October 18, 2011

Massachusetts Institute of Technology

Department of Physics - Physics 8.09 - Fall 2011

Exam I

General comments

- This is a ‘closed-book’ exam. No books, notes or other reference material may be used except the attached equation sheet.
- The exam has three problems.
- Budget your time carefully!
- Good luck!

Problem 1 (10 points)

A point mass m moves under the influence of an unknown force field on the following trajectory:

$$x = a \sin \omega t, \quad y = b \cos \omega t$$

(a) (2 points) Formulate the force field $\vec{F}(\vec{r})$ that gives rise to such a motion.

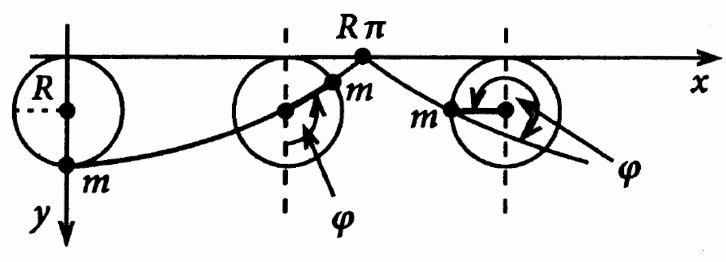
(b) (2 points) Show explicitly that this force field is conservative.

(c) (2 points) Determine the respective potential.

(d) (2 points) Show explicitly that the angular momentum is conserved.

(e) (2 points) Show explicitly that the total energy as the sum of the kinetic energy and potential energy is conserved.

Problem 2 (20 points)



A cycloid pendulum is realized by projecting the path of a point mass on the edge of a circular wheel as the wheel rolls along a straight line. This gives rise to the following functional dependence of x and y:

$$x = R\varphi + R\sin\varphi = R(\varphi + \sin\varphi) \quad y = 2R - R(1 - \cos\varphi) = R(1 + \cos\varphi)$$

(a) (3 points) Formulate the boundary conditions and transformation equations of this problem.

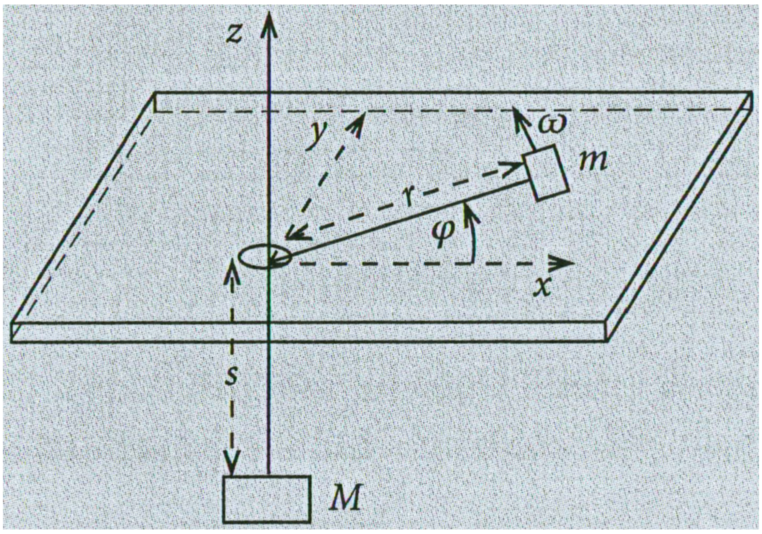
(b) (4 points) Show that the Lagrange function can be formulated as follows:
$L = T - V = mR(1 + \cos\varphi)(R\dot{\varphi}^2 + g)$

(c) (4 points) Determine the equation of motion.

(d) (5 points) Show explicitly using the equation of motion that the total energy is conserved and that this energy is equal to the total mechanical energy, i.e. the sum of kinetic energy and potential energy.

(e) (4 points) Determine the angular velocity ω and show that it does not depend on the amplitude.
Hint: Use the following substitution in the exact solution of the equation of motion in part (d), $u = \sin(\varphi/2)$ by first using the following replacements of $\sin\varphi = 2\sin(\varphi/2)\cos(\varphi/2)$ and $(1 + \cos\varphi) = 2\cos^2(\varphi/2)$ in your result from part (c).

Problem 3 (20 points)



A point mass m is rotating on a table. The point mass m is connected by a massless string of total length l ($l = r + s$) through a hole in the table to another point mass M underneath the table.

(a) (4 points) Formulate the boundary conditions and transformation equations for both masses m and M.

(b) (4 points) Determine the Lagrange function.

(c) (4 points) Determine the equations of motion.

(d) (2 points) Show explicitly using the equations of motions that the angular momentum of this system is conserved.

(e) (3 points) Show explicitly using the equations of motions that the total energy is conserved and that this energy is equal to the total mechanical energy, i.e. the sum of kinetic energy and potential energy.

(f) (2 points) Consider the case of $\ddot{s} = 0$. Determine the angular velocity for this special case.

(g) (1 points) Consider the case of $\dot{\varphi} = 0$. Determine the value of $\ddot{s}$ for this special case.

Equation sheet

- Nabla operator: $\vec{\nabla} = \left(\frac{\partial}{\partial x} \; , \; \frac{\partial}{\partial y} \; , \; \frac{\partial}{\partial z}\right)$
- Curl of a vector field $\vec{A}$: $\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \; , \; \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \; , \; \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$
- Divergence of vector field $\vec{A}$: $\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
- Lagrange equation for holonomic boundary conditions (generalized variable q_j with $j = 1, ..., 3N - k$ with N particles and k boundary conditions):
  $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \left(\frac{\partial L}{\partial q_j}\right) = 0$
- Small angle approximation ($\theta \simeq 0$): $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$