December 01, 2011

Massachusetts Institute of Technology

Department of Physics - Classical Mechanics 8.09 - Fall 2011

Homework set No. 12 (Due: Monday, December 12, 2011)

Reading:

1. Goldstein, Chapter 8 and 9

Problems:

1. (10 points) Determine generator functions for the following transformations:

   (a) (2 points) $Q_i = q_i$, $P_i = p_i$

   (b) (2 points) $Q_1 = q_1$, $P_1 = p_1$ and $Q_2 = p_2$, $P_2 = -q_2$

   (c) (2 points) The generator function F_1 for: $Q = p/t$, $P = -qt$

   (d) (2 points) For which parameters a, b is there a generator function F_4 for: $Q = q^a \cos(bp)$, $P = q^a \sin(bp)$?

   (e) (2 points) For which parameters k, l, m, n is there a generator function F_1 for: $Q = q^k p^l$, $P = q^m p^n$?

2. (10 points) Assume that we deal with only time independent canonical transformations, i.e. $\sum_{i=1}^{n}(p_i dq_i - P_i dQ_i) = dF$. Show explicitly that the following transformations are canonical:

   (a) (2 points) $Q = \arctan\left(\frac{q}{p}\right)$, $P = \frac{1}{2}\left(q^2 + p^2\right)$

   (b) (2 points)
   $p_1 = \sqrt{k_1 P_1}\sin Q_1 + \sqrt{k_2 P_2}\sin Q_2$, $p_2 = \sqrt{k_1 P_1}\sin Q_1 - \sqrt{k_2 P_2}\sin Q_2$
   $q_1 = -\sqrt{P_1/k_1}\cos Q_1 - \sqrt{P_2/k_2}\cos Q_2$, $q_2 = -\sqrt{P_1/k_1}\cos Q_1 + \sqrt{P_2/k_2}\cos Q_2$

   (c) (4 points) Based on the transformation in part b), solve the equations of motion for:
   $H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{k_1^2}{4}(q_1 + q_2)^2 + \frac{k_2^2}{4}(q_1 - q_2)^2$

3. (10 points)

   (a) (5 points) Show that the gauge transformation of the Lagrange function
   $L(q,\dot{q},t) \rightarrow L'(q,\dot{q},t) = \text{L}(q,\dot{q},t) + \frac{d}{dt}f(q,t) = L + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t}$
   is a canonical transformation. Determine the generator function F_2 and the new Hamilton function K.

(b) (5 points) Show that the gauge transformation of the electromagnetic field

$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla} f(\vec{r}, t)$ and $\phi \rightarrow \phi' = \phi - \frac{\partial f(\vec{r},t)}{\partial t}$

is a canonical transformation for the coordinate and momentum of a charged particle. Determine the generator function $F_2(\vec{r}, \vec{P}, t$.

Hint: Review the Hamilton function for the example on page 20-21, Lecture 18!

4. (10 points) A particle of mass $m = 1/2$ is moving along the x-axis inside a potential $V(x) = \exp(x)$.

   (a) (2 points) Determine the generator function $F_2(x, P)$ which transforms the Hamilton function $H = p^2 + e^x$ to the new Hamilton function $K = P^2/4$.

   (b) (2 points) Determine the transformation equations.

   (c) (2 points) Determine $x(t)$ and $p(t)$.

5. (10 points)

   (a) (4 points) For which parameters k, l, m, n are the transformations

   $Q = q^k p^l$ and $P = q^m p^n$

   canonical?

   (b) (6 points) Proof that the invariance of the following Poisson brackets

   $[Q_i, Q_j]_{q,p} = [q_i, q_j]_{q,p} = 0$

   $[P_i, P_j]_{q,p} = [p_i, p_j]_{q,p} = 0$

   $[Q_i, P_j]_{q,p} = [q_i, p_j]_{q,p} = \delta_{ij}$

   are equivalent to general Poisson brackets

   $[f, g]_{q,p} = [F, G]_{Q,P}$

   with two arbitrary observables $f(q, p, t)$ and $g(q, p, t)$ and

   $F(Q, P, t) = f\{q(Q, P, t), p(Q, P, t), t\}$

   $G(Q, P, t) = g\{q(Q, P, t), p(Q, P, t), t\}$