December 19, 2011

Massachusetts Institute of Technology

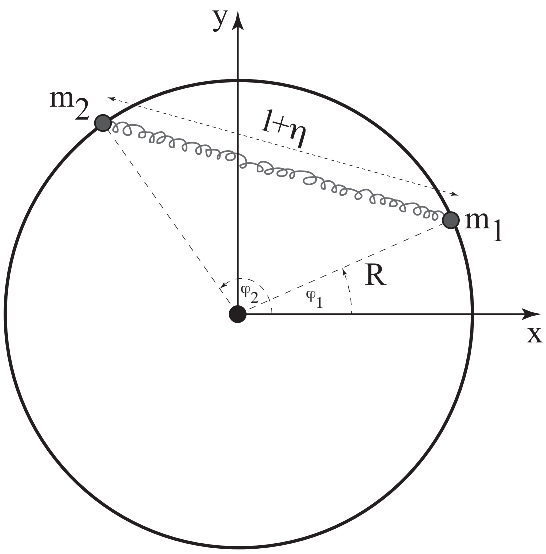
Department of Physics - Physics 8.09 - Fall 2011

Final Exam

General comments

- This is a ‘closed-book’ exam. No books, notes or other reference material may be used except the attached equation sheet.
- The exam has five problems.
- Budget your time carefully!
- Good luck!

Problem 1 (20 points): Lagrange mechanics



Two masses m_1 and m_2 are connected by a spring as shown above of unstretched length l and a spring constant k. The change of the unstretched length l is quantified by η. The motion of both masses is constrained to a circle of radius R without any friction and without the impact of a gravitational field.
Hint: It is useful to employ the following center-of-mass coordinates: χ and ψ with $M = m_1 + m_2$

$$\varphi_1 = \chi - \tfrac{m_2}{M}\psi \qquad \varphi_2 = \chi + \tfrac{m_1}{M}\psi$$

(a) (4 points) Formulate the boundary conditions of this problem and write down explicitly the transformation equations using the center-of-mass coordinates χ and ψ.

(b) (4 points) Determine the Lagrange function.

(c) (4 points) Determine the equations of motion.

(d) (4 points) Show explicitly using the equations of motion that the total energy is conserved and that this energy is equal to the total mechanical energy, i.e. the sum of kinetic energy and potential energy.

(e) (4 points) Solve the equations of motion using a linear approximation and discuss two extreme limits: $l > 2R$ and $l = \sqrt{2}R$.
Hint: First determine the stationary solution ψ_0 for $\dot{\psi} = 0$ and then use for the solution $\psi = \psi_0 + \delta(t)$ with $\delta(t)$ very small.

Problem 2 (20 points): Central Force Motion

The Earth-Moon separation changes due to tidal torques and the concomitant energy dissipation in the Earth's crust. The moon is in an approximately circular orbit around the earth, whose radius $a(t)$ changes slowly with time. The total energy E, and angular momentum L, of the system have two parts:

1. The orbital motion of the moon. For this problem treat the moon as a point mass m, in orbit around the earth which is considered as a fixed object (i.e. ignore the orbital motion of the earth around the sun and around the earth-moon center-of-mass). The Keplerian orbital angular frequency is then $\Omega_K = \sqrt{GM/a^3}$, where G is the universal gravitational constant, and M is the mass of the earth.

2. The earth rotates at an angular frequency ω_E, around an axis which is assumed to be perpendicular to the orbital plane. It can be treated as a sphere of uniform density with mass M, radius R, and moment of inertia $I = 2MR^2/5$.

(a) (5 points) Write down the total energy of the system, E.
Hint: Use the Virial theorem (Equation sheet) to express the kinetic energy contribution in terms of the potential energy. This is a critical step for the remaining problem!

(b) (5 points) Write down the total angular momentum of the system, L.

(c) (4 points) Assume that L is a constant of motion, but that E is continuously decreasing due to tidal dissipation. Write an expression for E in terms of constants of the problem (G, L, M, m, R) and the orbital separation a.
Hint: Use your explicit result from part a) and b)

(d) (2 points) Find how much the orbital separation, a, changes when E changes by a small amount, i.e. compute the derivative dE/da using your result from part c).

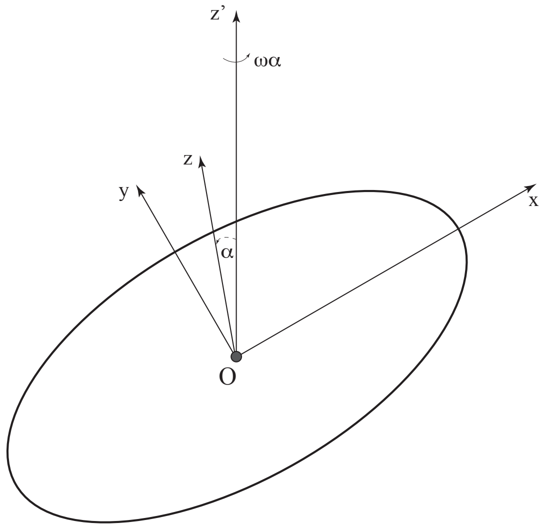
(e) (2 points) Energy is lost from the system due to tidal dissipation at a rate

$$\frac{dE}{dt} = I\omega_E\dot{\omega}_E = I\omega_E^2\left(\frac{\dot{\omega}_E}{\omega_E}\right) \equiv -\frac{I\omega_E^2}{\tau_{diss}},$$

where τ_{diss} is the tidal dissipation time-scale. Use this information to find an expression for the fractional rate of change of the earth-moon separation, i.e. find $\dot{a}/a$ based on your result in part d) and the expression above for dE/dt.

(f) (2 points) Find an analytic expression for the time-scale, $\tau_a \equiv a/\dot{a}$, for changes in the orbital separation, in terms of τ_{diss} and dimensionless ratios of m/M, a/R and Ω_K/ω_E. At this point, you may simplify your expression by noting that $\omega_E \gg \Omega_K$. Is the orbit expanding or shrinking?

Problem 3 (20 points): Rigid Body



A thin uniform disk of radius a and mass m is rotating freely on a frictionless bearing with uniform angular velocity ω about a fixed vertical axis passing through its center, and inclined at an angle α to the symmetry axis of the disc.

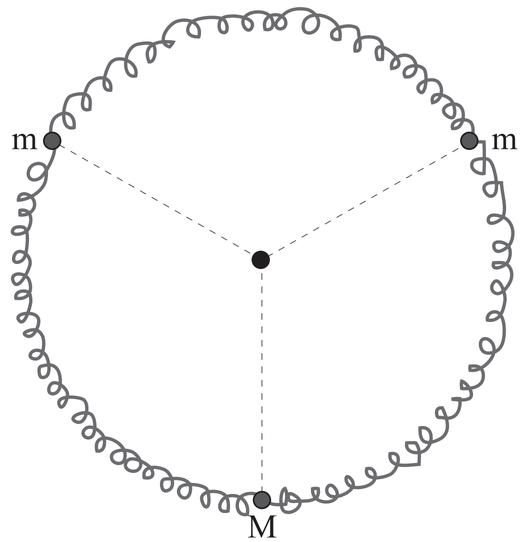
(a) (5 points) Determine the principal moments for the body system (x, y, z) as shown above.

(b) (5 points) Formulate the angular velocity vector ω in the body system (x, y, z).

(c) (5 points) Determine the angular momentum $\vec{L}$ in the body system (x, y, z).

(d) (5 points) What is the magnitude and direction of the torque τ relative to the body system (x, y, z)?

Problem 4 (20 points): Linear oscillations



Three point particles, two of mass m and one of mass M, are constrained to lie on a horizontal circle r. They are mutually connected by springs, each of constant k, that follow the arc of the circle and that are of equal length when the system is at rest as shown above. Assume that any motion of the masses involved stretch the spring with respect to their equilibrium position by only a small amount.

(a) (5 points) Determine the Lagrange function of this coupled spring system using the angular displacements θ_1, θ_2 and θ_3 with respect to the respective equilibrium locations.

(b) (5 points) Determine the equations of motion.

(c) (5 points) Formulate the equations of motion for the following coordinate transformations α, β and γ:

$$\alpha = \theta_1 + \frac{M}{m}\theta_2 + \theta_3$$
$$\beta = \theta_1 - \theta_3$$
$$\gamma = \theta_1 - 2\theta_2 + \theta_3$$

(d) (5 points) Determine the frequency for each of the above modes ω_α, ω_β and ω_γ.

Problem 5 (20 points): Hamilton mechanics

Consider a mechanical system with the following Hamilton function $H = \frac{1}{2m}p^2q^4 + \frac{k}{2q^2}$ and a generator function $F_1(q, Q) = -\sqrt{mk}\frac{Q}{q}$.

(a) (5 points) Determine the transformation equations $p = p(Q, P)$ and $q = q(Q, P)$.

(b) (5 points) Show explicitly using Poisson brackets that the transformations in a) are canonical.

(c) (5 points) Formulate the new Hamilton function $K = K(Q, P)$.

(d) (5 points) Determine the canonical equations for K and formulate the solutions for Q and P.

Equation sheet

- Nabla operator: $\vec{\nabla} = \left(\frac{\partial}{\partial x} \; , \; \frac{\partial}{\partial y} \; , \; \frac{\partial}{\partial z} \right)$
- Curl of a vector field $\vec{A}$: $\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \; , \; \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \; , \; \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$
- Divergence of vector field $\vec{A}$: $\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
- Lagrange equation for holonomic boundary conditions (generalized variable q_j with $j = 1, ..., 3N - k$ with N particles and k boundary conditions):
  $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \left(\frac{\partial L}{\partial q_j} \right) = 0$
- Small angle approximation ($\theta \simeq 0$): $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$
- Moment of inertia tensor: $I_{ij} = \int \rho(x_1, x_2, x_3) \left[x_k x_k \delta_{ij} - x_i x_j \right] dV$
- Effective potential for central force motion: $V_{eff} = V(r) + \frac{L^2}{2mr^2}$
- Virial theorem: $k\bar{V} = 2\bar{T}$ with $V(r) \sim r^k$
- Hamilton equations: $\dot{q} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial q}$
- Generator function F_1 of canonical transformation: $q_i, p_i \;\rightarrow\; Q_i, P_i$

$$p = \frac{\partial F_1}{\partial q} \;\; P = -\frac{\partial F_1}{\partial Q} \;\; K = H + \frac{\partial F_1}{\partial t}$$