MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 20, 2007

Prof. Alan Guth

PROBLEM SET 8

DUE DATE: Tuesday, November 27, 2007

READING ASSIGNMENT: Barbara Ryden, *Introduction to Cosmology*, Chapter 8 (Dark Matter). Also "Inflation and the New Era of High-Precision Cosmology," by Alan Guth, available at

http://web.mit.edu/physics/alumniandfriends/physicsjournal_fall_02_cosmology.pdf.

For this week it will be sufficient to read the first 5 pages, but it may be more coherent for you to read all 12 pages at once. In addition, you may find Ryden's Chapter 10 useful for understanding nucleosynthesis, but it will not be tested independently.

PROBLEM 1: BIG BANG NUCLEOSYNTHESIS *(8 points)*

The calculations of big bang nucleosynthesis depend on a large number of measured parameters. Below you are asked to qualitatively describe the effects of changing some of these parameters. Include a sentence or two to explain each of your answers.

(a) Suppose an extra neutrino species is added to the calculation. Would the predicted helium abundance go up or down?

(b) Suppose the weak interactions were stronger than they actually are, so that the thermal equilibrium distribution between neutrons and protons were maintained until $kT \approx 0.25$ MeV. Would the predicted helium abundance be larger or smaller than in the standard model?

(c) Suppose the proton-neutron mass difference were larger than the actual value of 1.29 MeV/c^2. Would the predicted helium abundance be larger or smaller than in the standard calculation?

(d) The standard theory of big bang nucleosynthesis assumes that the matter in the universe was distributed homogeneously during the era of nucleosynthesis, but the alternative possibility of inhomogeneous big-bang nucleosynthesis has been discussed since the 1980s. Inhomogeneous nucleosynthesis hinges on the hypothesis that baryons became clumped during a phase transition at $t \approx 10^{-6}$ second, when the hot quark soup converted to a gas of mainly protons, neutrons, and in the early stages, pions. The baryons would then be concentrated in small nuggets, with a comparatively low density outside of these nuggets. After the phase transition but before nucleosynthesis, the neutrons would have the opportunity to diffuse away from these nuggets, becoming more or less uniformly distributed in space. The protons, however, since they are charged, interact electromagnetically with the plasma that fills the universe, and therefore have a much shorter mean free path than the neutrons. Most of the protons, therefore, remain concentrated in the nuggets. Does this scenario result in an increase or a decrease in the expected helium abundance?

PROBLEM 2: THE DEUTERIUM BOTTLENECK *(10 points)*

The "deuterium bottleneck" plays a major role in the description of big bang nucleosynthesis: all of the nuclear reactions involved in nucleosynthesis depend on deuterium forming at the start, but deuterium does not become stable until the temperature reaches a rather low value. In this problem we will explore the statistical mechanics of the deuterium bottleneck.

An ideal gas of classical nonrelativistic particles of type X, in thermal equilibrium, has a number density given by

$$n_X = g_X \left(\frac{m_X kT}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{m_X c^2}{kT}\right) \exp\left(\frac{\mu_X}{kT}\right) . \qquad (1)$$

Here g_X is the number of spin degrees of freedom associated with the particle (like the factor $g = 2$ that we encountered with photons), m_X is the mass of the particle, T is the temperature, and μ_X is the chemical potential of the particle. ($\hbar = h/2\pi$, c, and k have their usual meanings: Planck's constant, the speed of light, and the Boltzmann constant.) You may or may not be familiar with chemical potential, but it will suffice for you to know that it is a concept introduced to treat quantities that are conserved or at least effectively conserved over the time scales of interest. Such quantities can have any value in thermal equilibrium, since the value is determined by the initial conditions and cannot be changed. For each such conserved quantity Q_i one introduces a chemical potential μ_i. The chemical potential of particle X is given by

$$\mu_X = \sum_i \mu_i q_i^X , \qquad (2)$$

where q_i^X is the amount of quantity Q_i contained in one particle of type X. The chemical potentials μ_i are then adjusted to produce the desired values for each of the conserved quantities Q_i. (In the grand canonical ensemble, which gives the probability distribution that leads to Eq. (1), each possible state for the system as a whole is assigned a probability proportional to $\exp(-E/kT)\exp(\sum_i \mu_i Q_i)$, where E is the energy of the state and Q_i is the amount of quantity i in the state.) Note that Eq. (2) implies that for any allowed reaction, such as

$$A + B \longleftrightarrow C , \qquad (3)$$

we are guaranteed that

$$\mu_A + \mu_B = \mu_C , \qquad (4)$$

since the conserved quantities must balance on the two sides of the equation.

(a) I mentioned in lecture that our textbook writes Eq. (1) incorrectly, omitting the chemical potential factor. See for example Eqs. (10.11) and (10.12). The author does, however, have a footnote about this (p. 156), which concludes that "in most cosmological contexts, as it turns out, the chemical potential is small enough to be safely neglected." We can check this statement by using the author's formula to calculate the proton density at 3 minutes into the big bang, at the time of Steven Weinberg's Fifth Frame, from chapter 5 of *The First Three Minutes*. At that time the temperature was $T = 10^9$ K. To compare with the right answer, we make use of the fact that the ratio of the number density n_b of baryons to the number density n_γ of photons is estimated from WMAP data* as

$$\eta \equiv \frac{n_b}{n_\gamma} = (6.1 \pm 0.2) \times 10^{-10} \ . \tag{5}$$

According to Weinberg, at that time 14% of the baryons were neutrons, with 86% protons. At the risk of appearing impertinent toward the author (but physicists are known for their impertinence), I will phrase the question this way: By how many kilo-orders of magnitude is the author's formula for n_p in error?† (Be prepared to have your calculators overflow — if they do, calculate the logarithm of the answer.)

(b) For deuterium production, the relevant reaction is

$$n + p \longleftrightarrow D \ , \tag{6}$$

so Eq. (4) tells us that $\mu_n + \mu_p = \mu_D$. This equality implies that if we form the ratio

$$\frac{n_D}{n_p \, n_n} \ , \tag{7}$$

expressing each number density as in Eq. (1), then the chemical potential factors will cancel out. (This is how the formula is normally used, and this is how Ryden uses it on p. 180. From here on her treatment is correct, but we will proceed with slightly more detail.) To describe the bookkeeping for the reaction of Eq. (6), we need to define our variables. I am using n_n, n_p, and n_D to mean the number densities of free neutrons, free protons, and deuterium nuclei. n_b denotes the total baryon number density, so

$$n_b = n_n + n_p + 2n_D \ . \tag{8}$$

Finally, I will use $n_n^{\rm TOT}$ and $n_p^{\rm TOT}$ to denote the total number densities of neutrons and protons respectively, whether free or bound inside deuterium. We assume that deuterium production happens fast enough so that there is no further change in the neutron-proton balance while deuterium if forming, so the ratio

$$f \equiv \frac{n_n^{\rm TOT}}{n_b} \tag{9}$$

is fixed. We will describe the extent to which the reaction has proceeded by specifying the fraction x of neutrons that remain free,

$$x \equiv \frac{n_n}{n_n^{\rm TOT}} \ . \tag{10}$$

Using these definitions, write the equation that equates the ratio $n_D/(n_p \, n_n)$ to a function of temperature, using Eq. (1) for each of the number densities. The deuteron is spin-1, with $g = 3$, and the proton and neutron are each spin-$\frac{1}{2}$, with $g = 2$. You may approximate $m_n = m_p = m_D/2$. Manipulate this formula so that it has the form

$$F(\eta, f, x) = G(T) \ ,$$

where F and G are functions that you must determine. You will need the binding energy of deuterium,

$$B = (m_p + m_n - m_D)c^2 \approx 2.22 \text{ MeV}. \tag{11}$$

This formula determines x as a function of T, or vice versa, but we will not try to write the function explicitly in either case.

(c) Using your result in part (b), and taking $f = 0.14$ from Weinberg's book, find the value of x, the fraction of neutrons that have been bound in deuterium, at the time of the Fifth Frame, when $T = 10^9$ K. You will probably want to solve the equation numerically. Two significant figures will be sufficient.

(d) Again using your result from part (b), and assuming that $f = 0.14$ is still accurate, find the temperature at which $x = \frac{1}{2}$, i.e., the temperature for which half of the neutrons have become combined into deuterium. Again you will presumably find the answer numerically, and 2 significant figures will be sufficient. What is the value of kT at this temperature. Qualitatively, what feature of the calculation causes this number to be small compared to B.

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* D.N. Spergel et al., "Wilkinson Microwave Anisotropy Probe (WMAP) three year results: implications for cosmology," *Astrophys. J. Suppl.* **170,** 377 (2007), also available at http://arxiv.org/abs/astro-ph/0603449. They actually write it as $6.116^{+0.197}_{-0.249} \times 10^{-10}$, but I don't think that we have any need for the extra digits.

† I have exchanged email with Ryden about this, and she said she would fix it in the next edition.

PROBLEM 3: THE HORIZON PROBLEM *(8 points)*

The success of the big bang predictions for the abundances of the light elements suggests that the universe was already in thermal equilibrium at one second after the big bang. At this time, the region which later evolves to become the observed universe was many horizon distances across. Try to estimate how many. You may assume that the universe is flat.

PROBLEM 4: THE FLATNESS PROBLEM *(7 points)*

Although we now know that $\Omega_0 = 1$ to an accuracy of a few percent, let us pretend that the value of Ω today is 0.1. It nonetheless follows that at 10^{-37} second after the big bang (about the time of the grand unified theory phase transition), Ω must have been extraordinarily close to one. Writing $\Omega = 1 - \delta$, estimate the value of δ at $t = 10^{-37}$ sec (using the standard cosmological model).