Massachusetts Institute of Technology
Physics Department

8.13/8.14 2001/2002

Junior Physics Laboratory Experiment #37

Determination of the Size, Distance and Mass of the Sun

by High-Resolution Solar Spectroscopy and the Cavendish Experiment

1 PREPARATORY QUESTIONS

1. Describe three different ways (including this one) to measure the distance to the sun.
2. By how much does the sun-earth distance vary over the course of a year?
3. Using the data tabulated by Allen (1973), plot the sun's temperature against the radius from the center to the corona. Make a separate plot of the temperature against the distance from the base of the chromosphere from $-5,000$ to $+5,000$ km. Mark the photosphere where most of the visible light is emitted. Mark the region where the Fraunhöfer lines are formed.
4. Imagine an experiment performed in the Shuttle in which two lead (density = 11.3 g cm^{-3}) spheres are set in orbit about one another under their mutual gravitation. Calculate their minimum orbital period.
5. Estimate the difference in wavelength of a Fraunhöfer line at 6000 Å in light coming from the east and west limbs of the sun due to the solar rotation (see Allen 1973 for the properties of the sun).

Note 1: Appendix A is an explanation of astronomical terms with diagrams. You may need also to consult a general text on astronomy for more complete discussions, e.g. Pasachoff (1981).

Note 2: If you encounter a cloudy day, do the Cavendish experiment so you will have a measured value of G to use with the value of the solar distance obtained on a sunny day to determine the mass of the sun.

2 WHAT YOU WILL MEASURE

1. The Doppler shifts in the wavelengths of several Fraunhöfer absorption lines in light coming from various positions along a chord of the solar disk.

2. The track of a sun spot across the solar disk using your own or data down loaded from the World Wide Web.

3. The sun's angular radius.

4. The gravitational constant G.

3 SAFETY

There are no special hazards in this experiment except for the fact that the penthouse of Building 6 is an isolated area. A person injured while working alone on the roof might not be able to summon help. For your safety it is therefore important that you not work alone on this experiment.

To protect your eyes from permanent injury never view the sun directly through a telescope or field glass without a very dark filter securely in place.

Please do not

1. **attempt to clean any of the optical surfaces;**

2. **adjust any of the optical components inside the spectrograph;**

3. **adjust the large fixed mirror of the heliostat or move the heliostat base.**

Proper alignment of these components requires night work with a laser. If you think there is a problem of misalignment, ask your professor or Jay Kirsch for help and advice.

You may and must adjust:

1. the adjustable mirror on the heliostat to send the sunlight down the pipe.

2. the telescope lens and aperture control at the entrance pipe—to focus the sun's image on the slit mirror with an aperture appropriate to the operation you are performing, i.e. focusing (wide open) or photographing the spectrum (diameter ∼1 cm);

3. the orientation of the slit mirror—to make the slit horizontal (perpendicular to the direction of echelle dispersion;

4. the projection lens that casts the sun's image on the drawing board on the cement floor

4 LOGISTICS OF THE EXPERIMENT

The ideal schedule for this experiment, based on the assumption that you are blessed with at least 3 clear days is as follows:

Day 1—Measure the positions on the solar disc of whatever sunspots there are. Learn how the spectrograph works and take some practice Polaroid echellograms.

Day 2—Measure the sun spot positions. Take the sharpest possible echellograms of Fraunhöfer and terrestrial absorption lines as near noon as possible with the image of the sun centered on the slit.

Day 3—Measure the sun spot positions if it's clear. Practice the Cavendish experiment.

Day 4—Measure the sun spot positions if it's clear. Take the best possible data on the value of G.

Dated images of the sun can be downloaded from the Web at `www.sel.noaa.gov/images`.

A very nice interactive solar spectral atlas (model or images) is available at `mesola.obspm.fr/form_spectre.html`.

5 INTRODUCTION

The semi-major axis of the earth's elliptical orbit around the sun is defined as one astronomical unit (1 AU). Recent measurements have yielded the value $(1.495979 \pm 0.000001) \times 10^{13}$ cm (Allen 1976). This quantity is of fundamental importance in astronomy because it is the basic size parameter of the solar system and the first rung on the cosmic distance ladder. (The next rungs are the distances to nearby stars that have annual parallaxes of the order of 1 arc second, which can be measured to accuracies of about ± 0.001 arc second with the refined methods of optical astrometry.)

It is not easy to measure the sun's distance accurately. Direct triangulation from the earth is not very accurate because the sun's brightness makes it difficult to measure the parallax relative to the background stars with sufficient accuracy using even the largest possible geographical baselines.

Nevertheless, there are many ways to measure the sun's distance, and it is amusing to figure out how you might do it while limiting yourself to the technologies available to, say, Hipparchus, Galileo, Newton, Newcombe, or a modern astronomer. For example, a 19th century method capable of yielding high accuracy is based on the measurement of the positions of an asteroid that passes within a few million miles of earth. A recent method capable of extremely high accuracy is based on the measurement of the round-trip times of radio signals to artificial solar satellites. Neither of these methods involves a direct measurement of the sun.

In this experiment you will determine the sun's linear radius and distance by direct observations of the sun. Your measurements will yield the following quantities:

1. the differences in velocity toward earth of various places on the solar surface due to solar rotation;
2. the sun's rotational angular velocity (magnitude and direction);
3. the sun's angular radius.

From the measured differences in the components of velocities of points on the solar surface and the angular velocity you can compute the sun's linear radius. From the linear radius and the angular radius you can compute the sun's distance which, because of the ellipticity of the earth's orbit and depending on the time of the year, is somewhere within a factor of (1 ± 0.016722) of the semimajor axis of the earth's orbit, otherwise known as the astronomical unit.

The velocity differences are measured by observing the Doppler shifts in the wavelengths of Fraunhöfer absorption lines in the solar spectrum with a high resolution echelle spectrograph originally constructed for this purpose by B. Lacy as his senior thesis project in 1975 under the supervision of Professor David Frisch. The key element of the spectrograph is an echelle grating made on an interferometrically controlled ruling engine developed by Dean George Harrison and operated in the basement of Building 6 in the 1940's. The Lacy/Frisch spectrograph, with improvements, is now set up in the penthouse on the roof of Building 6.

The sun's angular radius and angular velocity are measured by tracking the motions of images of the sun across the sky and sun spots across the sun's disk.

Given the length of the year, the sun's distance (representing an approximation to the earth's semi-major axis), and the the gravitational constant G, one can estimate the sun's mass. The gravitational constant will be measured with the Cavendish apparatus set up in room 4-304.

There is available for checkout from the stockroom a 6-inch portable astronomical telescope with a very dark filter consisting of two sheets of aluminized Mylar. With the filter carefully fitted over the front end of the telescope you can view the surface of the sun directly through the eyepiece. Alternatively, you can examine the bright image of the sun projected through the eyepiece onto the special sun screen which can be attached to the telescope with the threaded aluminum mounting rod provided.

6 THE SOLAR SPECTRUM

Most of the visible light of the sun is produced in a narrow region of the sun's atmosphere called the photosphere. Its spectrum is approximately that of a blackbody with a temperature of 5,800 K. As the light travels outward from the photosphere is passes through a cooler region called the chromosphere where the kinetic temperature reaches its minimum value of approximately 4,200 K. Here selective absorption occurs at the resonant frequencies of the atoms and ions of the various elements, thereby imposing on the continuum spectrum a forest of dark lines of which the most prominent were discovered by Fraunhöfer in 1814. Moving outward the temperature rises slowly to about 10,000 K near a height of 2,000 km above the base of the chromosphere. It then rises steeply through the so called transition zone to values of several millions of degrees in the corona, which is the source of solar X rays and the solar wind.

The most prominent Fraunhöfer lines are those of hydrogen, calcium, sodium, iron,silicon, and magnesium. Absorption in the earth's atmosphere imposes additional lines on the solar spectrum as observed from the ground. (There are spectacular absorption doublets due to molecular oxygen just barely visible in the far red part of the spectrum. You may want to try to photograph them and see if you can explain the surprising result.)

The sun rotates, and the earth moves in orbit about the sun. Thus the absorbing material at any given point in the chromosphere has a component of velocity along our line of sight that depends on 1) the magnitude and direction of the sun's angular velocity vector, 2) the sun's radius, 3) the solar latitude and longitude of the given point, and 4) the motion of the earth. Thus the Fraunhöfer lines have position-dependent Doppler shifts. It is these Doppler shifts you will measure, together with the direction and magnitude of the angular velocity of rotation of the sun, and its angular radius as viewed from the earth. From these quantities and the elements of the earth's orbital motion you can deduce the linear radius of the sun. And from the linear radius and the angular radius, you can deduce the sun's distance.

7 THEORY OF THE ECHELLE SPECTROGRAPH

Sunlight, reflected from a heliostat, is focused by the lens L through the aperture stop A to form an image of the sun on a slit in a front-surface mirror SM as shown schematically in Figure 1. Light from the portion of this image that falls on the slit passes through SM and enters the spectrograph where it first strikes the parabolic mirror M1 located at its focal distance from SM. Spherical light waves (i.e. "Huygens wavelets") diverging from any given point at SM are reflected by M1 into plane waves traveling toward the echelle grating EG. Reflections from the multiple steps of the echelle form cylindrical waves which interfere constructively in certain vertically dispersed directions that depend on the wavelength. A second grating disperses the beam horizontally and reflects it toward the parabolic mirror M2. Parallel rays falling on M2 are focused to a point in an image of the original slit at the focal plane FP of the spectrograph where they can be observed visually on a frosted screen or through a magnifying eyepiece, or recorded photographically with a Polaroid film pack. The multi-line absorption spectrum appears as a pattern of horizontal dark slit images as illustrated schematically in Figure 2.

To understand the optics of any spectrograph it is essential to realize that an observed spectrum "line" is actually a monochromatic image of the illuminated portion of the slit. Widen, lengthen, or rotate the illuminated portion of the

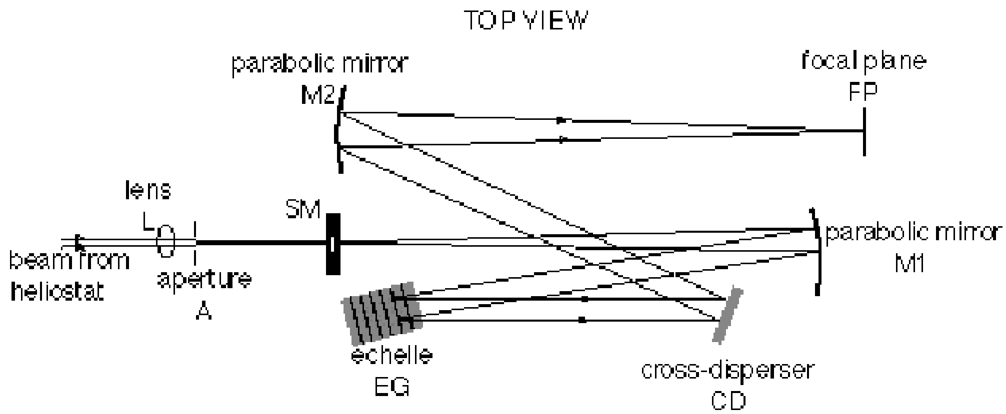

Figure 1: Schematic top view of the echelle spectrograph located in the penthouse of Building 6.

slit and you widen, lengthen, or rotate the spectrum line. The width of a line also depends on the sharpness of the focus, the angular resolution of the optical. system, and the intrinsic spread in wavelength of the spectrum feature, i.e. the emission or absorption feature.

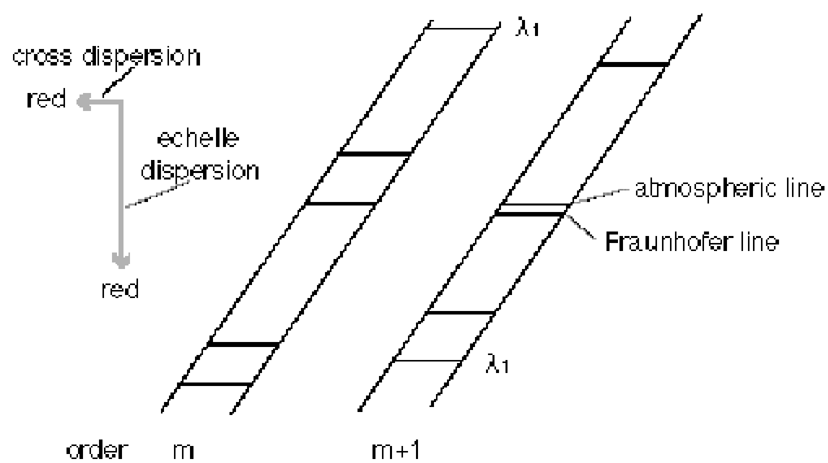

Figure 2: Schematic illustration of two orders of spectra in an echellogram. Note how the λ_1 spectrum lines appear at different vertical positions in the m and $m+1$ order of echelle dispersion, but at the same horizontal position in the first order of cross dispersion. Note also that the orientation of the spectrum lines (which are, in effect, images of the slit) depends on the orientation of the slit.

The echelle geometry is illustrated in Figure 3. The echelle grating is ruled with step-like grooves that are uniform and parallel to within a very small fraction of λ. Each groove reflects a narrow rectangular piece of an incident plane wave, and this piece spreads about the specular reflection direction according to the principles of Fraunhöfer diffraction. The amplitudes of the resulting cylindrical wavelets interfere to form plane waves with maximum intensities in directions such that the differences in path length along the reflected rays from successive grooves is an integral number of wavelengths.

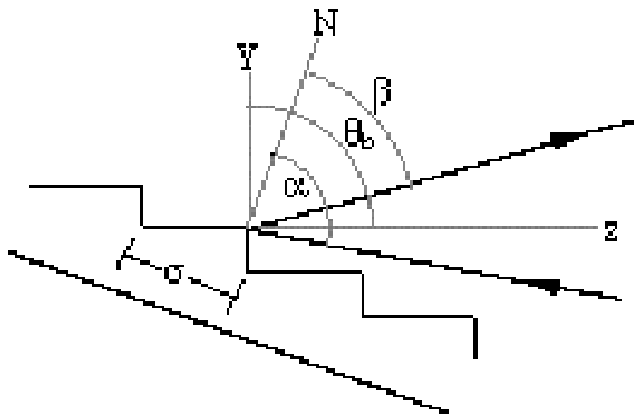

Figure 3: Schematic diagram of rays reflected from one step of an echelle grating. (After Chaffee and Schroeder, 1976).

This condition is expressed by the equation

$$\sigma(\sin\alpha + \sin\beta) = m\lambda, \tag{1}$$

where α and β are the angles in the y-z plane between the grating normal N and the directions of incidence and reflection, respectively, σ is the distance between grooves, and m is an integer called the "order" of the interference.

The angular dispersion is defined by the differential relation

$$d\beta/d\lambda = \frac{m}{\sigma\cos\beta} = \frac{\sin\alpha + \sin\beta}{\lambda\cos\beta}. \tag{2}$$

An echelle spectrograph is generally arranged so that $\alpha \approx \beta \approx \theta_{\rm b}$, where $\theta_{\rm b}$ is called the blaze angle. Under these conditions equation (2) becomes

$$d\beta/d\lambda = \frac{2}{\lambda}\tan\theta_{\rm b}. \tag{3}$$

Most commercial echelle gratings have blaze angles such that $\tan\theta_{\rm b} = 2$, i.e. $\theta_{\rm b} = 63.5°$. At $\lambda = 6,000$ Å such a grating has a dispersion of 0.0382 deg Å.

For given angles of incidence and reflection, the condition expressed by equation (1) can be met by the set of discrete wavelengths defined by the relation

$$m\lambda_m = \text{constant} \tag{4}$$

The difference between two wavelengths which are diffracted at the same angle in successive orders is called the **free spectral range** defined by the equation

$$\delta\lambda = \lambda/m = \lambda^2/(2\sigma\sin\theta_{\rm b}). \tag{5}$$

For $m = 100$ and $\lambda = 6{,}000$ Å, $\delta\lambda \approx 60$ Å.

If the cross disperser were omitted from the optical system, a single vertical band of monochromatic horizontal images of the slit would be formed in the focal plane at positions corresponding to the values of β that are related to the wavelengths by equation (1) and over a range of β determined by the Fraunhöfer diffraction from the individual steps of the echelle. The problem would then be that the wavelengths of the images would be jumbled. For example, one might find a 90th order red line at the same position as a 120th order blue line. The cross-disperser, which is an ordinary low-dispersion reflection grating, spreads the various wavelengths in the horizontal direction, perpendicular to the high dispersion direction of the echelle. The result is a series of slanted spectral bands each one of which corresponds to a given order of echelle interference, with a vertical spread in angle determined by the Fraunhöfer diffraction of the individual echelle steps. Each band has a useful spread in wavelength approximately equal to the "free spectral range" of the echelle. The horizontal spread in angle of the entire pattern of bands is determined by the Fraunhöfer diffraction from the grooves of the cross disperser grating.

Meanwhile, back at the slit mirror SM, most of the light in the image of the sun formed by L is reflected backward by SM and about 5% of that light is reflected downward by each surface of the plate glass beam-splitter through a camera lens which casts a double image of the sun (and slit) onto a drawing board on which the features of the sun and slit can be traced, as shown in Figure 4.

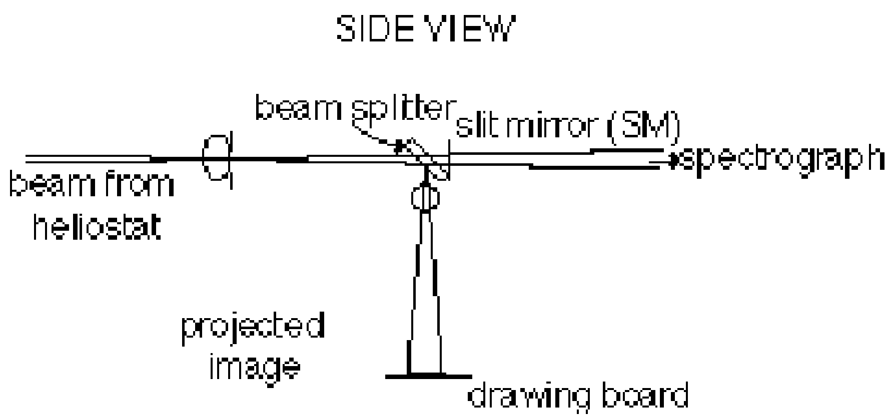

Figure 4: Side view of the front end of the spectrograph showing the scheme for projecting an image of the sun onto a drawing board with a superposed image of the slit.

8 THE HELIOSTAT

To compensate for the Earth's rotation and prevent the sun's image from drifting off the slit during the measurements, the experimental arrangement includes a heliostat, a device with two mirrors and a clock drive located under the sheet metal hood on the north side of the

Building 6 penthouse. The adjustable mirror is rotated by the clock drive about a "right ascension" axis (see Appendix A for an explanation of astronomical terms) which is aligned parallel to the Earth's rotation axis. The fixed mirror is oriented so that a ray parallel to the right ascension axis is reflected in a direction parallel to the line from the spectrograph slit S to the first mirror M1. To set the heliostat loosen the set screws on the right ascension axis and on the perpendicular "declination" axis. Orient the adjustable mirror so that sunlight is reflected to the fixed mirror and thence down the center of the pipe to the lens which forms the image of the sun on the slit. Then gently tighten the set screws and turn on the motor. With the incident beam of sunlight coming more or less squarely down the center of the pipe, the final position of the image on the slit can be adjusted by moving the telescope lens L located at the inside end of the pipe.

The altitude of the right ascension axis was set equal to the local latitude (42°) with the aid of a bubble level and a machinist's protractor; its azimuth was set to zero (north) by reference to the Infinite Corridor the azimuth of which is 24°32′, according to the office of MIT's Chief Architect.

9 OBSERVATIONS

9.1 Determine the Sun's angular diameter

With the projected image of the sun on the drawing board (doubled by virtue of reflections from both the front and back side of the 45° plate glass beam-splitter) held steady by the motion of the heliostat, trace with pen or pencil on paper taped to the drawing board the images of the sun's circumference, the slit, and any sunspots that happen to be in view. (Be careful not to confuse the shadows of dust on the lenses and images of dust on the slit mirror with real sunspots—you can tell the difference by jiggling the apparatus). Then, without moving the paper record, turn the heliostat drive motor off for a measured length of time, and then back on. Record the new positions of the sun's circumference, slit and spots on the same paper. From your sketches and knowledge of the earth's angular velocity you can derive the angular diameter of the sun, the orientation of the slit with respect to the east-west direction, and the positions of the sunspots in NSEW coordinates. Repeat these measurements several times to achieve good accuracy during each of your lab sessions. Don't forget the dependence on declination of the angular velocity of the drift motion.

Using your own records of sunspot positions or images of the sun and spots down-loaded from the WEB at `www.sel.noaa.gov/images`., determine the angular velocity of the sun's rotation. Note that the tracks are approximately (why only approximately?) parts of ellipses. Thus you must measure spot positions in a well determined coordinate system onseveral days, preferable spaced over a substantial fraction of the sun's rotation period, in order to determine the direction and magnitude of the angular velocity vector in space. Note that the quantity you are trying to determine is the sidereal period of rotation, i.e. the period of sun's rotation with respect to the distant stars.

9.2 Determine the Doppler Shifts

The goal of this part of the experiment is to obtain a sharply focused Polaroid photograph of a Fraunhöfer line and a nearby terrestrial atmosphere absorption line with the sun's image focused on the slit mirror and the east-west direction in that image as close to horizontal as possible.

If you are blessed with a clear sky near noon (say between 10:30 and 1:30) adjust the heliostat to send sunlight down the entrance pipe. Inside the penthouse, open wide the aperture stop next to the primary lens in the wooden trough, and adjust the lens L to cast an image of the sun onto the center of the slit mirror SM in the shutter mount. Confirm that the beam reflected from the slit mirror and the 45° mirror goes vertically down through the camera lens onto the drawing board. Adjust the position of the lens L and/or the heliostat mirrors so that the slit bisects the image. Open the shutter by rotating the shutter control ring full clockwise and depressing the shutter trigger lever gently. Sunlight passing through the slit should now strike the collimator mirror M1 and be reflected squarely onto the echelle grating EG.

Remove the plastic cover from the grating, and replace it when you are finished for the day. The echelle grating should be oriented so as to reflect its brightest diffracted beams squarely onto the cross disperser CD (approximately 22° on the angle indicator). The cross disperser, in turn, should be oriented to reflect the first-order diffracted beam squarely onto the camera mirror M2. And, finally, the camera mirror should be oriented and positioned to reflect the beam down the stovepipe light shield and to focus the chromatically dispersed images of the sun onto the film plane FP. There, on a ground glass screen, you should see a set of brilliant spectrum bands, each one corresponding to a different interference order of the echellogram, and each consisting of monochromatic images of the sun smeared into a vertical continuum by diffraction by the echelle, and spread horizontally from left (red) to right (blue) by the cross disperser.

With the aperture wide open and the ground glass screen in place, adjust the position of the film-pack holder for sharpest focus. Identify the Fraunhöfer sodium D-lines in absorption with the aid of the solar spectrum atlas provided at `mesola.obspm.fr/form_spectre.html`. To obtain measurements of the Doppler shift with the high accuracy required for a useful determination of the tangential velocity of the solar equator **it is essential that you record a portion of the spectrum that has a Fraunhöfer line lying immediately adjacent to a terrestrial atmospheric absorption line.** The latter will serve as a comparison line relative to which the Doppler shift can be measured as a tiny difference in the separation between the east and west ends of the slit images that constitute the observed lines. You will find a couple of excellent solar-terrestrial line pairs in the red part of the spectrum around 6280 Å which is covered by the portions of the solar spectrum atlas available at the web site listed above. Starting with the sodium D lines in the middle of the camera frame you can get to this particular place in the red part of the spectrum by rotating the cross disperser so as to move the echellogram 6 orders toward the red, and then tilting the echelle grating so as to move down approximately one camera frame.

To obtain the sharpest possible focus after your adjustment with a wide aperture you should reduce the aperture stop to a diameter of ∼1 cm so as to restrict the angular divergence of the light emerging from the slit into the spectrograph. Insert the Polaroid film pack, close the shutter by depressing the shutter control lever, gently rotate the shutter speed control ring to the desired shutter speed ("5" = 1/5 s, "25" = 1/25 s, etc.), turn out the lights, pull the dark slide nearly out, pull the shutter trigger string gently and steadily until you hear the shutter operate, push the dark slide back, pull the white Polaroid tab and the black Polaroid film out, let the film develop for 45 s, peel the photograph away from the developer, and examine your results. With a 1 cm aperture the proper exposure will probably be about 1/25 second.

Try to obtain a sharply focused echellogram (watch out for vibration—pull the shutter trigger string very gently) as close to local noon as possible when the image of the sun's equator on the slit mirror is nearly horizontal, i.e. perpendicular to the dispersion direction of the echelle. You should see adjacent lines (broad Fraunhöfer and narrow atmospheric lines) that are very slightly inclined with respect to one another and look like those shown schematically in Figure 2. A fairly dark exposure (i.e. under exposure) is required to obtain Fraunhöfer lines dense enough to be accurately measured under the traveling microscope. However, since a dark exposure may obscure the ends of the line due to the effects of limb darkening (the surface brightness at the limb is 1/3 of that at the center), it is wise to take and strong over exposure to reveal the exact length of the line. With this datum you will be able to extrapolate the measured separations between the Fraunhöfer and atmosphere lines to the exact limb of the sun.

When you obtain a good echellogram, immediately trace the image of the sun's circumference and the slit. Then turn off the heliostat drive for a suitable interval with the switch mounted on the front of the spectrograph box, and trace the shifted image of the sun's circumference. This will give you a "double exposure" drawing from which you can determine the location and orientation of the slit on the solar image in celestial coordinates. These data, taken nearly contemporaneously with the echellogram, are essential to your determination of the tangential velocity from the measured Doppler shift.

On a cloudy day, or at least once during your solar lab sessions, you should explore the operation of the echelle spectrograph by shooting a laser beam forward through the optical system. **Never allow a beam of laser light to enter your eye directly or after specular reflection or passage through an optical system. View laser light only after it has been diffusely reflected from some object such as a piece of paper or dust in the air.** Open the shutter as described above, i.e. by rotating the shutter control to the right and gently depressing the shutter trigger lever. Set up a laser in the wooden trough so its beam passes through the slit to the collimator mirror and on through the system. Note the various diffracted beams reflected from the echelle grating. Adjust the echelle and cross disperser to place one of the diffracted beams in the middle of the film pack holder, and note how the other diffracted beams, corresponding to different orders of echelle diffraction, can be brought into position by changing the angle of the echelle. Then replace the laser with a sodium vapor lamp and observe the sodium D lines in emission by focussing the light from a sodium vapor lamp onto the slit. Bring the brightest order of the spectrum to the center of the film holder by adjusting the echelle angle ($\sim 22°$) and turning the cross disperser in its mount. Examine the spectrum at the focal plane by casting it onto a piece of ground glass or by viewing it through the eyepiece which is mounted on an aluminum plate that fits in the film pack holder. Adjust the sliding support of the holder for best focus. Photograph the echellogram of the sodium lines. **Measure all the relevant dimensions of the spectrograph for use in your analysis of the spectrograph performance.**

10 ANALYSIS

Determine the dispersion of the spectrograph at the wavelength of your Doppler shift measurement by measuring the separations (in the direction of high dispersion) between various identified lines in your Polaroid echellograms. Determine the free spectral range. Compare your results with the predictions from equations (3), (4), and (5).

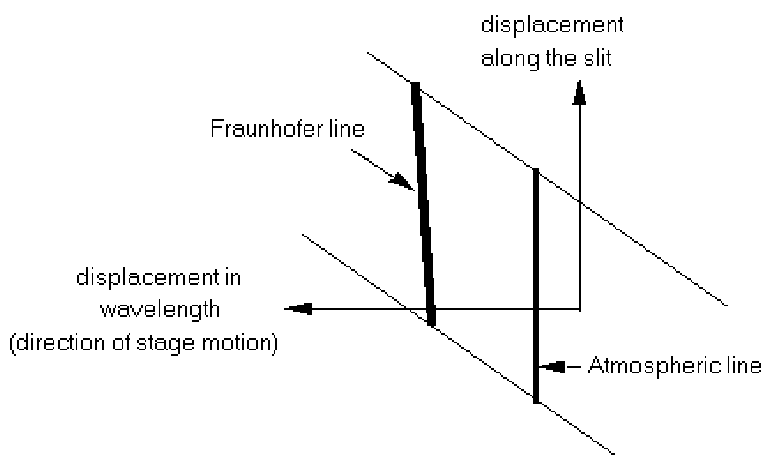

Figure 5: Schematic illustration of the echellogram measurement.

Measure the separation between adjacent solar and atmospheric lines as a function of position along the chord of the sun, i.e. the slit image. With care you can do this to about ± 0.001 cm with the traveling stage microscope set up in 4-310 near the window. Arrange your measurements to record the separation between adjacent Fraunhöfer and atmospheric lines at positions in a sequence spaced at one mm intervals along the slit images, as illustrated in Figure 5. **Be sure to orient the photograph so the spectrum lines are perpendicular to the motion of the stage caused by turning the crank on the right hand end of the apparatus. To achieve the accuracy necessary for a significant measure of the Doppler shifts you must exploit fully the precision of the lead screw by mitigating the effects of backlash and by repeating each measurement at least five times to reduce random error.** Plot the separation as a function of position along the slit (i.e. along the spectrum "lines"). To obtain good accuracy it is essential to derive a best value for the shift from the ensemble of data by statistical analysis. It turns out that the Doppler shift is approximately a linear function of the position along any diameter of the sun's image (can you demonstrate that?). Therefore you can fit a straight line by linear regression to your plot of separation against position along the slit, and use the slope and its error together with the measured width of the spectrum to determine the total change in separation between the end points of the Fraunhöfer and atmosphere reference lines.

Convert your results into the wavelength shift and the corresponding difference in the line-of-sight velocities between the solar surface points corresponding to the ends of the line: use the value of the spectral dispersion (Angstroms per cm on the photograph) obtained through measurement of the separation between lines of known wavelength that have been identified by reference to the portions of the solar spectrum atlas appended to this lab guide.

From the tracks of sunspots deduce as much as you can about the celestial direction and magnitude of the sun's rotational angular velocity. Appendix A explains some of the astronomical lore involved in the description of celestial positions and motions.

A LabVIEW file called `SIZE, DISTANCE, AND MASS OF THE SUN`, is available on the of the Junior Lab Server PC (folder Junior Lab Software Mirror) and can be downloaded to any of the local workstations. When you double click on this program, two programs are actually loaded into memory. `Calculate Sun Rotational Velocity from Sunspot Tracks` computes the predicted track of a sunspot for a set of specified trial parameters that describe the angular velocity of the sun's rotation and the position of the sunspot on the sun. Trial and error comparison of computed tracks with the data from NOAA should enable you to determine the sun's angular velocity. `Calculate Sun Distance from Doppler Measurement` does the computations required to convert your measurements into a distance to the sun. The following two sections describe the theory of these programs.

11 MOTION OF A SUNSPOT ACROSS THE FACE OF THE SUN

In the following we use bold letters to denote vectors and lower case bold letters to denote unit direction vectors. Thus $\mathbf{R} = R\mathbf{r}$ denotes a displacement vector of magnitude R in the direction $\mathbf{r}$.

Given measurements of the track of a sunspot across the face of the sun it is possible, in principle, to deduce the magnitude and direction of the sun's angular velocity of rotation. In fact, the mathematician Euler did it many years ago. However, a direct solution of the problem is complicated and difficult. It is much easier to solve the inverse problem, i.e. to calculate the track of a sun spot given its latitude and longitude on the sun's surface at a specified epoch and the sun's angular velocity vector ω. One can then adjust the assumed values of the given quantities until the computed track matches the measured track.

Call $\mathbf{r}(t)$ the direction of the sunspot as seen from the sun's center at time t. It's representation in solar coordinates is $[\cos\lambda\cos\phi, \cos\lambda\sin\phi, \sin\lambda]$, where λ and ϕ are the solar latitude and longitude of the spot. The representation in celestial coordinates is obtained by successive orthogonal transformations representing rotations of the coordinate system about the z, x, z, and x axes by the angles $-\Omega t$, $-i$, $-\xi$, $-i_{\mathrm{e}}$, respectively. Here Ω is the magnitude of the angular velocity at the solar latitude of the spot, i and ξ are the ecliptic inclination and longitude of the ascending node of the sun's equator, and $i_{\mathrm{e}} = 23.45°$ is the ecliptic inclination of the earth's equator. The transformations, readily done on a computer, yield the three components of $\mathbf{r}(t)$ in celestial coordinates.

We call $\mathbf{e}(t)$, $\mathbf{n}(t)$, $\mathbf{s}(t)$ the triad of orthogonal unit vectors that define the plane tangent to the celestial sphere ("plane of the sky") at the position of the sun, where $\mathbf{s}$ is the direction to the sun's center, and $\mathbf{e}$ and $\mathbf{n}$ point to the east and north directions, as illustrated in Figure 6b. The latter are defined by

$$\mathbf{e} = (\mathbf{p}\times\mathbf{s})/|\mathbf{p}\times\mathbf{s}|, \tag{6}$$

and

$$\mathbf{n} = (\mathbf{s}\times\mathbf{e}), \tag{7}$$

where $\times$ denotes the cross product and $\mathbf{p}$ is the direction of the celestial pole, i.e. the earth's spin axis whose representation in celestial coordinates is simply (0,0,1). In this experiment the projection of $\mathbf{e}$ on the sun's image is determined from a measurement of the direction of drift of the image when the heliostat motor was turned off at the time of the spectroscopy. The projection of $\mathbf{n}$ on the image can be deduced by consideration of the effects of the mirror reflections involved in the optical train from the heliostat to the final image on the floor. The celestial representation of the unit vector $\mathbf{s}$ is derived by solution of Kepler's equation for the angular motion of the earth about the sun. This is also readily done by computer. The celestial representations of $\mathbf{e}$ and $\mathbf{n}$ are then computed from equations (6) and (7). (Note that the orbital elements that determine the angular motion are determined purely from

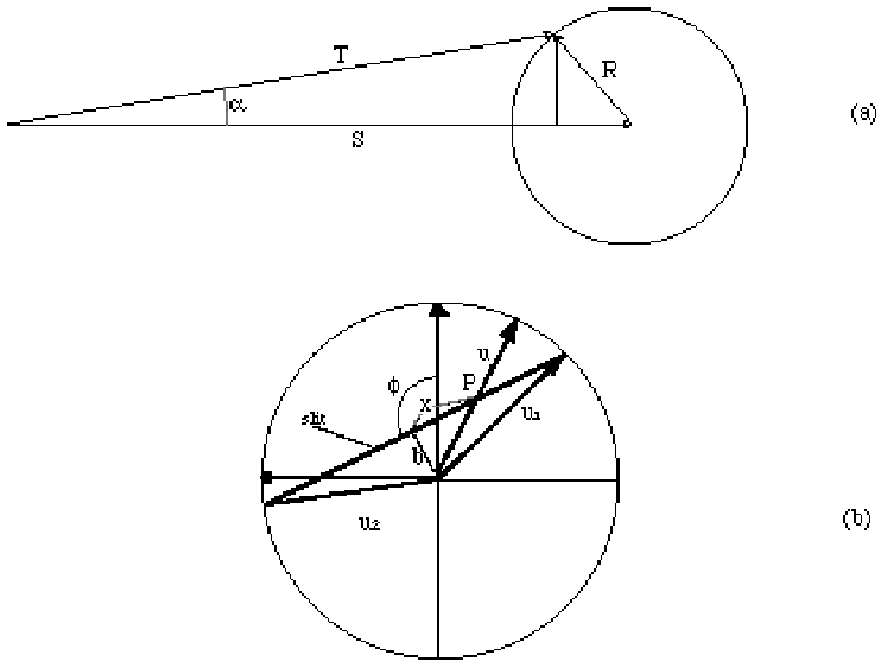

Figure 6: (a) Diagram of the displacement vectors from earth to sun and the displacement of the solar surface point from the sun's center. (b) Diagram of the sun's image represented by a unit circle on the plane of the sky, with the slit image superposed.

observations of the apparent motion of the sun against the background of the fixed stars. Such observations do not determine the semi-major axis that fixes the size of the earth's orbit.)

If the angular radius of the sun is ρ, then the parallel projection onto the plane of the sky of the sun spot position is a point with angular coordinates $\rho(\mathbf{r} \cdot \mathbf{e})$ and $\rho(\mathbf{r} \cdot \mathbf{n})$ in the east and north directions, respectively. These time-dependent quantities are the predicted coordinates of the sunspot track inside the circle that represents the image of the sun's circumference. The quantities Ω, ξ, and λ, are then to be adjusted until the predicted track matches the observed track.

Having found the parallel projection of the sunspot position onto the **e**-**n** plane of the sky we could make a slight adjustment for the fact that the projection is actually conical with the point of projection being our position of observation. However, the adjustment is small compared to the errors of observation in the present setup and can be ignored.

The computations can be carried out with the programs called `Calculate Sun Rotational Velocity from Sunspot Tracks` and `Calculate Sun Distance from Doppler Measurement` on the Junior Lab PC's. The code is downloadable to any machine (running LabVIEW 6i) from the Junior Lab Server.

12 DETERMINATION OF THE SUN'S DISTANCE FROM MEASUREMENTS OF THE DOPPLER SHIFT AT TWO POSITIONS ON THE SOLAR SURFACE

When the measurements and data reduction of this experiment are completed one has, in principle, the following data:

1. ΔV—the difference in the radial velocity relative to the earth of the two points on the limb of the sun corresponding to the intersections of the spectrograph slit with the image of the solar circumference (limb);
2. ϕ—the position angle of the slit (orientation angle of the slit on the solar image measured from north toward east);
3. b—the perpendicular displacement of the slit from the center of the solar image expressed as a fraction of the radius of the circular solar image;
4. Ω—the angular momentum vector of the solar rotation (derived by fitting trial values to the sun spot tracking data with the aid of the `Calculate Sun Rotational Velocity from Sunspot Tracks` program).
5. ρ—the angular radius of the sun.

We assume as given the six orbital elements of the earth's motion around the sun which can be determined by measurements of the angular position of the sun in the sky during the course of a year. Our problem is to determine the distance, S, from which one can determine the seventh orbital element called the semi-major axis, otherwise known as the astronomical unit.

Given the orbital elements (except the semi-major axis, which is in effect what this experiment is intended to measure), one can solve Kepler's equation to obtain the direction vector of the earth relative to the sun at any given time. The direction vector of the sun relative to the earth is the negative of that, so we will assume we have available:

6. $\mathbf{s}(t)$—the unit direction vector of the sun at time t.

Consider now a point on the surface of the sun which is imaged at a point P on the slit. Call $\mathbf{R} = R\mathbf{r}$ the displacement of the solar surface point from the center of the sun, where $R = S\sin\rho$ is the solar radius. The velocity of that point relative to earth is

$$\mathbf{V} = S\left[\sin(\rho)\cdot(\mathbf{\Omega}\times\mathbf{r}) + (d\mathbf{s}/dt)\right], \tag{8}$$

where $\times$ denotes the vector cross product. If we call $\mathbf{T} = T\mathbf{t}$ the displacement from earth to the solar surface point, where $\mathbf{t}$ is a unit vector, then the radial component, i.e., the component of velocity along the line of sight from earth, is

$$V_{\mathrm{s}} = \mathbf{V}\cdot\mathbf{t}. \tag{9}$$

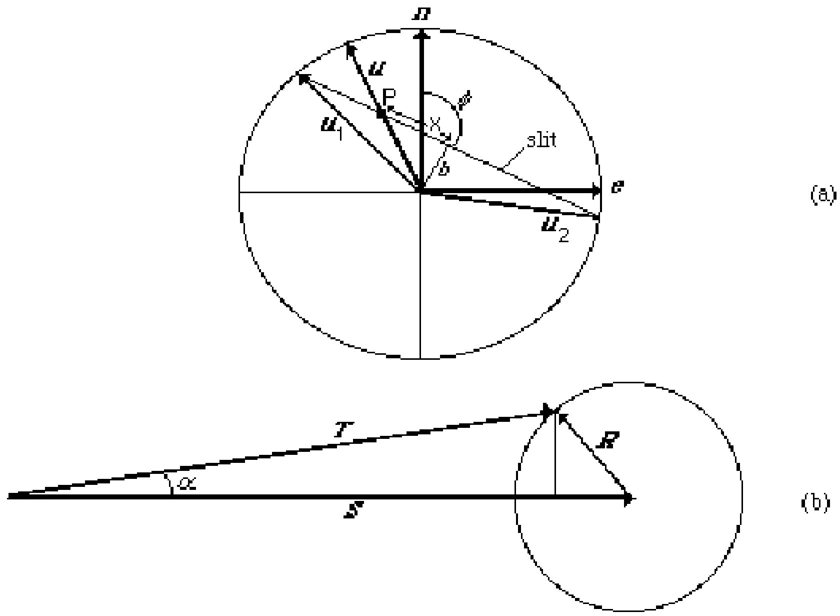

Figure 7: (a) Diagram of the sun's image on the plane of the sky, with the slit image superposed. (b) Diagram of the displacement vectors from earth to sun and the displacement of the solar surface point from the sun's center.

Suppose one determines, by measurement of Doppler shifts, the difference $\Delta V = (V_{\rm s1} - V_{\rm s2})$ between the radial velocities at two different points on the solar surface with direction vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to the sun's center. Then the solar distance can be computed as

$$S = \frac{V_{\rm s1} - V_{\rm s2}}{[\sin(\rho)(\boldsymbol{\Omega} \times \mathbf{r}_1) + (d\mathbf{s}/dt)] \cdot \mathbf{t}_1 - [\sin(\rho)(\boldsymbol{\Omega} \times \mathbf{r}_2) + (d\mathbf{s}/dt)] \cdot \mathbf{t}_2} \tag{10}$$

The problem is thus reduced to finding expressions for the unit direction vectors $\mathbf{t}$ and $\mathbf{r}$ representing the displacements of the solar surface point imaged at P from the earth and from the sun's center, respectively.

We first establish a coordinate system suitable for describing the relation between positions in the sun's image and positions on the sun's surface. An image of the sun may be considered as a projection of the sun's surface onto a plane perpendicular to the line of sight to the sun's center. That plane is called "the plane of the sky" at the position on the sky of the sun's center. Positions in that plane are conventionally specified with reference to a coordinate system with unit orthogonal vectors pointing north, $\mathbf{n}$, and east, $\mathbf{e}$, as illustrated in Figure 7a. If we call $\mathbf{p}$ the direction vector of the celestial pole (i.e. the Earth's spin axis), then

$$\mathbf{e} \;=\; (\mathbf{p} \times \mathbf{s})/|\mathbf{p} \times \mathbf{s}|, \tag{11}$$

and

$$\mathbf{n} = \mathbf{s} \times \mathbf{e}. \tag{12}$$

The third orthogonal unit vector is $\mathbf{s}$, the direction vector to the sun.

The directions of $\mathbf{e}$ and $\mathbf{n}$ on the sun's image are determined from a measurement of the direction of drift of the image when the heliostat motor was turned off at the time of the spectroscopy. Measurements on the image itself yield the quantities ϕ and b that specify the position angle and displacement of the slit from the center of the image, as illustrated in Figure 7a.

We now construct the unit vector $\mathbf{t} = \mathbf{T}/T$ which is in the direction of the displacement from earth to a point P on the sun's surface corresponding to the point P' in the sun's image at a position along the slit lying a distance x from the midpoint of the slit. We call $\mathbf{u}$ the unit vector in the $\mathbf{e}$-$\mathbf{n}$ plane in the direction from the image center to P. From Figure 7a we see that

$$\mathbf{u} = \frac{(-b\cos\phi + x\sin\phi)\mathbf{e} + (b\sin\phi + x\cos\phi)\mathbf{n}}{(b^2 + x^2)^{1/2}}, \tag{13}$$

where x ranges from $-(1-b^2)^{1/2}$ to $+(1-b^2)^{1/2}$ corresponding to points along the slit between the intersections of the slit image with the solar limb image. If we call α the angular displacement from the sun's center of the solar surface point corresponding to the point P on the slit, then the vector $\mathbf{T}$ is

$$\mathbf{T} = S\left(\cos\alpha - (\sin^2\rho - \sin^2\alpha)^{1/2}\right) \cdot \left((\sin\alpha)\mathbf{u} + (\cos\alpha)\mathbf{s}\right), \tag{14}$$

where $\alpha = (b^2 + x^2)^{1/2}\rho$, and the first term in square brackets is the distance to the surface point.

The displacement from the sun's center of the solar surface point corresponding to P is

$$\mathbf{R} = \mathbf{T} - \mathbf{S}, \tag{15}$$

so the unit vector is

$$\mathbf{r} = (\mathbf{T} - \mathbf{S})/|\mathbf{T} - \mathbf{S}|. \tag{16}$$

At the solar limb the expressions for $\mathbf{t}$ and $\mathbf{r}$ reduce to

$$\mathbf{t} = (\sin\rho)\mathbf{u} + (\cos\rho)\mathbf{s} \tag{17}$$

and

$$\mathbf{r} = (\cos\rho)\mathbf{u} - (\sin\rho)\mathbf{s}. \tag{18}$$

By substituting the extreme values of x in equations (13) and (14) one obtains the expressions for $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{t}_1$, and $\mathbf{t}_2$ required for the implementation of equation (10).

To put this all together in a single expression for S in terms of the measured quantities is a formidable job of algebra and not very enlightening. On the other hand, it is relatively simple to do it stepwise in a computer program, as in the application called `Calculate Sun Distance from Doppler Measuremen` which resides on the Junior Lab Server PC.

REFERENCES

[1] P. R. Bevington & D. K. Robinson 1992, *Data Reduction and Error Analysis for the Physical Sciences*, 2nd Edition, McGraw Hill.

[2] C. W. Allen 1976, *Astrophysical Quantities*, 3rd Edition The Athlone Press. (Junior Lab Library, Physics Library or CSR Reading Room).

[3] J. M. Becker, C. A. Bridges & L. B. Gilliam 1976, *A High Resolution Spectral Atlas of the Solar Irradiance from 380 to 700 Nanometers*, Air Force Geophysics Laboratory. (Junior Lab reference file, relevant portions copied in the appendix).

[4] F. H. Chaffee & D. J. Schroeder 1976, *Astronomical Applications of Echelle Spectroscopy*, Ann. Rev. Astron. Astrophys. **14**, 23. (Physics Library, Science Library, or CSR Reading Room).

[5] J. A. Hynek, ed. 1951, *Astrophysics*, McGraw-Hill. (Junior Lab Library).

[6] G. R. Harrison 1949, *The Production of Diffraction Gratings: II. The Design of Echelle Gratings and Spectrographs*, J. Opt. Soc. Amer. **39**, 522 (Junior Lab reference file).

[7] G. P. Kuiper, ed. 1970, *The Sun*, University of Chicago Press. (Junior Lab Library).

[8] B. A. Lacy 1975, *Spectrographic Determination of the Rotational Velocity of the Sun*, B.S. Thesis, MIT. (Junior Lab reference file).

[9] J. M. Pasachoff 1981, *Contemporary Astronomy*, Saunders.

SUGGESTED THEORETICAL TOPICS

1. Variation of the Doppler shift along an arbitrary chord on the solar disk.
2. Widths of the solar and atmospheric absorption lines.
3. Line formation in the solar atmosphere.
4. Another method for determination of the astronomical unit.
5. Orbit theory.

A CELESTIAL GEOMETRY

Celestial coordinates (RA and DEC) are the spherical coordinates used to specify the location of a celestial object (Figures A1 and A2). The north celestial pole (NCP) is the direction of the earth's rotation axis, and the celestial equator is the projection onto the sky of the plane of the earth's equator.

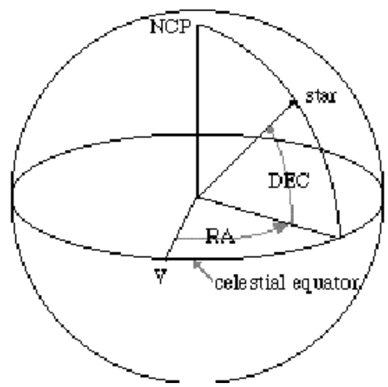

Figure A1: Diagram of the celestial coordinate system.

Right ascension (RA) is the celestial analog of geographic longitude. RA is measured eastward along the celestial equator from the vernal equinox ("V" in Figures A1 and A2) which is the ascending node of the plane defined by the sun's apparent motion (caused by the orbital motion of the earth around the sun) and the celestial equator. In catalogs of celestial objects RA is generally specified in units of hours, minutes and seconds from 0 to 24 hours, but it is often more conveniently specified in degrees from $0°$ to $360°$ with a decimal fraction.

Declination (DEC) is the celestial analog of geographic latitude. DEC is measured north from the celestial equator along a celestial meridian which is a great circle of constant RA. In catalogs DEC is generally specified in degrees, arc minutes ($'$) and arc seconds ($''$), but it is also often more conveniently specified in degrees from $-90°$ to $+90°$ with a decimal fraction.

The ecliptic is the intersection of the earth's orbital plane with the celestial sphere. To an observer on earth the sun appears to move relative to the background stars along the ecliptic with an angular velocity of about 1 degree per day. The angular velocity is not exactly constant due to the eccentricity of the earth's orbit ($e = 0.016722$). The period of the earth's orbit is 365.256 days.

The inclination (i_e) of the earth's equator to the ecliptic is $23°27'$.

The ascending node of the ecliptic with respect to the celestial equator is the intersection of the ecliptic and the celestial equator (the vernal equinox) where the sun in its apparent motion crosses from south to north declinations on March 21.

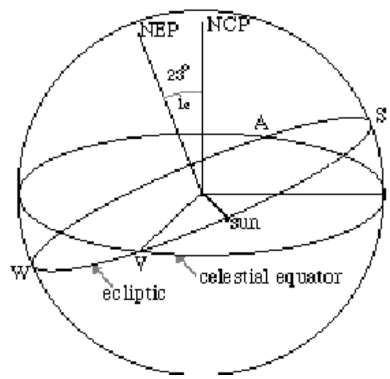

Figure A2: Diagram of the celestial sphere, showing the celestial north pole (NCP), the celestial equator, the north ecliptic pole (NEP) and the ecliptic. The points on the ecliptic labeled V, S, A, and W are, respectively, the vernal equinox, summer solstice, autumnal equinox, and winter solstice corresponding to the directions of the sun on March 21, June 21, September 21, and December 21. The point labeled "sun" is the direction of the sun on approximately April 21.

Precession of the equinoxes is the motion of the equinoxes along the ecliptic due to precession of the earth's rotational angular momentum about the ecliptic pole. The precession is caused by the torque of the gravitational attractions between the sun and moon and the earth's equatorial bulge. The period of the precession is approximately 25,000 years.

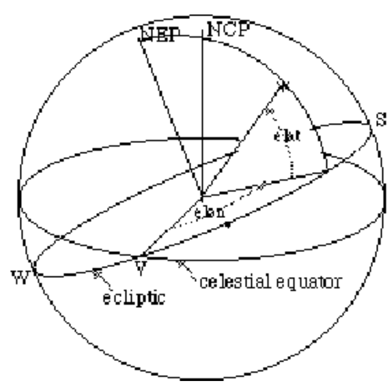

Figure A3: Diagram showing the relation between the ecliptic and celestial coordinate systems.

Ecliptic coordinates (Figure A3) are generally used to specify the positions and orientations of objects in the solar system.

Ecliptic longitude (elon) is measured along the ecliptic eastward from the vernal equinox.

Ecliptic latitude (elat) is measured along a great circle northward from the ecliptic.

The orientation of the orbit of a planet is specified by 1) the ecliptic longitude of the ascending node of the orbital plane and 2) the inclination of the orbit to the ecliptic. Similarly, the orientation of a planet's rotation or the rotation of the sun itself, as illustrated in Figures A4 and A5, is specified by the ecliptic longitude (ELON) of the ascending node of its equator and the inclination (INCL) of the equator to the ecliptic.

Figure A4 can be used to visualize the motions of the earth and sunspots relative to the center of the sun. (Stare at this diagram and concentrate your attention on the central one of the three images you see).

The direction of the earth relative to the sun moves along the ecliptic with an angular velocity parallel to the ecliptic pole, i.e. in the direction of the curled fingers of your right hand when the thumb points in the direction of EP.

Solar coordinates are used to specify the position of sun spots and other features on the sun's surface.

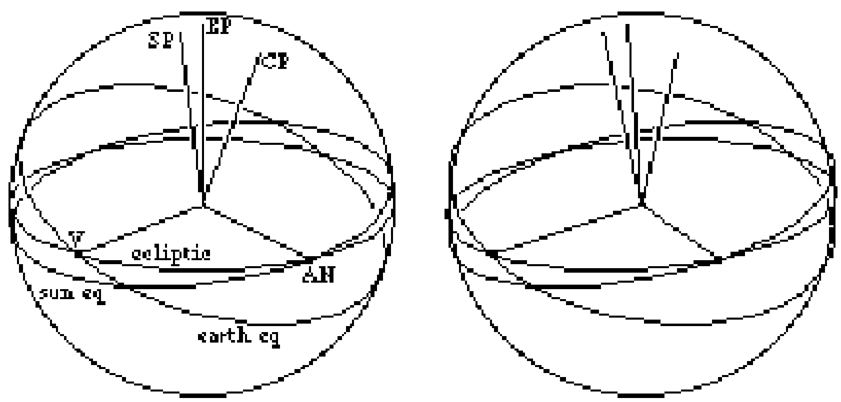

Figure A4: Stereoscopic diagram of the relations between celestial, ecliptic, and solar coordinates. The poles of the three systems are designated CP, EP, and SP, respectively. The respective equatorial circles are labeled earth, ecliptic, and sun.

Solar longitude (SLON) measured along the solar equator in the direction of solar rotation from the ascending node on the ecliptic.

Solar latitude (SLAT) measured along a great circle northward from the solar equator.

Unlike geographic longitude which is referenced to a fixed point (Greenwich) on the solid earth, solar longitude is referenced to a direction fixed in space. This is necessary because the sun's surface is fluid and has no permanent features. Thus the longitude of a sunspot, carried around by the general solar rotation, increases with time. Sunspots drift toward the solar equator from the time they are formed till they dissipate in the course of several solar rotations.

We turn, finally, to the practical problem of pointing a telescope to observe an object at

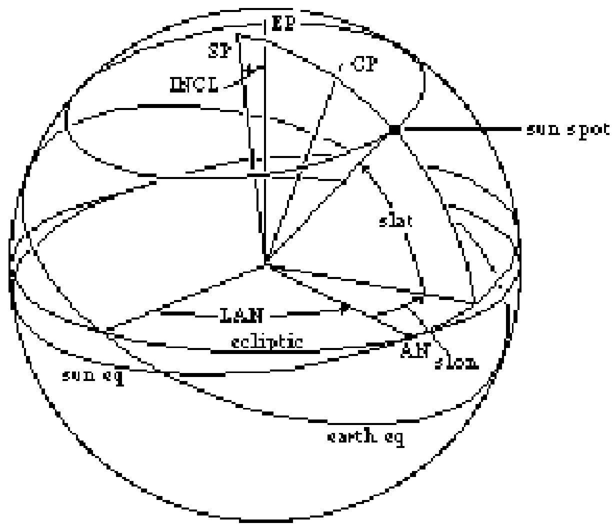

Figure A5: Diagram showing the relation between the solar and ecliptic coordinate system.

some given time of the day. Times will be expressed in degrees according to the relation 1 hour = 15°. We first define the following quantities:

<u>Hour circle of an object</u>. A great circle on the celestial sphere passing through the celestial poles and the position of the object.

<u>Hour angle</u> (HA). The angle, measured westward through 360°, between the celestial meridian of an observer and the hour circle of the celestial object. (The hour angle of the "mean sun" advances with time like the hands of a 24-hour clock, completing one revolution in one solar day. The hour angle of a fixed star completes one revolution in one sidereal day of 23.934 hours.)

<u>Universal time</u> (UT). The hour angle of the mean sun at Greenwich plus twelve hours.

<u>Greenwich sidereal time</u> (GST). The hour angle at Greenwich of the vernal equinox.

<u>Eastern standard time</u> (EST). UT + 15°n, where n is the number of time zones west of Greenwich. ($n = 5$ at Boston, so 12 noon EST is 17:00:00 UT.)

<u>Local sidereal time</u> (LST). The hour angle of the vernal equinox at the location of the observer.

Suppose the telescope is on a standard two-axis polar mounting at geodetic longitude LONG and latitude LAT with the RA axis (polar axis) parallel to the earth's rotation axis and the DEC axis perpendicular to the RA axis. This is the most common arrangement of telescopes and the way the heliostat is mounted. Given the RA and DEC of a star, properly corrected for the precession of the equinoxes from the epoch of the catalog to the present date, one proceeds as follows:

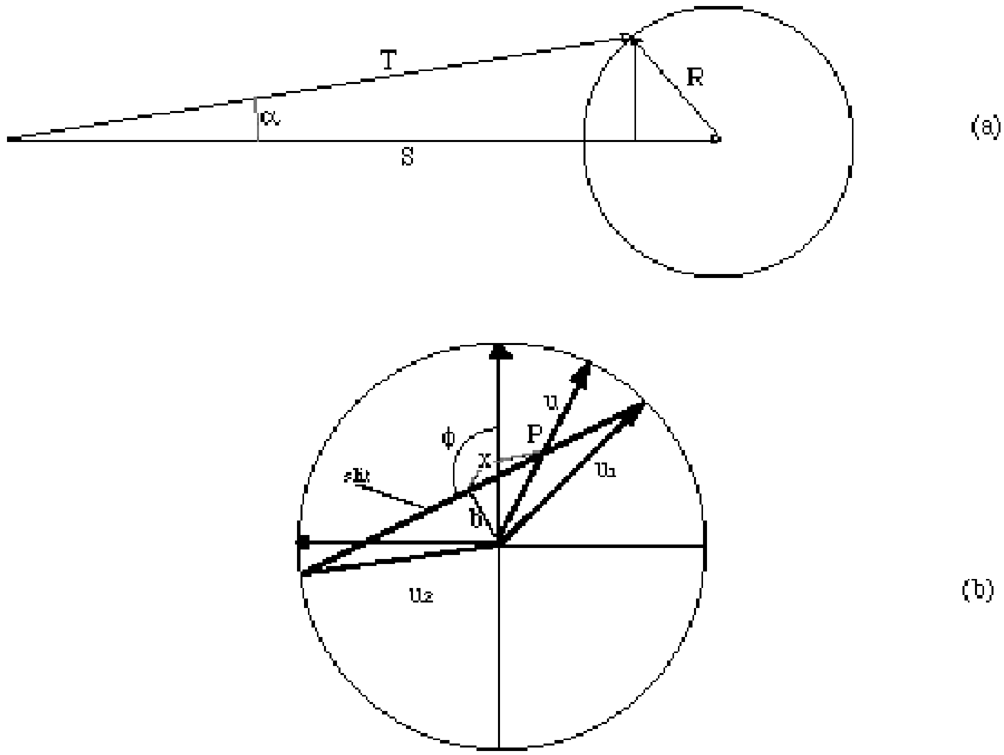

Figure A6: Diagram of the relations between the quantities involved in setting the position of a telescope about the polar (RA) axis.

1. Compute the local sidereal time according to the formulas (all numbers in units of degrees):

$$\text{LST} = \text{GST} - \text{LONG} \tag{A1}$$

$$\text{GST} = \text{GST(Jan 0.0)} + (360/365.25) \cdot D + \text{UT} \tag{A2}$$

   where D is the time in days from Jan 0.0 to the instant of observation.

   Note: The precise formula for the relation between GST and UT can be found in The Astronomical Almanac. For the year 1992 GST(Jan 0.0) = 98.87. For other years the value differs by less than 1°.

2. Rotate the telescope about the DEC axis through the angle DEC − LAT, where LAT is the local latitude (positive rotation is north). Then rotate the telescope about the RA axis through the angle HA = LST − RA (positive is west).

A.1 Interpretation of Diurnal Motion in a Sky Image

The objective lens (or mirror) of a telescope, like a pinhole, forms a conical projection of the sky on its image plane which may be a photographic film or, as in the solar experiment, the drawing board on the floor beneath the slit lens. Call **t** and **s** the unit vectors in the directions of the telescope axis and star, respectively, and β the angle between the two directions. Then $\beta = \arccos(\mathbf{t} \cdot \mathbf{s})$, and the linear displacement of the star from the center of the projection is $f \tan\beta$ If the celestial coordinates of the telescope and star are (α, δ) and $(\alpha + \Delta\alpha, \delta)$, respectively, then the cartesian coordinates of $\mathbf{t}$ and $\mathbf{s}$ are $[\cos\delta\cos\alpha, \cos\delta\sin\alpha, \sin\delta]$ and $[\cos\delta\cos(\alpha+\Delta\alpha), \cos\delta\sin(\alpha+\Delta\alpha), \sin\delta]$. Now

$$\cos\beta = \mathbf{t} \cdot \mathbf{s} = \cos^2\delta\cos(\Delta\alpha) + \sin^2\delta. \tag{A3}$$

In the present measurements, $\Delta\alpha \ll 1$ and $\beta \ll 1$, so

$$\begin{aligned} 1 - \beta^2/2 &\approx \cos^2\delta\left[1 - (\Delta\alpha)^2/2\right] + \sin^2\delta && \text{(A4)} \\ &\approx 1 - \cos^2\delta \cdot (\Delta\alpha)^2/2. && \text{(A5)} \end{aligned}$$

Thus, the angle β between the look direction and a direction at the same declination, but separated by the angle $\Delta\alpha$ in right ascension, can be expressed approximately as

$$\beta \approx (\Delta\alpha)\cos\delta, \tag{A6}$$

with an error of the order of $(\Delta\alpha)^2$. If the image of a star, formed with a lens of focal length f and initially at the center of the field, drifts for a short time T, then the motion of the star in right ascension is $\Delta\alpha = \omega T$, where ω is the angular velocity of the earth's rotation. Call a the measured linear drift in the image plane. Then

$$a = f\tan(\omega T\cos\delta) \approx f\omega T\cos\delta. \tag{A7}$$

Call θ the angular diameter of the sun and d the measured linear diameter of its image. Then

$$d = f\theta. \tag{A8}$$

Combining the last two equations, we find

$$\theta = (d/a)\omega T\cos\delta. \tag{A9}$$

Unit vectors $\mathbf{e}$ and $\mathbf{n}$ pointing east and north in the image plane are defined in terms of the look direction $\mathbf{t}$ and the direction of the celestial pole $\mathbf{p}$ by the equations

$$\mathbf{e} = \frac{\mathbf{p} \times \mathbf{t}}{|\mathbf{p} \times \mathbf{t}|}, \qquad \mathbf{n} = \mathbf{t} \times \mathbf{e}. \tag{A10}$$

The <u>position angle</u> of a given line in the image plane is the angle between the north direction and the given line, measured eastward. Thus the position angle of a direction vector pointing due west is 270°. At solar noon the position angle of the slit of the echelle spectrograph is close to 90°.

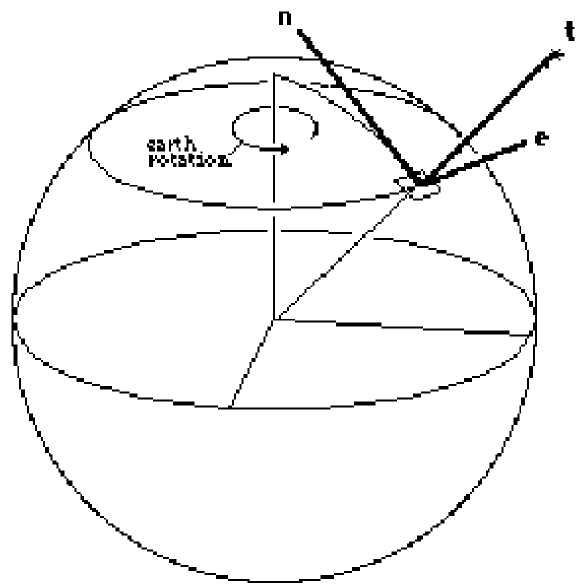

Figure A7: Diagram of the celestial sphere showing the local coordinate system used in measuring the relative positions of objects in a small piece of the sky.

B DETERMINATION OF THE GRAVITATIONAL CONSTANT—THE CAVENDISH EXPERIMENT

B.1 Introduction

According to Newton, two spherically symmetric bodies, A and B, with inertial masses, M_A and M_B, attract one another with a force of magnitude GM_AM_B/r^2 where r is the separation between the centers and G is the universal constant of gravity. The determination of G is obviously of fundamental importance in physics and astronomy. But gravity is the weakest of the forces, and the measurement of the gravity force between two bodies of measurable mass requires a delicate approach, with meticulous care to reduce perturbing influences such as air currents and electromagnetic forces. Henry Cavendish did it in 1798, a century after Newton's discovery of the law of universal gravitation. He used a torsion balance invented by one Rev. John Michell and, independently, by Charles Coulomb. Michell died shortly after completing his device, never having had the opportunity to apply it to the measurement of small forces for which he had devised it. It was passed on to Professor John Wollaston of Cambridge University and eventually to Cavendish who improved it and used it in a painstaking series of experiments to measure the mean density of the earth from which the value of G is readily derived. For the earth's mean density he found the value 5.48 g cm^3 with a stated uncertainty of 1 part in 14, which implies a value for G of $(6.70 \pm 0.48) \times 10^{-8}$ dynes cm^2 g^{-2}. The current best value is $(6.67259 \pm 0.00085) \times 10^{-8}$ dynes cm^2 g^{-2}. The large uncertainty, 128 ppm, compared to that of any of the other fundamental constants such as the elementary charge (0.30 ppm) and Planck's constant (0.60 ppm) reflects the fact that even today it is difficult to achieve an accurate measurement of G.

The difficulties are obscured by our common experience with gravity which is the weight of things. If A is an apple and E is the earth, then the weight of A is $W_{AE} = GM_AM_E/R_E^2 = M_Ag$, where R_E is the radius of the earth and g is the acceleration of gravity. The latter two quantities can be measured easily to high accuracy (how?). Thus if you could measure M_E then you could determine G, or vice versa. It is clearly impossible to measure M_E as you do ordinary things, i.e. by direct comparison with a standard weight on a balance.

The only recourse is to replace the earth with a body B that can be measured directly, and to measure the force W_{AB} it exerts on the test body A. Suppose the radius of B is R_B and its density is ρ_B so that $M_B = (4\pi/3)R_B^3\rho_B$. Then if $r \approx R_B$, we find $W_{AB} = GM_A(4\pi/3)R_B\rho_B$. To get an idea of the practical difficulties that must be overcome in the measurement we can estimate the ratio of the force between a lead ball and a small test body at its surface to the force of earth's gravity on the test body. The radius of the earth is 6.371×10^8 cm. We can use Cavendish's value for the earth's mean density. If the radius of the lead ball is 3 cm and its density is 11.3 g cm^{-3}, then the ratio of forces is $W_{AB}/W_{AE} = (\rho_B/\rho_E)(R_B/R_E) \approx 10^{-8}$. Thus he had to measure a force on a test body that was about one hundred-millionth of its weight!

The torsion balance in the Junior Lab is shown schematically in Figure B1. It consists of a horizontal brass beam on the ends of which are two brass balls each of of mass m separated by a distance l between their centers, as shown in detail in Figure B2. The beam is suspended from its balance point by a fine tungsten wire which allows the beam to rotate about a vertical axis, subject to a restoring torque that is proportional to the angular displacement, θ, of the beam from its equilibrium orientation. The idea of the experiment is to measure the angular twist $\Delta\theta$ of the beam when two lead balls, each of mass M, are shifted from the positions labeled 1 to the positions labeled 2. If the distance between the center of each brass ball to the center of the nearest lead ball in both configuration 1 and 2 is called b, then the angular twist is

$$\Delta\theta = \frac{2GMml}{b^2\kappa} \tag{B1}$$

where κ is the torsion constant. To measure this latter quantity we turn to the equation of motion of the torsion pendulum which is

$$\frac{d^2\theta}{dt^2} = -(\kappa/I)\theta - \beta\frac{d\theta}{dt}, \tag{B2}$$

where I is the moment of inertia of the pendulum, and β is the coefficient of damping. With the initial condition $\theta(t = 0) = 0$ the solution to equation (B2) is

$$\theta(t) = \theta_0 \exp(-\beta t/2)\sin(\omega t), \tag{B3}$$

where

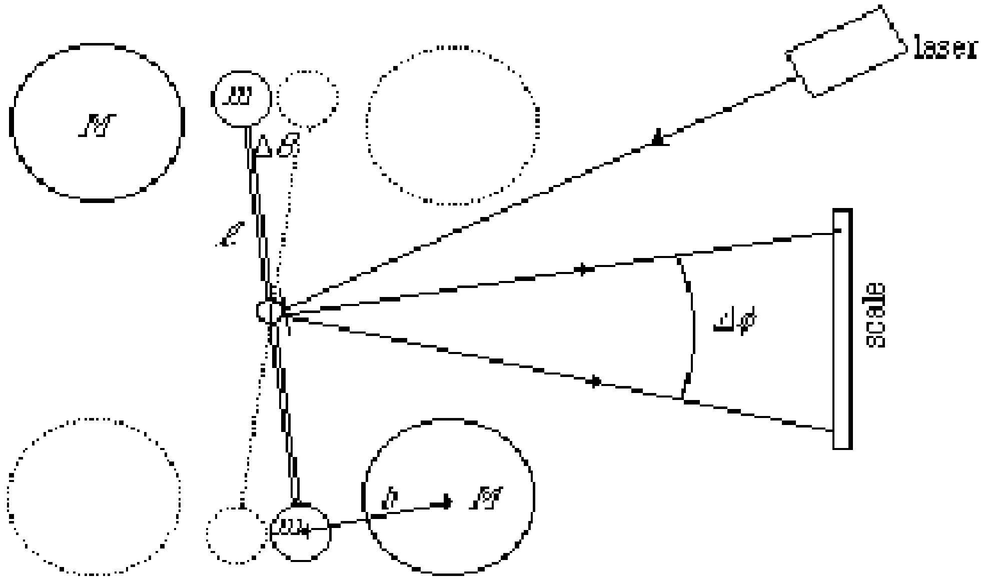

Figure B1: Schematic diagram of the torsion pendulum used in the Cavendish measurement of G.

$$\omega = \sqrt{\kappa/I - \beta^2/4}. \tag{B4}$$

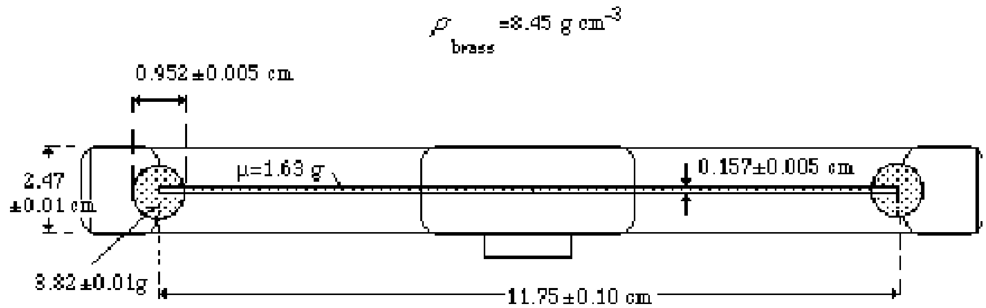

Figure B2: Details of the torsion balance beam, showing the two small brass balls mounted on the ends of a brass rod and suspended at the middle by a tungsten wire inside 1/2″ pipe.

Equation (B3) describes a damped harmonic motion about an equilibrium orientation with a period $T = 2\pi/\omega$ and a characteristic damping time of $2/\beta$. In a typical setup this damping time is more than twice the period so that to a good approximation $\kappa = I(2\pi/T)^2$. If the beam is light compared to the brass balls, the moment of inertia is given to good accuracy by the formula $I = ml^2/2$. Finally, one can determine the angular displacement caused by shifting the lead balls by measuring the angular deflection $\Delta\phi = 2\Delta\theta$ of a laser beam reflected from a mirror mounted on the beam of the torsion pendulum. Substituting these quantities into equation (B1) and rearranging we obtain an expression for G in terms of measurable quantities which is

$$G = \frac{b^2 l}{8M}\left(\frac{2\pi}{T}\right)^2 \Delta\phi. \tag{B5}$$

Note that the result is independent of the value of m.

The Junior Lab Cavendish pendulum is suspended by a fragile 1 mil tungsten wire. The wire and beam are contained within copper plumbing to shield them from air currents and electric forces from stray static charges. Even the window for the laser beam is covered with fine wire mesh and glass. To avoid having to take the device apart at the risk of breaking the tungsten wire we provide you with the value of the distance between the brass balls , namely $l = (11.75 \pm 0.10)$ cm. The other quantities are left for you to measure.

B.2 Experiment

Set up a laser so that its beam reflects from the mirror on the pendulum beam onto a meter stick mounted far enough away to facilitate an accurate measure of the angular displacement caused by shifting the lead balls. Ascertain whether the torsion pendulum is swinging freely about an equilibrium orientation near the center of its free range. If it isn't, make very gentle and cautious adjustment by twisting the fitting on the top of the pipe a few degrees at a time. (Take care not to snap the tungsten wire which is attached to the screw in the top fitting. That screw can be turned to raise or lower the pendulum in the pipe.) The pendulum can be very gently maneuvered from the outside by the magnetic force exerted on the brass balls by a bar magnet. Set the lead balls in contact with the pipes so they are as close to the unseen brass balls inside as possible.

When you get the pendulum swinging with a very small amplitude about a central equilibrium position, start recording and plotting the position of the reflected beam on the scale at regular intervals so that you have a record of the damped harmonic motion from which you can determine the period and damping time of the pendulum. After the amplitude has died to a very small value or zero, shift the lead balls to the other position and begin regular periodic reading and plotting of the laser spot position on the scale. Go back and forth several times in this way to improve the statistical accuracy of your measurement. To save time, you can make an accurate determination of the equilibrium position y_0 of the swinging pendulum by measuring the positions y_1, y_3 of two successive maximum positive displacements and the one maximum negative displacement y_2 in between during a little more than one period of oscillation. The equilibrium position is approximately $y_0 = [(y_1 + y_3)/2 + y_2]/2$.

Before shutting down, check that you have measured all the relevant quantities.

B.3 Analysis

Determine the period and characteristic damping time (the time for the amplitude to decrease by $1/e$). Compute the angular deflection from the displacement between the two equilibrium positions of the laser spot on the scale. Compute the value of G from equation (B5) with these and the other measured and given quantities. Make a careful estimate of the uncertainty.

Using your value of G and the well known value of the acceleration of gravity at the earth's surface (which you can readily measure to high accuracy in a simple experiment), compute the mass of the earth. Using the period of the earth's orbit and the value of the astronomical unit derived from your measurement of the Doppler shifts of the Fraunhöfer lines, compute the mass of the sun. And finally, given the period of the sun's orbit around the center of the galaxy ($\sim 2 \times 10^8$ yr) and its distance from the center ($\sim 3 \times 10^4$ lt yr), estimate the mass of the galaxy.

B.4 Refinements

Consider the following corrections to the simple analysis above:

1. the effect of damping on the pendulum period.
2. the effect of the attractive forces between the lead balls and
   (a) the opposite brass balls (weight = 7.60 g),
   (b) the brass beam (weight = 1.625 g).
3. error in the calculation of the moment of inertia of the brass beam and ball.
4. other things that come to mind.

Derive the value in cm of the unit of astronomical distance called the parsec, which is the distance of a star with an annual parallax of one arc second. (Note: The distance to the nearest star beyond the sun is about 1.3 parsec).

B.5 Suggested Theoretical Topics

6. Damped harmonic motion.
7. Corrections to equation (B5)