MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 26, 2005
Prof. Alan Guth

PROBLEM SET 4

DUE DATE: Thursday, November 3, 2005

READING ASSIGNMENT: Barbara Ryden, *Introduction to Cosmology*, Chapter 6.

PROBLEM 1: CIRCULAR ORBITS IN A SCHWARZSCHILD METRIC *(10 points)*

(This was originally Problem 3 on Problem Set 3, but was held over.)

The Schwarzschild metric, which describes the external gravitational field of any spherically symmetric distribution of mass, is given by

$$c^2 d\tau^2 = -ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \ ,$$

where M is the total mass of the object, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, and $\phi = 2\pi$ is identified with $\phi = 0$. We will be concerned only with motion outside the Schwarzschild horizon $R_{\text{Sch}} = 2GM/c^2$, so we can take $r > R_{\text{Sch}}$. (This restriction allows us to avoid the complications of understanding the effects of the singularity at $r = R_{\text{Sch}}$.) In this problem we will use the geodesic equation to calculate the behavior of circular orbits in this metric. We will assume a perfectly circular orbit in the x-y plane: the radial coordinate r is fixed, $\theta = 90°$, and $\phi = \omega t$, for some angular velocity ω.

(a) Use the metric to find the proper time interval $d\tau$ for a segment of the path corresponding to a coordinate time interval dt. Note that $d\tau$ represents the time that would actually be measured by a clock moving with the orbiting body. Your result should show that

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2GM}{rc^2} - \frac{r^2\omega^2}{c^2}} \ .$$

Note that for $M = 0$ this reduces to the special relativistic relation $d\tau/dt = \sqrt{1 - v^2/c^2}$, but the extra term proportional to M describes an effect that is new with general relativity— the gravitational field causes clocks to slow down, just as motion does.

(b) Show that the geodesic equation of motion (Eq. (6.38)) for one of the coordinates takes the form

$$0 = \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial r}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{1}{2}\frac{\partial g_{tt}}{\partial r}\left(\frac{dt}{d\tau}\right)^2 \ .$$

(c) Show that the above equation implies

$$r\left(\frac{d\phi}{d\tau}\right)^2 = \frac{GM}{r^2}\left(\frac{dt}{d\tau}\right)^2 ,$$

which in turn implies that

$$r\omega^2 = \frac{GM}{r^2} .$$

Thus, the relation between r and ω is exactly the same as in Newtonian mechanics. *[Note, however, that this does not really mean that general relativity has no effect. First, ω has been defined by $d\phi/dt$, where t is a time coordinate which is not the same as the proper time τ that would be measured by a clock on the orbiting body. Second, r does not really have the same meaning as in the Newtonian calculation, since it is not the measured distance from the center of motion. Measured distances, you will recall, are calculated by integrating the metric, as for example in Problem 1. Since the angular ($d\theta^2$ and $d\phi^2$) terms in the Schwarzschild metric are unaffected by the mass, however, it can be seen that the circumference of the circle is equal to $2\pi r$, as in the Newtonian calculation.]*

PROBLEM 2: GEODESICS IN A FLAT UNIVERSE

(This problem is not required, but can be done for 5 points extra credit. It was originally Problem 5 on Problem Set 3, but was held over.)

According to general relativity, in the absence of any non-gravitational forces a particle will travel along a spacetime geodesic. In this sense, gravity is reduced to a distortion in spacetime.

Consider the case of a flat (*i.e.*, $k = 0$) Robertson–Walker metric, which has the simple form

$$ds^2_{\text{ST}} = -c^2dt^2 + R^2(t)\left[dx^2 + dy^2 + dz^2\right] .$$

Since the spatial metric is flat, we have the option of writing it in terms of Cartesian rather than polar coordinates. Now consider a particle which moves along the x-axis. (Note that the galaxies are on the average at rest in this system, but one can still discuss the trajectory of a particle which moves through the model universe.)

(a) Use the geodesic equation to show that the coordinate velocity computed with respect to proper time (*i.e.*, $dx/d\tau$) falls off as $1/R^2(t)$.

(b) Use the expression for the spacetime metric to relate dx/dt to $dx/d\tau$.

(c) The physical velocity of the particle relative to the galaxies that it is passing is given by

$$v = R(t)\frac{dx}{dt} \quad .$$

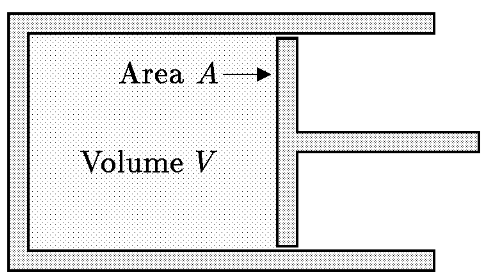
Show that the momentum of the particle, defined relativistically by

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

falls off as $1/R(t)$. (This implies, by the way, that if the particle were described as a quantum mechanical wave with wavelength $\lambda = h/|\vec{p}\,|$, then its wavelength would stretch with the expansion of the universe, in the same way that the wavelength of light is redshifted.)

PROBLEM 3: GAS PRESSURE AND ENERGY CONSERVATION *(10 points)*

In this problem we will pursue the implications of the conservation of energy. Consider first a gas contained in a chamber with a movable piston, as shown below:



Let U denote the total energy of the gas, and let p denote the pressure. Suppose that the piston is moved a distance dx to the right. (We suppose that the motion is slow, so that the gas particles have time to respond and to maintain a uniform pressure throughout the volume.) The gas exerts a force pA on the piston, so the gas does work $dW = pAdx$ as the piston is moved. Note that the volume increases by an amount $dV = Adx$, so $dW = pdV$. The energy of the gas decreases by this amount, so

$$dU = -pdV \ . \tag{1}$$

It turns out that this relation is valid whenever the volume of a gas is changed, regardless of the shape of the volume.

Now consider a homogeneous, isotropic, expanding universe, described by a scale factor $R(t)$. Let u denote the energy density of the gas that fills it. (Remember that $u = \rho c^2$, where ρ is the mass density of the gas.) We will consider a fixed coordinate volume V_{coord}, so the physical volume will vary as

$$V_{\rm phys}(t) = R^3(t) V_{\rm coord} \ . \tag{2}$$

The energy of the gas in this region is then given by

$$U = V_{\rm phys} u \ . \tag{3}$$

(a) Using these relations, show that

$$\frac{d}{dt}\left(R^3 \rho c^2\right) = -p \frac{d}{dt}(R^3) \ , \tag{4}$$

and then that

$$\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) \ , \tag{5}$$

where the dot denotes differentiation with respect to t.

(b) The scale factor evolves according to the relation

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2} \ . \tag{6}$$

Using (5) and (6), show that

$$\ddot{R} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) R \ . \tag{7}$$

This equation describes directly the deceleration of the cosmic expansion. Note that there are contributions from the mass density ρ, but also from the pressure p.

(c) So far our equations have been valid for any sort of a gas, but let us now specialize to the case of black-body radiation. For this case we know that $\rho = aT^4$, where a is a constant and T is the temperature. We also know that as the universe expands, RT remains constant. Using these facts and Eq. (5), find an expression for p in terms of ρ.

PROBLEM 4: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION *(10 points)*

A radiation-dominated universe behaves differently from a matter-dominated universe because the pressure of the radiation is significant. In this problem we explore the role of pressure for several fictitious forms of matter.

(a) For the first fictitious form of matter, the mass density ρ decreases as the scale factor $R(t)$ grows, with the relation

$$\rho(t) \propto \frac{1}{R^6(t)} .$$

What is the pressure of this form of matter? *[Hint: the answer is proportional to the mass density.]*

(b) Find the behavior of the scale factor $R(t)$ for a flat universe dominated by the form of matter described in part (a). You should be able to determine the function $R(t)$ up to a constant factor.

(c) Now consider a universe dominated by a different form of fictitious matter, with a pressure given by

$$p = \frac{1}{2}\rho c^2 .$$

As the universe expands, the mass density of this form of matter behaves as

$$\rho(t) \propto \frac{1}{R^n(t)} .$$

Find the power n.

PROBLEM 5: TIME EVOLUTION OF A UNIVERSE WITH MYSTERIOUS STUFF *(5 points)*

Suppose that a model universe is filled with a peculiar form of matter for which

$$\rho \propto \frac{1}{R^5(t)} .$$

Assuming that the model universe is flat, calculate

(a) The behavior of the scale factor, $R(t)$. You should be able to find $R(t)$ up to an arbitrary constant of proportionality.

(b) The value of the Hubble parameter $H(t)$, as a function of t.

(c) The physical horizon distance, $\ell_{p,\mathrm{horizon}}(t)$.

(d) The mass density $\rho(t)$.

Total points for Problem Set 4: 35, plus up to 5 points extra credit (of which 10 points plus 5 extra credit points were held over from Problem Set 3)