MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 9, 2007
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PROBLEM SET 7

DUE DATE: Friday, November 16, 2007

READING ASSIGNMENT: Barbara Ryden, *Introduction to Cosmology*, Chapters 6 and 7. These sections are intended to help you understand the lecture material, so there will be no quiz questions based specifically on this material.

PROBLEM 1: THE REDSHIFT OF THE COSMIC MICROWAVE BACKGROUND *(10 points)*

It was mentioned in Lecture Notes 7 that the black-body spectrum has the peculiar feature that it maintains its form under uniform redshift. That is, as the universe expands, even if the photons do not interact with anything, they will continue to be described by a black-body spectrum, although at a temperature that decreases as the universe expands. Thus, even though the cosmic microwave background (CMB) has not been interacting significantly with matter since 350,000 years after the big bang, the radiation today still has a black-body spectrum. In this problem we will demonstrate this important property of the black-body spectrum.

The spectral energy density $\rho(\nu, T)$ for the thermal (black-body) radiation of photons at temperature T was stated in Lecture Notes 7 as Eq. (7.69), which we can rewrite as

$$\rho(\nu, T) = \frac{16\pi^2 \hbar \nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} \ , \tag{1}$$

where $h = 2\pi\hbar$ is the original Planck's constant. $\rho(\nu, T)\, d\nu$ is the energy per unit volume carried by photons whose frequency is in the interval $[\nu, \nu + d\nu]$. (In Lecture Notes 7 it was called ρ_ν, to distinguish it from ρ_λ, the energy per unit volume per unit interval of wavelength. Here, for simplicity, we drop the subscript ν.) In this problem we will assume that this formula holds at some initial time t_1, when the temperature had some value T_1. We will let $\tilde{\rho}(\nu, t)$ denote the spectral distribution for photons in the universe, which is a function of frequency ν and time t. Thus, our assumption about the initial condition can be expressed as

$$\tilde{\rho}(\nu, t_1) = \rho(\nu, T_1) \ . \tag{2}$$

The photons redshift as the universe expands, and to a good approximation the redshift is the only physical effect that causes the distribution of photons to evolve. Thus, using our knowledge of the redshift, we can calculate the spectral distribution $\tilde{\rho}(\nu, t_2)$ at some later time $t_2 > t_1$. It is not obvious that $\tilde{\rho}(\nu, t_2)$ will be a thermal distribution, but in fact we will be able to show that

$$\tilde{\rho}(\nu, t_2) = \rho\big(\nu, T(t_2)\big) \ , \tag{3}$$

where in fact $T(t_2)$ will agree with what we already know about the evolution of T in a radiation-dominated universe:

$$T(t_2) = \frac{R(t_1)}{R(t_2)} T_1 \ . \tag{4}$$

To follow the evolution of the photons from time t_1 to time t_2, we can imagine selecting a region of comoving coordinates with coordinate volume V_c. Within this comoving volume, we can imagine tagging all the photons in a specified infinitesimal range of frequencies, those between ν_1 and $\nu_1 + d\nu_1$. Recalling that the energy of each such photon is $h\nu$, the number dN_1 of tagged photons is then

$$dN_1 = \frac{\tilde{\rho}(\nu_1, t_1)\, R^3(t_1)\, V_c\, d\nu_1}{h\nu_1} \ . \tag{5}$$

(a) We now wish to follow the photons in this frequency range from time t_1 to time t_2, during which time each photon redshifts. At the latter time we can denote the range of frequencies by ν_2 to $\nu_2 + d\nu_2$. Express ν_2 and $d\nu_2$ in terms of ν_1 and $d\nu_1$, assuming that the scale factor $R(t)$ is given.

(b) At time t_2 we can imagine tagging all the photons in the frequency range ν_2 to $\nu_2 + d\nu_2$ that are found in the original comoving region with coordinate volume V_c. Explain why the number dN_2 of such photons, on average, will equal dN_1 as calculated in Eq. (5).

(c) Since $\tilde{\rho}(\nu, t_2)$ denotes the spectral energy density at time t_2, we can write

$$dN_2 = \frac{\tilde{\rho}(\nu_2, t_2)\, R^3(t_2)\, V_c\, d\nu_2}{h\nu_2} \ , \tag{6}$$

using the same logic as in Eq. (5). Use $dN_2 = dN_1$ to show that

$$\tilde{\rho}(\nu_2, t_2) = \frac{R^3(t_1)}{R^3(t_2)}\, \tilde{\rho}(\nu_1, t_1) \ . \tag{7}$$

Use the above equation to show that Eq. (3) is satisfied, for $T(t)$ given by Eq. (4).

PROBLEM 2: THE GREISEN-ZATSEPIN-KUZMIN (GZK) CUTOFF, *(10 points)*

Very shortly after the CMB was discovered, it was pointed out* that the existence of the radiation would impose a cutoff on very high energy cosmic rays. Protons with an energy above about 6×10^{19} eV would have a high cross section for scattering off the photons of the CMB, limiting the range that they could travel to something like 50 Mpc. Since there are no known sources within this distance, there is a prediction that we should not see cosmic rays higher than this energy.

(a) Using the formulas for the energy density and number density of black-body radiation, calculate the average energy of a photon for radiation with an arbitrary temperature T. Your answer should be in the form of a dimensionless number times kT. For $T = 2.725$ K, the temperature of the CMB, what is this energy, in MeV?

(b) The cross section for proton-photon scattering has a strong enhancement when the particles have just enough energy to create a very short-lived particle called the $\Delta(1232)$, which has a rest energy of 1232 MeV. The Δ then decays immediately (in about 10^{-23} second) to a proton and π^0 particle, or a neutron and a π^+ particle:

$$p + \gamma \longrightarrow \Delta \begin{array}{l} \nearrow\ p + \pi^0 \\ \searrow\ n + \pi^+ \end{array}$$

Suppose that photons with an energy E_γ of 3 times the mean are plentiful enough to scatter the cosmic ray protons. What energy E_p must the proton have so that it is possible to create a $\Delta(1232)$ when it collides head-on with a photon of energy E_γ? The mass of the proton is given by $m_p c^2 = 938.27$ MeV.

[Hint: one cannot expect that $E_p + E_\gamma = 1232$ MeV, since the conservation of momentum implies that the final Δ must have nonzero momentum, and hence nonzero kinetic energy. One could solve the conservation of energy and momentum equations simultaneously, but it is easiest to remember that the square of the energy-momentum four-vector is Lorentz-invariant:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \quad \Longrightarrow \quad p^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = \text{Lorentz-invariant quantity.}$$

By applying this Lorentz-invariance to the total energy-momentum vector, you can deduce that

$$|\vec{p}_{\text{tot}}|^2 - \frac{E_{\text{tot}}^2}{c^2} = -\frac{E_{\text{rest}}^2}{c^2} \ ,$$

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* K. Greisen, *Phys. Rev. Lett.* **16,** 748 (1966); G.T. Zatsepin and V.A. Kuzmin, *Pis'ma Zh. Eksp. Teor. Fiz.* **4,** 114 (1966) [*JETP Lett.* **4,** 78 (1966)].

where E_{rest} is the total energy in the rest frame of the system. When there is just enough energy to produce a Δ particle, the energy in the rest frame must be 1232 MeV. In doing the calculation, you may use the fact that $m_p c^2 \gg E_\gamma$, and that $E_p \gg m_p c^2$.]

PROBLEM 3: MASS DENSITY OF VACUUM FLUCTUATIONS *(10 points)*

The energy density of vacuum fluctuations will be discussed qualitatively in lecture. In this problem we will calculate in detail the energy density associated with quantum fluctuations of the electromagnetic field. To keep the problem finite, we will not consider all of space at once, but instead we will consider the electromagnetic field inside a cube of side L, defined by coordinates

$$0 \le x \le L \ ,$$
$$0 \le y \le L \ ,$$
$$0 \le z \le L \ .$$

Our goal, however, will be to compute the energy density in the limit as the size of the box is taken to infinity.

(a) The electromagnetic waves inside the box can be decomposed into a Fourier sum of sinusoidal normal modes. Suppose we consider only modes that extend up to a maximum wave number k_{max}, or equivalently modes that extend down to a minimum wavelength λ_{min}, where

$$k_{\text{max}} = \frac{2\pi}{\lambda_{\text{min}}} \ .$$

How many such modes are there? I do not expect an exact answer, but your approximations should become arbitrarily accurate when $\lambda_{\text{min}} \ll L$. (These mode counting techniques are probably familiar to many of you, but in case they are not I have attached an extended hint after part (c).)

(b) When the electromagnetic field is described quantum mechanically, each normal mode behaves exactly as a harmonic oscillator: if the angular frequency of the mode is ω, then the quantized energy levels have energies given by

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega \ ,$$

where $\hbar$ is Planck's original constant divided by 2π, and n is an integer. The integer n is called the "occupation number," and is interpreted as the number of photons in the specified mode. The minimum energy is not zero, but instead is $\frac{1}{2}\hbar\omega$, which is the energy of the quantum fluctuations of the electromagnetic field. Assuming that the mode sum is cut off at $\lambda_{\rm min}$ equal to the Planck length (as defined in the Lecture Notes), what is the total mass density of these quantum fluctuations?

(c) How does the mass density of the quantum fluctuations of the electromagnetic field compare with the critical density of our universe?

Extended Hint:

The electromagnetic fields inside a closed box can be expanded as the sum of modes, each of which has a sinusoidally varying time dependence, but the precise form of these modes depends on the nature of the boundary conditions on the walls of the box. Physically reasonable boundary conditions, such as total reflection, are in fact difficult to use in calculations. However, it is known that in the limit of an infinite-sized box, the nature of the boundary conditions will not make any difference. We are therefore free to choose the simplest boundary conditions that we can imagine, and for this purpose we will choose periodic boundary conditions. That is, we will assume that the fields and their normal derivatives on each wall are fixed to precisely match the fields and their normal derivatives on the opposite wall.

To begin, we consider a wave in one dimension, moving at the speed of light. Such waves are most easily described in terms of complex exponentials. If $A(x,t)$ represents the amplitude of the wave, then a sinusoidal wave moving in the positive x-direction can be written as

$$A(x,t) = \mathrm{Re}\left[Be^{ik(x-ct)}\right] ,$$

where B is a complex constant and k is a real constant. Defining $\omega = c|k|$, waves in either direction can be written as

$$A(x,t) = \mathrm{Re}\left[Be^{i(kx-\omega t)}\right] ,$$

where the sign of k determines the direction. To be periodic with period L, the parameter k must satisfy

$$kL = 2\pi n ,$$

where n is an integer. So the spacing between modes is $\Delta k = 2\pi/L$. The density of modes dN/dk (i.e., the number of modes per interval of k) is then one divided by the spacing, or $1/\Delta k$, so

$$\frac{dN}{dk} = \frac{L}{2\pi} \quad \text{(one dimension)} .$$

In three dimensions, a sinusoidal wave can be written as

$$A(\vec{x},t) = \mathrm{Re}\left[Be^{i(\vec{k}\cdot\vec{x}-\omega t)}\right] ,$$

where $\omega = c|\vec{k}|$, and

$$k_x L = 2\pi n_x \ , \quad k_y L = 2\pi n_y \ , \quad k_z L = 2\pi n_z \ ,$$

where n_x, n_y, and n_z are integers. Thus, in three-dimensional $\vec{k}$-space the allowed values of $\vec{k}$ lie on a cubical lattice, with spacing $2\pi/L$. In counting the modes, one should also remember that for photons there is an extra factor of 2 associated with the fact that electromagnetic waves have two possible polarizations for each allowed value of $\vec{k}$.

PROBLEM 4: BRIGHTNESS VS. REDSHIFT WITH A POSSIBLE COSMOLOGICAL CONSTANT *(10 points)*

In Lecture Notes 8, we derived the relation between the power output P of a source and the energy flux J, for the case of a closed universe:

$$J = \frac{P H_0^2 |\Omega_{k,0}|}{4\pi(1+z_S)^2 c^2 \sin^2 \psi_D} \ ,$$

where

$$\psi_D = \sqrt{|\Omega_{k,0}|} \int_0^{z_S} \frac{dz}{\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\text{rad},0}(1+z)^4 + \Omega_{\text{vac},0} + \Omega_{k,0}(1+z)^2}} \ .$$

Here z_S denotes the observed redshift, H_0 denotes the present value of the Hubble constant, $\Omega_{m,0}$, $\Omega_{\text{rad},0}$, and $\Omega_{\text{vac},0}$ denote the present contributions to Ω from nonrelativistic matter, radiation, and vacuum energy, respectively, and $\Omega_{k,0} \equiv 1 - \Omega_{m,0} - \Omega_{\text{rad},0} - \Omega_{\text{vac},0}$.

(a) Derive the corresponding formula for the case of an open universe. You can of course follow the same logic as the derivation in the lecture notes, but the solution you write should be complete and self-contained. (I.e., you should **NOT** say "the derivation is the same as the lecture notes except for ... .")

(b) Derive the corresponding formula for the case of a flat universe. Here there is of course no need to repeat anything that you have already done in part (a). If you wish you can start with the answer for an open or closed universe, taking the limit as $k \to 0$. The limit is delicate, however, because both the numerator and denominator of the equation for J vanish as $\Omega_{k,0} \to 0$.

PROBLEM 5: BRIGHTNESS VS. REDSHIFT WITH A POSSIBLE COSMOLOGICAL CONSTANT — NUMERICAL INTEGRATION *(EXTRA CREDIT, 8 pts)*

Calculate numerically the result from Problem 4 for the case of a flat universe in which the critical density is comprised of nonrelativistic matter and vacuum energy (cosmological constant). Specifically, calculate numerical values for $J/(PH_0^2)$ as a function of z, for $\Omega_{m,0} = 0.3$ and $\Omega_{\rm vac,0} = 0.7$. Compute a table of values for $z = 0.1, 0.2, 0.3, \ldots, 1.5$. Feel free to attach a computer printout of these results, but be sure that it is labeled well enough to be readable to someone other than yourself. (If you are not confident in the expression that you obtained in Problem 4 for the flat universe case, you can for equal credit do this problem for an open universe, with $\Omega_{m,0} = 0.3$ and $\Omega_{\rm vac,0} = 0.6$.) For pedagogical purposes you are asked to compute these numbers to 5 significant figures, although one does not need nearly so much accuracy to compare with data. For the fun of it, the solutions will be written to 15 significant figures. Note that the speed of light is now *defined* to be 299,792,458 m/s.

PROBLEM 6: PLOTTING THE SUPERNOVA DATA *(EXTRA CREDIT, 7 pts)*

The original data on the Hubble diagram based on Type Ia supernovae are found in two papers. One paper is authored by the High Z Supernova Search Team,* and the other is by the Supernova Cosmology Project.† More recent data from the High Z team, which includes many more data points, can be found in Riess *et al.*, http://arXiv.org/abs/astro-ph/0402512.¶ (By the way, the lead author Adam Riess was an MIT undergraduate physics major about 15 years ago.)

You are asked to plot the data from either the 2nd or 3rd of these papers, and to include on the graph the theoretical predictions for several cosmological models.

The plot will be similar to the plots contained in these papers, and to the plot on p. 121 of Ryden's book, showing a graph of (corrected) magnitude m vs. redshift z. Your graph should include the error bars. If you plot the Perlmutter *et al.* data, you will be plotting "effective magnitude" m vs. redshift z. The magnitude is related to the flux J of the observed radiation by $m = -\frac{5}{2}\log_{10}(J) + \text{const}$. The value of the constant in this expression will not be needed. The word "corrected" refers both to corrections related to the spectral sensitivity of the detectors and to the brightness

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* http://arXiv.org/abs/astro-ph/9805201, later published as Riess *et al.*, *Astronomical Journal* **116**, 1009 (1998).

† http://arXiv.org/abs/astro-ph/9812133, later published as Perlmutter *et al.*, *Astrophysical Journal* **517**:565–586 (1999).

¶ Published as *Astrophysical Journal* **607**:665-687 (2004).

of the supernova explosions themselves. That is, the supernova at various distances are observed with different redshifts, and hence one must apply corrections if the detectors used to measure the radiation do not have the same sensitivity at all wavelengths. In addition, to improve the uniformity of the supernova as standard candles, the astronomers apply a correction based on the duration of the light output. Note that our ignorance of the absolute brightness of the supernova, of the precise value of the Hubble constant, and of the constant that appears in the definition of magnitude all combine to give an unknown but constant contribution to the predicted magnitudes. The consequence is that you will be able to move your predicted curves up or down (i.e., translate them by a fixed distance along the m axis). You should choose the vertical positioning of your curve to optimize your fit, either by eyeball or by some more systematic method.

If you choose to plot the data from the 3rd paper, Riess *et al.* 2004, then you should see the note at the end of this problem.

For your convenience, the magnitudes and redshifts for the Supernova Cosmology Project paper, from Tables 1 and 2, are summarized in a text file on the 8.286 web page. The data from Table 5 of the Riess *et al.* 2004 paper, mentioned above, is also posted on the 8.286 web page.

For the cosmological models to plot, you should include:

(i) A matter-dominated universe with $\Omega_m = 1$.

(ii) An open universe, with $\Omega_{m,0} = 0.3$.

(iii) A universe with $\Omega_{m,0} = 0.3$ and a cosmological constant, with $\Omega_{\text{vac},0} = 0.7$. (If you prefer to avoid the flat case, you can use $\Omega_{\text{vac},0} = 0.6$. Or, if you want to compare directly with Figure 4 of the Riess *et al.* (2004) paper, you should use $\Omega_{m,0} = 0.29$, $\Omega_{\text{vac},0} = 0.71$.)

You may include any other models if they interest you. You can draw the plot with either a linear or a logarithmic scale in z. I would recommend extending your theoretical plot to $z = 3$, if you do it logarithmically, or $z = 2$ if you do it linearly, even though the data does not go out that far. That way you can see what possible knowledge can be gained by data at higher redshift.

NOTE FOR THOSE PLOTTING DATA FROM RIESS ET AL. 2004:

Unlike the Perlmutter *et al.* data, the Riess *et al.* data is expressed in terms of the distance modulus, which is a direct measure of the luminosity distance. The distance modulus is defined both in the Riess *et al.* paper and in Ryden's book (p. 120) as

$$\mu = 5\log_{10}\left(\frac{d_L}{1\ \text{Mpc}}\right) + 25\ ,$$

where Ryden uses the notation $m - M$ for the distance modulus, and d_L is the luminosity distance. The luminosity distance, in turn, is really a measure of the observed brightness of the object. It is defined as the distance that the object would have to be located to result in the observed brightness, if we were living in a static Euclidean universe. More explicitly, if we lived in a static Euclidean universe and an object radiated power P in a spherically symmetric pattern, then the energy flux J at a distance d would be

$$J = \frac{P}{4\pi d^2} \ .$$

That is, the power would be distributed uniformly over the surface of a sphere at radius d. The luminosity distance is therefore defined as

$$d_L = \sqrt{\frac{P}{4\pi J}} \ .$$

Thus, a specified value of the distance modulus μ implies a definite value of the ratio J/P.

In plotting a theoretical curve, you will need to choose a value for H_0. Riess *et al.* do not specify what value they used, but I found that their curve is most closely reproduced if I choose $H_0 = 66$ km-sec^{-1}-Mpc^{-1}. This seems a little on the low side, since the value is usually estimated as 70–72 km-sec^{-1}-Mpc^{-1}, but Riess *et al.* emphasize that they were not concerned with this value. They were concerned with the relative values of the distance moduli, and hence the shape of the graph of the distance modulus vs. z. In their own words, from Appendix A, "The zeropoint, distance scale, absolute magnitude of the fiducial SN Ia or Hubble constant derived from Table 5 are all closely related (or even equivalent) quantities which were arbitrarily set for the sample presented here. Their correct value is not relevant for the analyses presented which only make use of differences between SN Ia magnitudes. Thus the analysis are independent of the aforementioned normalization parameters."

Total points for Problem Set 7: 40, plus up to 15 points extra credit.