HOMEWORK PROBLEM SET 10

PROBLEM 1

a) The cartesian coordinates of m_1 and m_2 are:

$$1 \longrightarrow (l_1 \sin\theta_1, \; -l_1 \cos\theta_1)$$
$$2 \longrightarrow (l_1 \sin\theta_1 + l_2 \sin\theta_2, \; -l_1\cos\theta_1 - l_2\cos\theta_2)$$

The velocities are:

$$(l_1 \cos\theta_1 \dot\theta_1, \; l_1 \sin\theta_1 \dot\theta_1)$$
$$(l_1 \cos\theta_1 \dot\theta_1 + l_2 \cos\theta_2 \dot\theta_2, \; l_1 \sin\theta_1 \dot\theta_1 + l_2 \sin\theta_2 \dot\theta_2)$$

The lagrangian of the system, in the small oscillation limit, is:

$$\boxed{L = \frac{1}{2}(m_1+m_2) l_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 \left(l_2^2 \dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2\right) + - \frac{1}{2}(m_1+m_2) g l_1 \theta_1^2 - \frac{1}{2} m_2 g l_2 \theta_2^2}$$

where we neglected some constant pieces.

The equations of motion are:

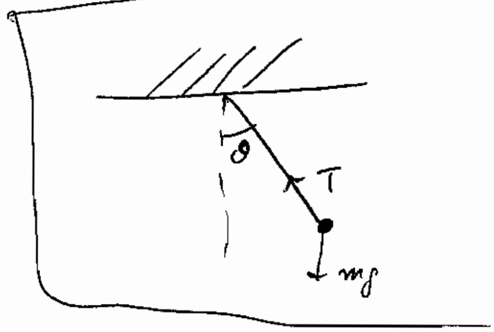
$$\boxed{\begin{aligned} &\ddot\theta_1 + \left(\frac{m_2}{m_1+m_2}\right)\frac{l_2}{l_1}\ddot\theta_2 + \frac{g}{l_1}\theta_1 = 0 \\ &\ddot\theta_1 + \frac{l_2}{l_1}\ddot\theta_2 + \frac{g}{l_1}\theta_2 = 0 \end{aligned}}$$

If we now plug the solutions: $\theta_i = \theta_{i0}\, e^{i\omega t}$ we get the equation:

$$\boxed{\frac{m_1}{m_1+m_2} l_1 l_2 \omega^4 - g(l_1+l_2)\omega^2 + g = 0}$$

Solving for ω we find the normal modes:

$$\boxed{\omega_{1,2}^2 = \frac{g}{2m_1 l_1 l_2}\left[(m_1+m_2)(l_1+l_2) \pm \sqrt{(m_1+m_2)^2(l_1+l_2)^2 - 4(m_1+m_2)m_1 l_1 l_2}\right]}$$



b) The forces acting on the mass m are: gravity and the tension T

We then have: $\boxed{T - mg\cos\vartheta = mr\dot{\vartheta}^2}$

If the string is shortened by dr, the work done by T is:

$$dW = \vec{T}\cdot d\vec{r} = -T\,dr = -(mg\cos\vartheta + mr\dot{\vartheta}^2)\,dr \simeq$$

$$\simeq -mg\,dr + \left(\frac{1}{2}mg\vartheta^2 - mr\dot{\vartheta}^2\right)dr = -mg\,dr + dE$$

change in the energy related to the oscillation

If we take the average: $\boxed{d\overline{E} = \left(\frac{1}{2}mg\overline{\vartheta^2} - mr\overline{\dot{\vartheta}^2}\right)dr}$ (1)

If we consider the small oscillation as a simple harmonic motion:

$$\vartheta = \vartheta_0\cos(\omega t + \delta) \Longrightarrow \boxed{\overline{\vartheta^2} = \frac{1}{2}\vartheta_0^2\,;\ \overline{\dot{\vartheta}^2} = \omega^2\,\overline{\vartheta^2}} \quad (2)$$

We can now write the average energy:

$$\boxed{\overline{E} = \frac{1}{2}mr^2\overline{\dot{\vartheta}^2} + \frac{1}{2}mgr\overline{\vartheta^2} = mgr\overline{\vartheta^2}} \quad (3)$$

We have now to put together equation (1) (2) and (3):

$$\boxed{d\overline{E} = -\frac{1}{2}\overline{E}\,\frac{dr}{r}} \Longrightarrow \boxed{\overline{E}\,r^{1/2} = \text{const}}$$

or equivalently: $\boxed{\vartheta_0^4 r^3 = \text{const}}$

PROBLEM 2

a) The system has 4 degrees of freedom (positions of the four masses along the circle).

Then there are <u>four normal modes</u>

b) We choose as generalised coordinates $(\xi_1, \xi_2, \xi_3, \xi_4)$, where ξ_i is the length of the arc of the mass i from its initial equilibrium position.

The kinetic energy is: $\boxed{T = \frac{1}{2} m\left(\dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2 + \dot{\xi}_4^2\right)}$

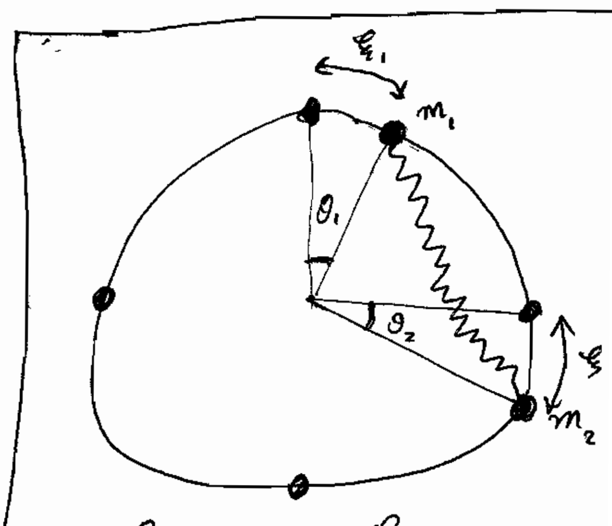
To compute the potential energy V we need to evaluate the length of each spring as a function of ξ_i.

At equilibrium the length of each spring is: $2b\sin\left(\frac{\pi}{4}\right)$

At generic ξ_i the spring connecting the masses i and $i+1$ has length:

$$2b\sin\left[\frac{1}{2}\left(\frac{\xi_{i+1}}{b} - \frac{\xi_i}{b} + \frac{\pi}{2}\right)\right]$$

See the figure for the spring between the masses 1 and 2. For the other springs is the same.



$$\theta_1 = \frac{\xi_1}{b} \; ; \; \theta_2 = \frac{\xi_2}{b}$$

The potential energy for each spring is:

$$V_i = \frac{1}{2} K \Delta x_i^2$$

where: $\Delta x_i = 2b\sin\left[\frac{1}{2}\left(\frac{\xi_{i+1}}{2} - \frac{\xi_i}{b} + \frac{\pi}{2}\right)\right] - 2b\sin\left(\frac{\pi}{4}\right)$

Since we are considering small oscillation we have: $\frac{\xi_i}{b} \ll 1$

We can Taylor expand the sin function and we get:

$$\Delta x_i \simeq \frac{1}{\sqrt{2}}\left(\xi_{i+1} - \xi_i\right)$$

and we finally get for the potential energy:

$$\boxed{V = \frac{1}{2} K\left(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 - \xi_1\xi_2 - \xi_2\xi_3 - \xi_3\xi_4 - \xi_4\xi_1\right)}$$

We can now write the matrices V and T

$$T = \begin{pmatrix} m & & & \\ & m & 0 & \\ & 0 & m & \\ & & & m \end{pmatrix} \qquad V = \begin{pmatrix} K & -\frac{K}{2} & 0 & -\frac{K}{2} \\ -\frac{K}{2} & K & -\frac{K}{2} & 0 \\ 0 & -\frac{K}{2} & K & -\frac{K}{2} \\ -\frac{K}{2} & 0 & -\frac{K}{2} & K \end{pmatrix}$$

and the frequencies of the small oscillations are given by the equation:

$$\boxed{\det\left(\omega^2 T - V\right) = 0}$$

We get the four solutions:

$$\boxed{\begin{matrix} \omega_1 = 0 & \omega_2 = 0 \\ \omega_3 = \sqrt{\frac{K}{m}} & \omega_4 = \sqrt{\frac{2K}{m}} \end{matrix}}$$

PROBLEM 3

a) The equation of motion for a central force field are:

$$\begin{cases} m\ddot{r} - m r \dot{\theta}^2 = f(r) \\ m r^2 \dot{\theta} = l = \text{const} \end{cases}$$

We can then find: $\dot{\theta} = \frac{l}{m r^2} \Longrightarrow \ddot{\theta} = \frac{d}{dt}\dot{\theta} = \frac{d}{dt}\left(\frac{l}{m r^2}\right) = -\frac{2l}{m r^3}\dot{r}$ (1)

We compute now the first and second derivative of r for the particular orbit:

$$\left.\begin{cases} r = a(1+\cos\theta) \\ \dot{r} = -a\sin\theta\,\dot{\theta} \\ \ddot{r} = -a\cos\theta\,\dot{\theta}^2 - a\sin\theta\,\ddot{\theta} = -a\left(\cos\theta\,\dot{\theta}^2 + \sin\theta\,\ddot{\theta}\right) \end{cases}\right\} \quad (2)$$

By substituting this expression into the equation of motion along the radial direction:

$$f(r) = -ma\left(\cos\theta\,\dot{\theta}^2 + \sin\theta\,\ddot{\theta}\right) - m r \dot{\theta}^2 \qquad (3)$$

We can now substitute the value of $\dot{\theta}$ and $\ddot{\theta}$ from eq. (1) into eq (3)

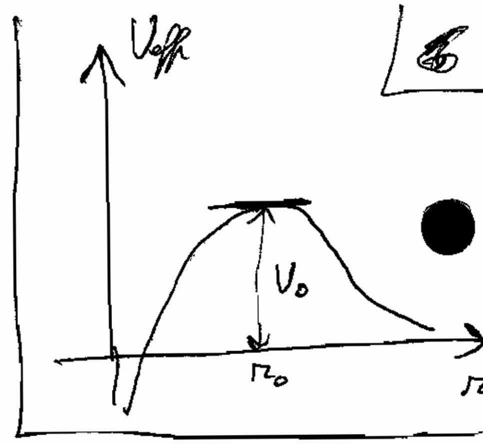
$$f(r) = -ma\left(\cos\theta\left(\frac{l}{m r^2}\right)^2 - \sin\theta\,\frac{2l}{m r^3}\dot{r}\right) - \frac{m r\, l^2}{m^2 r^4}$$

We now take $\dot{r}$ from equation (2) and then we get:

$$f(r) = -\frac{2a^2 l^2 \sin^2\theta + l^2 a r\cos\theta + l^2 r^2}{m r^5}$$

If we now use the orbit equation to eliminate the θ-dependence: $\boxed{f(r) = -\frac{3 l^2 a}{m r^4}}$

b) We call the potential: $V(r) = -\frac{\alpha}{r^4}$



The effective potential is: $V_{eff}(r) = \frac{\ell^2}{2mr^2} - \frac{\alpha}{r^4}$

If you plot this potential you find what is in the figure

The maximum for V_{eff} is given by: $\frac{dV_{eff}}{dr} = \frac{-\ell^2}{mr^3} + \frac{4\alpha}{r^5} = 0$

Therefore: $\boxed{r_0 = \sqrt{\frac{4m\alpha}{\ell^2}} \ ; \ V_0 = \frac{\ell^4}{16m^2\alpha}}$

Only the particles with $E > V_0$ can fall into the center.

If the angular momentum is: $\ell = m v_0 b$; the energy: $E = \frac{1}{2} m v_0^2$

The maximum value of b is given by the condition:

$$E = V_0 \Longrightarrow \frac{\ell^4}{16 m^2 \alpha} = \frac{m^4 v_0^4 b_{max}^4}{16 m^2 \alpha^2} = \frac{1}{2} m v_0^2$$

and therefore: $\boxed{b_{max} = \left(\frac{8\alpha}{m v_0^2}\right)^{1/4}}$

The cross section is:

$$\boxed{\sigma = \pi b_{max}^2 = 2\pi \sqrt{\frac{2\alpha}{m v_0^2}}}$$

PROBLEM 4

a) The equilibrium is given by: $\frac{dV}{dx} = 0$

We have: $\frac{dV}{dx} = \frac{c(a^2 - x^2)}{(x^2+a^2)^2}$

Therefore we have two equilibrium positions: $\boxed{x_1 = +a \qquad x_2 = -a}$

To study the stability of these positions we need the 2nd derivative of the potential:

$$\frac{d^2V}{dx^2} = \frac{2cx(x^2 - 3a^2)}{(x^2+a^2)^3}$$

To have a stable position we need $\left.\frac{d^2V}{dx^2}\right|_{\bar{x}_0} > 0 \Longrightarrow$ x_2 is a position of stable equilibrium

We now expand the potential around x_2; we define η as: $x = x_2 + \eta$

$$V(\eta) = \underbrace{V(x_2)}_{\text{constant}} + \underbrace{\cancel{\left.\frac{dV}{dx}\right|_{x_2}}}_{\text{Equilibrium}} + \frac{1}{2}\left.\frac{d^2V}{dx^2}\right|_{x_2} \eta^2 + \cdots$$

If we neglect the constant term $V(x_2)$ we get: $V(\eta) \simeq \frac{1}{2}\left(\frac{c}{2a^3}\right)\eta^2$

The equation of motion for η is: $\boxed{m\ddot{\eta} \simeq -\frac{c}{2a^3}\eta}$

This is a simple harmonic motion, with frequency: $\omega = \sqrt{\frac{c}{2a^3 m}}$

The period is: $\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2ma^3}{c}}}$

b) ~~Ata~~ At time $t=0$ the energy of the particle is:

$$E = \frac{1}{2} m v^2 + V(-a) = \frac{1}{2} m v^2 - \frac{c}{2a}$$

(1) It oscillates if $E < 0$

this corresponds to $\boxed{v < \sqrt{\frac{c}{ma}}}$

(2) It can get to $x = -\infty$ if $E > V(-\infty) = 0$

$$\boxed{v > \sqrt{\frac{c}{ma}}}$$

(3) It now has to pass over the point $x_2 = a$, where V has a maximum.

We now need $E > V(a) = \frac{c}{2a}$

and therefore $\boxed{v > \sqrt{\frac{2c}{ma}}}$

PROBLEM 5

a) In the system where the inertial tensor is diagonal:

$$I_{xx} = I_{yy} = \frac{ma^2}{12} \qquad I_{zz} = \frac{ma^2}{6}$$

b) The angular momentum is given by : $\vec{L} = \tilde{I} \cdot \vec{w}$

In the body system we have: $\vec{w} = w \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$ at the time $t=0$.

Then in the body system and at $t=0$ we have:

$$\left(\vec{L}\right)_{\substack{body \\ t=0}} = \frac{ma^2 w}{12} \begin{pmatrix} \sin\theta \\ 0 \\ 2\cos\theta \end{pmatrix}$$

At a generic time t we have that in the laboratory system we have:

$$\left(\vec{L}\right)_{laboratory} = \begin{pmatrix} \cos wt & -\sin wt & 0 \\ \sin wt & \cos wt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \frac{ma^2 w}{12} \begin{pmatrix} \sin\theta \\ 0 \\ 2\cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos wt \cos\theta & -\sin wt & -\cos wt \sin\theta \\ \sin wt \cos\theta & \cos wt & -\sin wt \sin\theta \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \frac{ma^2 w}{12} \begin{pmatrix} \sin\theta \\ 0 \\ 2\cos\theta \end{pmatrix} =$$

$$= \frac{ma^2 w}{12} \begin{pmatrix} -\sin\theta\cos\theta\cos(wt) \\ -\sin\theta\cos\theta\sin(wt) \\ (1+\cos^2\theta) \end{pmatrix}$$

The torque in the laboratory frame is simply given by:

$$\vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{lab} = \left(\frac{m a^2 \omega^2}{12}\right) \sin\theta \cos\theta \begin{pmatrix} \sin(\omega t) \\ -\cos(\omega t) \\ 0 \end{pmatrix}$$