MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe September 19, 2007
Prof. Alan Guth

PROBLEM SET 3

DUE DATE: Thursday, September 27, 2007

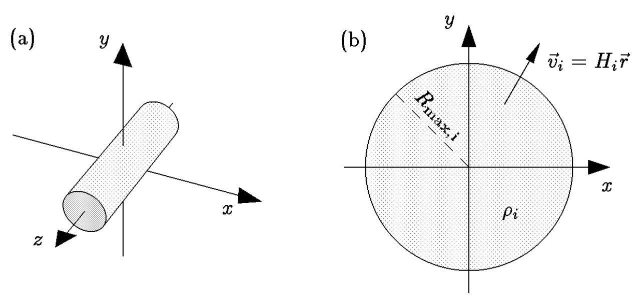
READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes*, Chapter 3; Barbara Ryden, *Introduction to Cosmology*, Chapter 4.

FIRST QUIZ: The first of three quizzes for the term will be given on Tuesday, October 2, 2007.

PROBLEM 1: A CYLINDRICAL UNIVERSE *(10 points)*

The following problem originated on Quiz 2 of 1994, where it counted 30 points.

The lecture notes showed a construction of a Newtonian model of the universe that was based on a uniform, expanding, sphere of matter. In this problem we will construct a model of a cylindrical universe, one which is expanding in the x and y directions but which has no motion in the z direction. Instead of a sphere, we will describe an infinitely long cylinder of radius $R_{\rm max,i}$, with an axis coinciding with the z-axis of the coordinate system:



3D Perspective

2D Cross Section
(Different scale from (a))

We will use cylindrical coordinates, so

$$r = \sqrt{x^2 + y^2}$$

and

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} \; ; \qquad \hat{r} = \frac{\vec{r}}{r} \; ,$$

where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the usual unit vectors along the x, y, and z axes. We will assume that at the initial time t_i, the initial density of the cylinder is ρ_i, and the initial velocity of a particle at position $\vec{r}$ is given by the Hubble relation

$$\vec{v}_i = H_i \vec{r} \; .$$

a) By using Gauss' law of gravity, it is possible to show that the gravitational acceleration at any point is given by

$$\vec{g} = -\frac{A\mu}{r}\hat{r} \; ,$$

where A is a constant and μ is the total mass per length contained within the radius r. Evaluate the constant A.

b) As in the lecture notes, we let $r(r_i, t)$ denote the trajectory of a particle that starts at radius r_i at the initial time t_i. Find an expression for $\ddot{r}(r_i, t)$, expressing the result in terms of r, r_i, ρ_i, and any relevant constants. (Here an overdot denotes a time derivative.)

c) Defining

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} \; ,$$

show that $u(r_i, t)$ is in fact independent of r_i. This implies that the cylinder will undergo uniform expansion, just as the sphere did in the case discussed in the lecture notes. As before, we define the scale factor $R(t) \equiv u(r_i, t)$.

d) Express the mass density $\rho(t)$ in terms of the initial mass density ρ_i and the scale factor $R(t)$. Use this expression to obtain an expression for $\ddot{R}$ in terms of R, ρ, and any relevant constants.

e) Find an expression for a conserved quantity of the form

$$E = \frac{1}{2}\dot{R}^2 + V(R) \; .$$

What is $V(R)$? Will this universe expand forever, or will it collapse?

PROBLEM 2: A FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION *(5 points)*

Consider a ***flat*** universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$R(t) = bt^{3/4} \; ,$$

where b is a constant.

(a) For this universe, find the value of the Hubble "constant" $H(t)$.

(b) Find the physical value of the horizon distance, $\ell_{p,\text{horizon}}(t)$.

(c) What is the mass density of the universe, $\rho(t)$? (In answering this question, you will need to know that the equation for $\dot{R}/R$, Eq. (4.24) in Lecture Notes 4, holds for all forms of matter, while the equation for $\ddot{R}$, Eq. (4.17), requires modification if the matter has a significant pressure. Eq. (4.17) is therefore not applicable to this problem.)

PROBLEM 3: ENERGY AND THE FRIEDMANN EQUATION *(10 points)*

The Friedmann equation,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2} , \tag{1}$$

was derived in Lecture Notes 4 as a first integral of the equations of motion. The equation was first derived in a different form,

$$E = \frac{1}{2}\dot{R}^2 - \frac{4\pi}{3}\frac{G\rho_i}{R} = \text{constant}, \tag{2}$$

where $k = -2E/c^2$. In this form the equation looks more like a conservation of energy relation, although the constant E does not have the dimensions of energy. There are two ways, however, in which the quantity E can be connected to the conservation of energy. It is related the energy of a test particle that moves with the Hubble expansion, and it is also related to the total energy of the entire expanding sphere of radius $R_{\max}$, which was discussed in Lecture Notes 4 as a method of deriving the Friedmann equations. In this problem you will derive these relations.

First, to see the relation with the energy of a test particle moving with the Hubble expansion, define a physical energy E_{phys} by

$$E_{\text{phys}} \equiv m r_i^2 E , \tag{3}$$

where m is the mass of the test particle and r_i is its initial radius. Note that the gravitational force on this particle is given by

$$\vec{F} = -\frac{GmM(r_i)}{r^2}\hat{r} = -\vec{\nabla}V_{\text{eff}}(r) , \tag{4}$$

where $M(r_i)$ is the total mass initially contained within a radius r_i of the origin, r is the present distance of the test particle from the origin, and the "effective" potential energy $V_{\text{eff}}(r)$ is given by

$$V_{\text{eff}}(r) = -\frac{GmM(r_i)}{r} . \tag{5}$$

The motivation for calling this quantity the "effective" potential energy will be explained below.

(a) Show that E_{phys} is equal to the "effective" energy of the test particle, defined by

$$E_{\text{eff}} = \frac{1}{2}mv^2 + V_{\text{eff}}(r) . \tag{6}$$

We understand that E_{eff} is conserved because it is the energy in an analogue problem in which the test particle moves in the gravitational field of a point particle of mass $M(r_i)$, located at the origin, with potential energy function $V_{\text{eff}}(r)$. In this analogue problem the force on the test particle is exactly the same as in the real problem, but in the analogue problem the energy of the test particle is conserved.

We call (6) the "effective" energy because it is really the energy of the analogue problem, and not the real problem. The true potential energy $V(r,t)$ of the test particle is defined to be the amount of work we must supply to move the particle to its present location from some fixed reference point, which we might take to be $r = \infty$. We will not bother to write $V(r,t)$ explicitly, since we will not need it, but we point out that it depends on the time t and on $R_{\max}$, and when differentiated gives the correct gravitational force at any radius. By contrast, $V_{\text{eff}}(r)$ gives the correct force only at the radius of the test particle, $r = R(t)r_i$. The true potential energy function $V(r,t)$ gives no conservation law, since it is explicitly time-dependent, which is why the quantity $V_{\text{eff}}(r)$ is useful.

To relate E to the total energy of the expanding sphere, we need to integrate over the sphere to determine its total energy. These integrals are most easily carried out by dividing the sphere into shells of radius r, and thickness dr, so that each shell has a volume

$$dV = 4\pi r^2\, dr . \tag{7}$$

(b) Show that the total kinetic energy K of the sphere is given by

$$K = c_K M R^2_{\max,i}\left\{\frac{1}{2}\dot{R}^2(t)\right\} , \tag{8}$$

where c_K is a numerical constant, M is the total mass of the sphere, and $R_{\max,i}$ is the initial radius of the sphere. Evaluate the numerical constant c_K.

(c) Show that the total potential energy of the sphere can similarly be written as

$$U = c_U M R_{\max,i}^2 \left\{ -\frac{4\pi}{3} G \frac{\rho_i}{R} \right\} . \tag{9}$$

(*Suggestion:* calculate the total energy needed to assemble the sphere by bringing in one shell of mass at a time from infinity.) Show that $c_U = c_K$, so that the total energy of the sphere is given by

$$E_{\text{total}} = c_K M R_{\max,i}^2 E . \tag{10}$$

PROBLEM 4: EVOLUTION OF A FLAT UNIVERSE WITH $R(t) = bt^{1/2}$ *(10 points)*

The following problem was taken from Quiz 2 of 1990. Each part counted 10 points, so the problem was 70% of the whole exam. For the quiz, students were told that they could express the answers either in terms of the original given variables, or in terms of the answer to any previous part, whether or not they had answered that part correctly. For this problem set, however, you should carry out the algebra necessary to express each answer in terms of given variables.

The following questions all pertain to a flat universe, with a scale factor given by

$$R(t) = bt^{1/2} ,$$

where b is a constant and t is the time. We will learn later that this is the behavior of a radiation-dominated universe.

(a) Find the Hubble constant $H(t)$.

(b) Find the horizon distance $\ell_{\text{hor}}(t)$. Your answer should give the horizon distance in physical units (e.g., centimeters) and not coordinate units (e.g., "notches").

(c) Suppose a light pulse is emitted by one galaxy at time t_e, and received at a second galaxy at time t_r. Find the coordinate separation ℓ_c between the two galaxies. (Note that the coordinate separation is a quantity measured in "notches", not centimeters.)

(d) Find the physical separation between the two galaxies of part (c), as it would be measured at the time of observation t_r.

(e) Find the physical separation between the two galaxies of part (c), as it would be measured at the time of emission t_e.

(f) Find the redshift z of the radiation received by the second galaxy in part (c).

(g) Suppose the first galaxy in part (c) is spherical, with diameter w. Find the apparent angular size θ (measured from one edge to the other) of the galaxy as it would be observed from the second galaxy. You may assume that $\theta \ll 1$.

PROBLEM 5: EVOLUTION OF A CLOSED, MATTER-DOMINATED UNIVERSE *(5 points)*

It was shown in Lecture Notes 5 that the evolution of a closed, matter-dominated universe can be described by introducing the time-parameter θ, with

$$ct = \alpha(\theta - \sin\theta) ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta) ,$$

where α is a constant with the units of length.

(a) Use these expressions to find H, the Hubble "constant," as a function of α and θ. *(Hint: You can use the first of the equations above to calculate $d\theta/dt$.)*

(b) Find ρ, the mass density, as a function of α and θ.

(c) Find Ω, where $\Omega \equiv \rho/\rho_c$, as a function of α and θ.

PROBLEM 6: EVOLUTION OF AN OPEN, MATTER-DOMINATED UNIVERSE *(10 points)*

The following problem originated on Quiz 2 of 1992, where it counted 30 points.

The equations describing the evolution of an open, matter-dominated universe were given in Lecture Notes 5 as

$$ct = \alpha\,(\sinh\theta - \theta)$$

and

$$\frac{R}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1) ,$$

where α is a constant with units of length. The following mathematical identities, which you should know, may also prove useful on parts (e) and (f):

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2} \quad , \quad \cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$$

$$e^\theta = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots .$$

a) Find the Hubble "constant" H as a function of α and θ.

b) Find the mass density ρ as a function of α and θ.

c) Find the mass density parameter Ω as a function of α and θ.

d) Find the physical value of the horizon distance, $\ell_{p,\text{horizon}}$, as a function of α and θ.

e) For very small values of t, it is possible to use the first nonzero term of a power-series expansion to express θ as a function of t, and then R as a function of t. Give the expression for $R(t)$ in this approximation. The approximation will be valid for $t \ll t^*$. Estimate the value of t^*.

f) Even though these equations describe an open universe, one still finds that Ω approaches one for very early times. For $t \ll t^*$ (where t^* is defined in part (e)), the quantity $1 - \Omega$ behaves as a power of t. Find the expression for $1 - \Omega$ in this approximation.

Total points for Problem Set 3: 50.