December 01, 2011

Massachusetts Institute of Technology

Department of Physics - Classical Mechanics 8.09 - Fall 2011

Homework set No. 13 (Practice Problems for Final Exam)

Reading:

1. Goldstein, Chapter 3

Problems:

1. The classical interaction between two inert gas atoms, each of mass m, is given by the potential

$$V(r) = -\frac{2A}{r^6} + \frac{B}{r^{12}}, \qquad A, B > 0, \qquad r = |\mathbf{r_1} - \mathbf{r_2}|$$

   (a) Give the Hamiltonian for the system of the two atoms

   (b) Describe completely the lowest energy classical state(s) of this system

   (c) If the energy is slightly higher than the lowest (part (b)), what are the possible frequencies of the motion of the system?

2. The Poisson bracket is defined by

$$[a, b] = \sum_k \left(\frac{\partial a}{\partial q_k} \frac{\partial b}{\partial p_k} - \frac{\partial a}{\partial p_k} \frac{\partial b}{\partial q_k} \right)$$

   (a) Show that for a dynamical quantity a(q,p,t)

$$\frac{da}{dt} = [a, H] + \frac{\partial a}{\partial t}$$

   A two dimensional oscillator has energies

$$T(\dot{x}, \dot{y}) = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right)$$

$$V(x, y) = \frac{1}{2} K \left(x^2 + y^2 \right) + Cxy$$

   where C and K are constants.

(b) Show by a coordinate transformation that this oscillator is equivalent to a nonisotropic harmonic oscillator

(c) Find two independent constants of the motion and verify using part (a)

(d) If $C = 0$ find a third constant of motion

(e) Show that for the isotropic oscillator the symmetric matrix

$$A_{ij} = \frac{p_i p_j}{2m} + \frac{1}{2} K x_i x_j$$

is a constant of the motion by expressing each element in terms of the known constants of motion.

3. The transformation equations between two sets of coordinates are

$$Q = \ln(1 + q^{1/2} \cos p)$$

$$P = 2(1 + q^{1/2} \cos p) q^{1/2} \sin p$$

(a) Show directly from these transformation equations that Q and P are canonical variables if q and p are.

(b) Show that the function that generates this transformation between the two sets of canonical variables is

$$F_3 = -[\exp(Q) - 1]^2 \tan p$$

4. A particle of mass m moves in one dimension q in a potential energy field $V(q)$ and is retarded by a damping force $-2m\gamma\dot{q}$ proportional to its velocity

(a) Show that the equation of motion can be obtained from the Lagrangian

$$L = \exp(2\gamma t) \left[\frac{1}{2} m \dot{q}^2 - V(q) \right]$$

and that the Hamilonian is

$$H = \frac{p^2 \exp(-2\gamma t)}{2m} + V(q) \exp(2\gamma t)$$

where $p = m\dot{q} \exp(2\gamma t)$ is the momentum conjugate to q

(b) For the generating function

$$F_2(q, P, t) = \exp(\gamma t) q P$$

find the transformed Hamiltonian $K(Q, P, t)$. For an oscillator potential

$$V(q) = \frac{1}{2} m \omega^2 q^2 \quad (1)$$

show that the transformed Hamiltonian yields a constant of motion

$$K = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 Q + \gamma Q P \quad (2)$$

(c) Obtain the solution $q(t)$ for the damped oscillator from the constant of the motion in (b) in the underdamped case $\gamma < \omega$. You may need the integral

$$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x \quad (3)$$

5. Suppose that a system with time-independent Hamiltonian $H_0(q, p)$ has imposed on it an external oscillating field, so that the Hamiltonian becomes $H = H_0(q, p) - \epsilon q \sin \omega t$, where ϵ and ω are given constants.

   (a) What is the physical interpretation of $\epsilon \sin \omega t$?

   (b) How are the canonical equations of motion modified?

   (c) Find a canonical transformation which restores the canonical form of the equation of motion. What is the "new" Hamiltonian?