PROBLEM 1: A CIRCLE IN A NON-EUCLIDEAN GEOMETRY (5 points)

Consider a universe described by the Robertson-Walker metric, Eq. (6.21),

\[ ds^2 = R(t)^2 \left\{ -dt^2 + \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\}. \]

This problem will involve only the geometry of space at some fixed time, so we can ignore the dependence of \( R \) on \( t \), and think of it as a constant. Consider a circle described by the equations

\[ z = 0 \]
\[ x^2 + y^2 = r_0^2, \]

or equivalently by the angular coordinates

\[ r = r_0 \]
\[ \theta = \pi/2. \]

(a) Find the circumference \( S \) of this circle. Hint: break the circle into infinitesimal segments of angular size \( d\phi \), calculate the arc length of such a segment, and integrate.

(b) Find the radius \( \rho \) of this circle. Note that \( \rho \) is the length of a line which runs from the origin to the circle \( (r = r_0) \), along a trajectory of \( \theta = \pi/2 \) and \( \phi = \text{constant} \). Hint: break the line into infinitesimal segments of coordinate length \( dr \), calculate the length of such a segment, and integrate. Consider the case of open and closed universes separately, and take \( k = \pm 1 \). (If you don't remember why we can take \( k = \pm 1 \), see the section called "Units" in Lecture Notes 4).

(c) Express the circumference \( S \) in terms of the radius \( \rho \). This result is independent of the coordinate system which was used for the calculation, since \( S \) and \( \rho \) are both measurable quantities. Since the space described by this metric is homogeneous and isotropic, the answer does not depend on where the circle is located or on how it is oriented. For the two cases of open and closed universes, state whether \( S \) is larger or smaller than the value it would have for a Euclidean circle of radius \( \rho \).

PROBLEM 2: VOLUME OF A CLOSED UNIVERSE (5 points)

Calculate the total volume of a closed universe. It will be easiest to use the metric in the form of Eq. (6.12):

\[ ds^2 = a^2 \left\{ d\psi^2 + \sin^2 \psi \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right\}. \]

We will continue to use the convention that \( k = \pm 1 \), so in this case \( k = 1 \) and \( a = R \).

Calculate the volume up into spherical shells of infinitesimal thickness, extending from \( \psi = 0 \) to \( \psi + d\psi \):

\[ \int dr \int d\psi = \sinh -1. \]

(b) Find the radius \( \rho \) of this circle. Note that \( \rho \) is the length of a line which runs from the origin to the circle \( (r = r_0) \), along a trajectory of \( \psi = \pi/2 \) and \( \theta = \text{constant} \). Hint: break the line into infinitesimal segments of coordinate length \( dr \), calculate the length of such a segment, and integrate. Consider the case of open and closed universes separately, and take \( k = \pm 1 \). (If you don't remember why we can take \( k = \pm 1 \), see the section called "Units" in Lecture Notes 4).

You will want the following integrals:

\[ \int \sqrt{1 + \sin^2 \psi} \]
For all practical purposes, the mass of the black hole can be treated as a point mass. The Schwarzschild geometry is given by the metric

\[ ds^2 = -c^2dt^2 + \frac{r^2}{GM}\left(\frac{dr^2}{r^2} + d\Omega^2\right) \]

where the radial coordinate $r$ depends on the mass of the black hole but for the Schwarzschild geometry, it can be treated as a point mass.

\[ r > r_s \]

This problem was taken from Quiz 2 of 1996, where it counted 50 points out of 100.

**Problem 3: Circular Orbits in a Schwarzschild Metric**

**Solution**

- **(10 points)**
  
  a) Use the metric to find the proper time interval $\tau$ required to complete one orbit.

  \[ \tau = \int \frac{dr}{\sqrt{\frac{r^2}{GM} - 1}} \]

  where $\tau$ is the proper time interval for a segment of the path.

  - **(15 points)**
    
    b) Compute the angular velocity $\omega$ at $r = r_s$.

    \[ \omega = \frac{\frac{dr}{dt}}{r^2} \]

    where $\omega$ is the angular velocity of circular motion at $r = r_s$.

  - **(10 points)**
    
    c) Calculate the time $\tau$ required to complete one orbit at $r = 2r_s$.

    \[ \tau_{2r_s} = \int \frac{dr}{\sqrt{\frac{r^2}{GM} - 1}} \]

    where $\tau_{2r_s}$ is the proper time interval for a segment of the path at $r = 2r_s$.

  **(3) Show that the above equation implies that $\omega$ is constant since $GM/c^2$ is a constant.**

  By comparing Eqs. (6.8) and (6.17), one can see that $\omega$ is indeed constant.

  **(5) Show that the proper time interval $\tau$ is independent of the location of the observer.**

  By considering different observers at different locations, the proper time interval $\tau$ remains constant.
Consider the case of a flat (i.e., $k = 0$) Robertson-Walker metric, which has
the simple form
\[
\left[ \varepsilon^p + \varepsilon^q \right] \varepsilon^{q} \varepsilon^{q} = \varepsilon^p
\]

Located by this relation, cosmologists define the angular size distance \( r \).

**PROBLEM 5: GEODESICS IN A FLAT UNIVERSE**

(This problem is not required, but can be done for 5 points extra credit.)

According to general relativity, in the absence of any non-gravitational forces,

\( \varepsilon^p = \varepsilon^q \varepsilon^q \) \( \varepsilon^q \varepsilon^q = \varepsilon^p \)

\[
\left[ \varepsilon^p + \varepsilon^q \right] \varepsilon^{q} \varepsilon^{q} = \varepsilon^p
\]

Find the Lagrangian for the geodesic, which for a flat universe has the simple form

\[
\frac{\partial \varepsilon}{\partial \varepsilon^0} = \varepsilon^p
\]

Cosmologists therefore define the luminosity distance by

\[
\ell_{\text{ang}} \equiv \frac{w}{\Delta \theta}
\]

\[
\ell_{\text{lum}} \equiv \sqrt{\frac{P}{4 \pi J}}
\]

What is the angular size distance \( \ell_{\text{ang}} \) of galaxy \( G \)?

Find the luminosity distance \( \ell_{\text{lum}} \) of galaxy \( G \). (Hint: the Robertson-Walker coordinates can be shifted so that the galaxy \( G \) is at the origin.)

Another common definition of distance is **angular size distance**, determined by measuring the apparent size of an object of known physical size. In a static, Euclidean space, a small sphere of diameter \( w \) at a distance \( \ell \) will subtend an angle \( \Delta \theta = w/\ell \):

\[
\text{Motivated by this relation, cosmologists define the angular size distance by}
\]

\[
\ell_{\text{ang}} \equiv \frac{w}{\Delta \theta}
\]

\[
\ell_{\text{lum}} \equiv \sqrt{\frac{P}{4 \pi J}}
\]

What is the luminosity distance \( \ell_{\text{lum}} \) of galaxy \( G \)?

**Problem 2: GEODESICS IN A FLAT UNIVERSE**

The light pulse left the galaxy at \( \psi_G \). Write down an equation which determines the time \( t_G \) at which the light pulse left the galaxy. (You may assume that the light pulse travels on a "null" trajectory, which means that \( d\tau = 0 \) for any segment of its path. Since you don't know \( R(t) \) you cannot solve this equation, so please do not try.)

b) What is the redshift \( z_G \) of the light from galaxy \( G \)? (Your answer may depend on \( t_G \), as well as \( \psi_G \) or any property of the function \( R(t) \).)

c) To estimate the number of galaxies that one expects to see in a given range of redshifts, it is necessary to know the volume of the region of space that corresponds to this range. Write an expression for the present value of the volume that corresponds to redshifts smaller than that of galaxy \( G \). (You may leave your answer in the form of a definite integral, which may be expressed in terms of \( \psi_G \), \( t_G \), \( z_G \), or the function \( R(t) \).)

d) There are a number of different ways of defining distances in cosmology, and generally they are not equal to each other. One choice is called **proper distance**, which corresponds to the distance that one could in principle measure with rulers. The proper distance is defined as the total length of a network of rulers that are laid end to end from here to the instant of time, each ruler just touching its neighbors on either side. Write down an expression for the proper distance \( \ell_{\text{prop}} \) of galaxy \( G \).

e) Another common definition of distance is **angular size distance**, determined by measuring the apparent size of an object of known physical size. In a static, Euclidean space, a small sphere of diameter \( w \) at a distance \( \ell \) will subtend an angle \( \Delta \theta = w/\ell \):

\[
\text{Motivated by this relation, cosmologists define the angular size distance by}
\]

\[
\ell_{\text{ang}} \equiv \frac{w}{\Delta \theta}
\]

\[
\ell_{\text{lum}} \equiv \sqrt{\frac{P}{4 \pi J}}
\]

f) A third common definition of distance is **luminosity distance**, which is determined by measuring the apparent brightness of an object for which the actual total power output is known. In a static, Euclidean space, the energy flux \( J \) received from a source of power \( P \) at a distance \( \ell \) is given by

\[
J \equiv \frac{P}{4 \pi \ell^2}
\]

Cosmologists therefore define the luminosity distance by

\[
\ell_{\text{lum}} \equiv \sqrt{\frac{P}{4 \pi J}}
\]
Since the spatial metric is flat, we have the option of writing it in terms of Cartesian rather than polar coordinates. Now consider a particle which moves along the \( x \)-axis. (Note that the galaxies are on the average at rest in this system, but one can still discuss the trajectory of a particle which moves through the model universe.)

(a) Use the geodesic equation to show that the coordinate velocity computed with respect to proper time (i.e., \( \frac{dx}{d\tau} \)) falls off as \( \frac{1}{R^2(t)} \).

(b) Use the expression for the spacetime metric to relate \( \frac{dx}{dt} \) to \( \frac{dx}{d\tau} \).

(c) The physical velocity of the particle relative to the galaxies that it is passing is given by

\[
v = R(t) \frac{dx}{dt}.
\]

Show that the momentum of the particle, defined relativistically by

\[
p = m v \sqrt{1 - \frac{v^2}{c^2}}
\]

falls off as \( \frac{1}{R^2(t)} \). (This implies, by the way, that if the particle were described as a quantum mechanical wave with wavelength \( \lambda = \frac{h}{|\vec{p}|} \), then its wavelength would stretch with the expansion of the universe, in the same way that the wavelength of light is redshifted.)

**Problem 6: The Klein Description of the G-B-L Geometry**

(This problem is not required, but can be done for 5 points extra credit.)

I stated in Lecture Notes 6 that the space invented by Klein, described by the distance relation

\[
\cosh [d(1,2) a] = 1 - x_1 x_2 - y_1 y_2 \sqrt{1 - x_1^2} \sqrt{1 - y_1^2} \sqrt{1 - x_2^2} \sqrt{1 - y_2^2},
\]

where

\[
x_2 + y_2 < 1,
\]

is a two-dimensional space of constant negative curvature. In other words, this is just a two-dimensional Robertson-Walker universe, which would be described by a two-dimensional version of the metric

\[
ds^2 = a^2 \left\{ \frac{dr^2}{1 - r^2} + r^2 d\theta^2 \right\}.
\]

The problem is to prove the equivalence.

(a) As a first step, show that if \( x \) and \( y \) are replaced by the polar coordinates defined by

\[
x = u \cos \theta \quad y = u \sin \theta,
\]

then the distance equation can be rewritten as

\[
\cosh [d(1,2) a] = 1 - u_1 u_2 \cos(\theta_1 - \theta_2) \sqrt{1 - u_1^2} \sqrt{1 - u_2^2}.
\]

(b) The next step is to derive the metric from the distance function above. Let

\[
\frac{d^2 - 1}{(\theta - \theta)^2} \cos \theta \sin \theta = \left[ \frac{p}{(\theta - \theta)} \right] \cosh
\]

insert these expressions into the distance function, expand everything to second order in the infinitesimal quantities, and show that

\[
ds^2 = a^2 \left\{ \frac{du^2}{1 - u^2} + u^2 d\theta^2 \right\}.
\]

(This part is rather messy, but you should be able to do it.)

(c) Now find the relationship between \( r \) and \( u \) and show that the two metric functions are identical. Hint: The coefficients of \( d\theta^2 \) must be the same in the two expressions.

Total points for Problem Set 3: 35, plus up to 10 points extra credit.