Massachusetts Institute of Technology
Department of Physics

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Solution to Problem set #4

1. (5 points)

The squared four momentum of the scattered parton is

$$(q + \xi P)^2 = m^2c^2.$$

By expanding this and multiplying with x^2/Q^2,

$$\begin{aligned} &q^2 + 2\xi q \cdot P + \xi^2 P^2 = m^2c^2 \\ &q^2 + 2\xi M\nu + \xi^2 M^2 = m^2c^2 \qquad (\because\ q \cdot P = M\nu, P^2 = M^2) \\ &\frac{x^2M^2c^2}{Q^2}\xi^2 + x\xi - x^2\left(1 + \frac{m^2c^2}{Q^2}\right) = 0. \end{aligned}$$

Solving for ξ,

$$\xi = \frac{-1 + \sqrt{1 + 4\frac{x^2M^2c^2}{Q^2}\left(1 + \frac{m^2c^2}{Q^2}\right)}}{2\frac{xM^2c^2}{Q^2}}.$$

The given approximate formula with $\varepsilon = 4\frac{x^2M^2c^2}{Q^2}$ and $\varepsilon' = \frac{m^2c^2}{Q^2}$ gives

$$\sqrt{1 + 4\frac{x^2M^2c^2}{Q^2}\left(1 + \frac{m^2c^2}{Q^2}\right)} \approx 1 + 2\frac{x^2M^2c^2}{Q^2}\left(1 + \frac{m^2c^2}{Q^2} - \frac{x^2M^2c^2}{Q^2}\right).$$

Employing this yields

$$\xi = x\left[1 + \frac{m^2c^2 - M^2c^2x^2}{Q^2}\right].$$

2. (5 points)

Sea quarks result when gluons produce quark-antiquark pairs. This process has the greatest probability of zero (or very low) energy and falls off as the energy increases. So the sea quark distribution peaks at x (momentum fraction) = 0.

The valence quarks on average would carry $\frac{1}{3}$ of the nucleon's momentum, were it not for the gluons. The data show that the gluons carry approximately $\frac{1}{2}$ of the momentum. So (xq_{v}) peaks at $x \approx 0.2$. Because of the interactions among the quarks, the distribution is broad, extending all the way from $x = 0$ to $x = 1$ ($x = 1$: single quark carries all the momentum!).

3. (15 points)

$$\begin{aligned}
\frac{1}{x}F_2^{\mathrm{p}}(x) &= \left(\frac{2}{3}\right)^2\left[u^{\mathrm{p}}(x)+\bar{u}^{\mathrm{p}}(x)\right] + \left(\frac{1}{3}\right)^2\left[d^{\mathrm{p}}(x)+\bar{d}^{\mathrm{p}}(x)\right] + \left(\frac{1}{3}\right)^2\left[s^{\mathrm{p}}(x)+\bar{s}^{\mathrm{p}}(x)\right] \\
&= \frac{1}{9}\left[4u_{\mathrm{v}}^{\mathrm{p}}(x)+d_{\mathrm{v}}^{\mathrm{p}}(x)\right] + \frac{4}{3}S(x) \quad \text{where } S(x) = \text{sea quark distribution} \\
\frac{1}{x}F_2^{\mathrm{n}}(x) &= \frac{1}{9}\left[4u_{\mathrm{v}}^{\mathrm{n}}(x)+d_{\mathrm{v}}^{\mathrm{n}}(x)\right] + \frac{4}{3}S(x) \\
&= \frac{1}{9}\left[4d_{\mathrm{v}}^{\mathrm{p}}(x)+u_{\mathrm{v}}^{\mathrm{p}}(x)\right] + \frac{4}{3}S(x) \quad \text{by isospin symmetry} \left\{\begin{array}{l} u^{\mathrm{p}}(x) = d^{\mathrm{n}}(x) \\ d^{\mathrm{p}}(x) = u^{\mathrm{n}}(x) \end{array}\right.
\end{aligned}$$

$$\int_0^1 F_2^{\mathrm{p}}(x)\mathrm{d}x \approx \frac{4}{9}\int_0^1 xu_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x + \frac{1}{9}\int_0^1 xd_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x,$$

since sea quarks carry very little momentum.

$$\int_0^1 F_2^{\mathrm{n}}(x)\mathrm{d}x \approx \frac{4}{9}\int_0^1 xd_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x + \frac{1}{9}\int_0^1 xu_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x.$$

Then,

$$\frac{\frac{4}{9}\int_0^1 xu_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x + \frac{1}{9}\int_0^1 xd_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x}{\frac{4}{9}\int_0^1 xd_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x + \frac{1}{9}\int_0^1 xu_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x} = \frac{0.18}{0.12} = \frac{3}{2}.$$

$$\begin{aligned}
&4\int_0^1 xu_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x + \int_0^1 xd_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x = \frac{3}{2}\left(4\int_0^1 xd_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x + \int_0^1 xu_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x\right) \\
&8\int_0^1 xu_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x + 2\int_0^1 xd_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x = 12\int_0^1 xd_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x + 3\int_0^1 xu_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x \\
&\Longrightarrow \boxed{\int_0^1 xu_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x = 2\int_0^1 xd_{\mathrm{v}}^{\mathrm{p}}(x)\mathrm{d}x}.
\end{aligned}$$

4. (10 points)
From Problem 3,

$$\frac{1}{x}F_2^{\mathrm{p}}(x) = \frac{1}{9}\left[4u_{\mathrm{v}}^{\mathrm{p}}(x) + d_{\mathrm{v}}^{\mathrm{p}}(x)\right] + \frac{4}{3}S(x)$$
$$\frac{1}{x}F_2^{\mathrm{n}}(x) = \frac{1}{9}\left[4d_{\mathrm{v}}^{\mathrm{p}}(x) + u_{\mathrm{v}}^{\mathrm{p}}(x)\right] + \frac{4}{3}S(x).$$

So

$$\frac{F_2^{\mathrm{n}}(x)}{F_2^{\mathrm{p}}(x)} = \frac{u_{\mathrm{v}}^{\mathrm{p}}(x) + 4d_{\mathrm{v}}^{\mathrm{p}}(x) + 12S(x)}{4u_{\mathrm{v}}^{\mathrm{p}}(x) + d_{\mathrm{v}}^{\mathrm{p}}(x) + 12S(x)}.$$

If at given value of x, $F_2^{\mathrm{p}}(x)$ was dominated by up quarks, and $F_2^{\mathrm{n}}(x)$ by down quarks (which have the same distribution as the up quarks in the protons),

$$\frac{F_2^{\mathrm{n}}(x)}{F_2^{\mathrm{p}}(x)} = \frac{1}{4}.$$

If $F_2^{\mathrm{p}}(x)$ was dominated by down quarks and $F_2^{\mathrm{n}}(x)$ by up quarks, then

$$\frac{F_2^{\mathrm{n}}(x)}{F_2^{\mathrm{p}}(x)} = 4.$$

If the sea quarks dominate, as they do at very low x,

$$\frac{F_2^{\mathrm{n}}(x)}{F_2^{\mathrm{p}}(x)} = 1.$$

Adding the same quantity to numerator and denominator will always bring the ratio close to 1. So

$$\frac{1}{4} \leq \frac{F_2^{\mathrm{n}}(x)}{F_2^{\mathrm{p}}(x)} \leq 4.$$

5. (10 points)
a)

$$x = \frac{Q^2}{2M\nu}$$
$$\implies x_{\min} = \frac{Q^2}{2ME} = \frac{4\mathrm{GeV}^2/\mathrm{c}^2}{2(0.938\mathrm{GeV}^2/\mathrm{c}^2)(600\mathrm{GeV})} \approx 0.003$$

b)
Normalizing constant A,

$$\int_0^1 q_{\mathrm{v}}(x)\mathrm{d}x = A\int_0^1 \frac{(1-x)^3}{\sqrt{(x)}}\mathrm{d}x = A\frac{32}{35} = 3 \quad \implies \quad A = \frac{105}{32}.$$

Integrating the parton distributions,

$$\int q_{\mathrm{v}}(x)\mathrm{d}x = \frac{105\sqrt{x}}{32}\left(2 - 2x + \frac{6}{5}x^2 - \frac{2}{7}x^3\right)$$
$$\int q_{\mathrm{s}}(x)\mathrm{d}x = 0.4\left(\ln x - 8x + 14x^2 - \frac{56}{3}x^3 + \frac{35}{2}x^4 - \frac{56}{5}x^5 + \frac{14}{3}x^6 - \frac{8}{7}x^7 + \frac{1}{8}x^8\right)$$
$$\int g(x)\mathrm{d}x = 4\left(\ln x - 6x + \frac{15}{2}x^2 - \frac{20}{3}x^3 + \frac{15}{4}x^4 - \frac{6}{5}x^5 + \frac{1}{6}x^6\right).$$

So, one finds

	Valence quarks	Sea quarks	Gluons
$x > 0.3$	0.30	0.005	0.126
$x > 0.03$	1.90	0.41	4.92
$x > 0.003$	2.64	1.25	13.51

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