MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 9, 2009
Prof. Alan Guth

PROBLEM SET 4

DUE DATE: Tuesday, October 20, 2009

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes*, Chapter 4. Barbara Ryden, *Introduction to Cosmology*, Chapters 4 and 5.

PROBLEM 1: EVOLUTION OF A CLOSED, MATTER-DOMINATED UNIVERSE *(10 points)*

It was shown in Lecture Notes 5 that the evolution of a closed, matter-dominated universe can be described by introducing the time parameter θ, sometimes called the development angle, with

$$ct = \alpha(\theta - \sin\theta) ,$$

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) ,$$

where α is a constant with the units of length.

(a) Use these expressions to find H, the Hubble expansion rate, as a function of α and θ. *(Hint: You can use the first of the equations above to calculate $d\theta/dt$.)*

(b) Find ρ, the mass density, as a function of α and θ.

(c) Find Ω, where $\Omega \equiv \rho/\rho_c$, as a function of α and θ. The relation is given in Lecture Notes 5 as Eq. (5.34), but you should show that you get the same answer by combining your answers from parts (a) and (b) of this question.

(d) Although the evolution of a closed, matter-dominated universe seems complicated, it is nonetheless possible to carry out the integration needed to compute the horizon distance. The integral becomes simple if one changes the variable of integration so that one integrates over θ instead of integrating over t. Show that the physical horizon distance $\ell_{p,\text{horizon}}$ for the closed, matter-dominated universe is given by

$$\ell_{p,\text{horizon}} = \alpha\theta(1 - \cos\theta) .$$

PROBLEM 2: EVOLUTION OF AN OPEN, MATTER-DOMINATED UNIVERSE *(10 points)*

The following problem originated on Quiz 2 of 1992, where it counted 30 points.

The equations describing the evolution of an open, matter-dominated universe were given in Lecture Notes 5 as

$$ct = \alpha\,(\sinh\theta - \theta)$$

and

$$\frac{a}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1) ,$$

where α is a constant with units of length. The following mathematical identities, which you should know, may also prove useful on parts (e) and (f):

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} \quad , \quad \cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

$$e^{\theta} = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots .$$

a) Find the Hubble expansion rate H as a function of α and θ.

b) Find the mass density ρ as a function of α and θ.

c) Find the mass density parameter Ω as a function of α and θ. As with part (c) of the previous problem, the answer to this part appears in Lecture Notes 5. However, you should show that you get the same answer by combining your answers to parts (a) and (b) of this question.

d) Find the physical value of the horizon distance, $\ell_{p,\text{horizon}}$, as a function of α and θ.

e) For very small values of t, it is possible to use the first nonzero term of a power-series expansion to express θ as a function of t, and then a as a function of t. Give the expression for $a(t)$ in this approximation. The approximation will be valid for $t \ll t^*$. Estimate the value of t^*.

f) Even though these equations describe an open universe, one still finds that Ω approaches one for very early times. For $t \ll t^*$ (where t^* is defined in part (e)), the quantity $1 - \Omega$ behaves as a power of t. Find the expression for $1 - \Omega$ in this approximation.

PROBLEM 3: THE CRUNCH OF A CLOSED, MATTER-DOMINATED UNIVERSE *(10 points)*

This is Problem 6.14 from Barbara Ryden's ***Introduction to Cosmology****, with some paraphrasing to make it consistent with the language used in lecture.*

Consider a closed universe containing only nonrelativistic matter. This is the closed universe discussed in Lecture Notes 5, and it is also the "Big Crunch" model discussed in Ryden's section 6.1. At some time during the contracting phase (i.e., when $\theta > \pi$), an astronomer named Elbbuh Niwde discovers that nearby galaxies have blueshifts ($-1 \le z < 0$) proportional to their distance. He then measures the present values of the Hubble expansion rate, H_0, and the mass density parameter, Ω_0. He finds, of course, that $H_0 < 0$ (because he is in the contracting phase) and $\Omega_0 > 1$ (because the universe is closed). In terms of H_0 and Ω_0, how long a time will elapse between Dr. Niwde's observation at $t = t_0$ and the final Big Crunch at $t = t_{\text{Crunch}} = 2\pi\alpha/c$? Assuming that Dr. Niwde is able to observe all objects within his horizon, what is the most blueshifted (i.e., most negative) value of z that Dr. Niwde is able to see? What is the lookback time to an object with this blueshift? (By lookback time, one means the difference between the time of observation t_0 and the time at which the light was emitted.)

PROBLEM 4: A CIRCLE IN A NON-EUCLIDEAN GEOMETRY

(5 points)

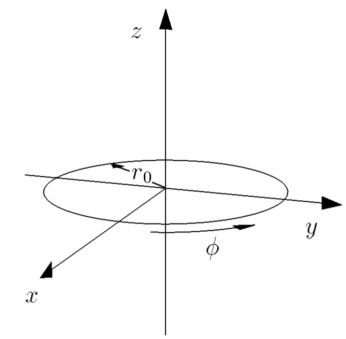
Consider a three-dimensional space described by the following metric:

$$ds^2 = R^2 \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right\} .$$

Here R and k are constants, where k will always have one of the values 1, -1, or 0. θ and ϕ are angular coordinates with the usual properties: $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$, where $\phi = 2\pi$ and $\phi = 0$ are identified. r is a radial coordinate, which runs from 0 to 1 if $k = 1$, and otherwise from 0 to ∞. (This is the Robertson-Walker metric of Eq. (6.27) of Lecture Notes 6, evaluated at some particular time t, with $R \equiv a(t)$. You should be able to work this problem, however, whether or not you have gotten that far. The problem requires only that you understand what a metric means.)

Consider a circle described by the equations

$$z = 0$$
$$x^2 + y^2 = r_0^2 \ ,$$



or equivalently by the angular coordinates

$$r = r_0$$
$$\theta = \pi/2 \ .$$

(a) Find the circumference S of this circle. Hint: break the circle into infinitesimal segments of angular size $d\phi$, calculate the arc length of such a segment, and integrate.

(b) Find the radius ρ of this circle. Note that ρ is the length of a line which runs from the origin to the circle ($r = r_0$), along a trajectory of $\theta = \pi/2$ and $\phi =$ constant. Hint: Break the line into infinitesimal segments of coordinate length dr, calculate the length of such a segment, and integrate. Consider the case of open and closed universes separately, and take $k = \pm 1$. (If you don't remember why we can take $k = \pm 1$, see the section called "Units" in Lecture Notes 4,). You will want the following integrals:

$$\int \frac{dr}{\sqrt{1 - r^2}} = \sin^{-1} r$$

and

$$\int \frac{dr}{\sqrt{1 + r^2}} = \sinh^{-1} r \ .$$

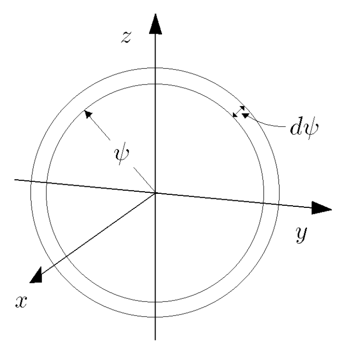
(c) Express the circumference S in terms of the radius ρ. This result is independent of the coordinate system which was used for the calculation, since S and ρ are both measurable quantities. Since the space described by this metric is homogeneous and isotropic, the answer does not depend on where the circle is located or on how it is oriented. For the two cases of open and closed universes, state whether S is larger or smaller than the value it would have for a Euclidean circle of radius ρ.

PROBLEM 5: VOLUME OF A CLOSED UNIVERSE *(5 points)*

Calculate the total volume of a closed universe, as described by the metric of Eq. (6.14) of Lecture Notes 6:

$$ds^2 = R^2 \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] \quad .$$

Break the volume up into spherical shells of infinitesimal thickness, extending from ψ to $\psi + d\psi$:



By comparing Eq. (6.14) with (6.9), the metric for the surface of a sphere, one can see that as long as ψ is held fixed, the metric for varying θ and ϕ is the same as that for a spherical surface of radius $a \sin\psi$. Thus the area of the spherical surface is $4\pi a^2 \sin^2\psi$. To find the volume, multiply this area by the thickness of the shell (which you can read off from the metric), and then integrate over the full range of ψ, from 0 to π.

Total points for Problem Set 4: 50.