_2 = z_2 - z_{02}$.

(e) (4 points) Determine the normal mode frequencies.

Equation sheet

- Moment of inertia tensor: $I_{ij} = \int \rho(x_1, x_2, x_3) \left[x_k x_k \delta_{ij} - x_i x_j\right] dV$
- Small angle approximation ($\theta \simeq 0$): $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$