MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 21, 2009
Prof. Alan Guth

# PROBLEM SET 5

**DUE DATE:** Tuesday, October 27, 2009

**READING ASSIGNMENT:** Steven Weinberg, *The First Three Minutes*, Chapter 5. Barbara Ryden, *Introduction to Cosmology*, Chapter 6. For Chapters 4, 5, and 6 of Ryden, the material parallels what we either have done or will be doing in lecture. For these chapters you should consider Ryden's book as an aid to understanding the lecture material, and not as a source of new material. On the upcoming quizzes, there will be no questions based specifically on the material in these chapters.

**UPCOMING QUIZZES:** Thursday, November 5, and Thursday, December 3, 2009.

**NOTE ABOUT EXTRA CREDIT:** This problem set contains 45 points of regular problems and 5 points extra credit, so it is probably worthwhile for me to clarify the operational definition of "extra credit". We keep track of the extra credit grades separately, and at the end of the course I will first assign provisional grades based solely on the regular coursework. I will consult with Leo Stein, and we will try to make sure that these grades are reasonable. Then I will add in the extra credit, allowing the grades to change upwards accordingly. Finally, Leo and I will look at each student's grades individually, and we might decide to give a higher grade to some students who are slightly below a borderline. Students whose grades have improved significantly during the term, and students whose average has been pushed down by single low grade, will be the ones most likely to be boosted.

The bottom line is that you should feel free to skip the extra credit problems, and you will still get an excellent grade in the course if you do well on the regular problems. However, if you are the kind of student who really wants to get the most out of the course, then I hope that you will find these extra credit problems challenging, interesting, and educational.

WARNING: Problem 5 would certainly be worth 10 points if it were not for extra credit. I am assigning fewer points for extra credit problems, because I would like students to attack these problems mainly for their intellectual interest, and not their grade-boosting potential.

**PROBLEM 1: SURFACE BRIGHTNESS IN A CLOSED UNIVERSE** *(10 points)*

The spacetime metric for a homogeneous, isotropic, closed universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + a^2(t) \left\{ \frac{dr^2}{1-r^2} + r^2 \left( d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\} ,$$

where I have taken $k = 1$. To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate $\psi$, related to $r$ by

$$r = \sin\psi \; .$$

Then

$$\frac{dr}{\sqrt{1-r^2}} = d\psi \; ,$$

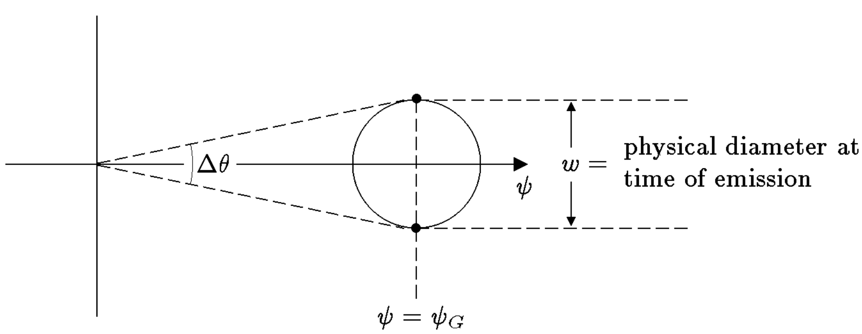
so the metric simplifies to

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + a^2(t) \left\{ d\psi^2 + \sin^2\psi \left( d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\} \; .$$

The form of $a(t)$ depends on the nature of the matter in the universe, but for this problem you should consider $a(t)$ to be an arbitrary function. You should simplify your answers as far as it is possible without knowing the function $a(t)$.

(a) Suppose that the Earth is at the center of these coordinates, and that we observe a spherical galaxy that is located at $\psi = \psi_G$. The light that we see was emitted from the galaxy at time $t_G$, and is being received today, at a time that we call $t_0$. At the time of emission, the galaxy had a power output $P$ (which could be measured, for example, in watts, where 1 watt = 1 joule/sec). The power was radiated uniformly in all directions, in the form of photons. What is the radiation energy flux $J$ from this galaxy at the earth today? Energy flux (which might be measured in joule-m$^{-2}$-sec$^{-1}$) is defined as the energy per unit area per unit time striking a surface that is orthogonal to the direction of the energy flow. [Hint: it is easiest to use a comoving coordinate system with the radiating galaxy at the origin.]

(b) Suppose that the physical diameter of the galaxy at time $t_G$ was $w$. Find the apparent angular size $\Delta\theta$ (measured from one edge to the other) of the galaxy as it would be observed from Earth today.



(c) The *surface brightness* $\sigma$ of the distant galaxy is defined to be the energy flux $J$ per solid angle subtended by the galaxy.* Calculate the surface brightness $\sigma$ of the galaxy described in parts (a) and (b). [Hint: if you have the right answer, it can be written in terms of $P$, $w$, and the redshift $z$, without any reference to $\psi_G$. The rapid decrease in $\sigma$ with $z$ means that high-$z$ galaxies are difficult to distinguish from the night sky.]

## PROBLEM 2: TRAJECTORIES AND DISTANCES IN AN OPEN UNIVERSE *(15 points)*

The spacetime metric for a homogeneous, isotropic, open universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1+r^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} ,$$

where I have taken $k = -1$. As in Problem 1, for the discussion of radial motion it is convenient to introduce an alternative radial coordinate $\psi$, which in this case is related to $r$ by

$$r = \sinh\psi \ .$$

Then

$$\frac{dr}{\sqrt{1+r^2}} = d\psi \ ,$$

so the metric simplifies to

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{d\psi^2 + \sinh^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ .$$

You should treat the function $a(t)$ as a given function. You should simplify your answers as far as it is possible without knowing explicitly the function $a(t)$.

a) Suppose that the Earth is at the origin of the coordinate system ($\psi = 0$), and that at the present time, $t_0$, we receive a light pulse from a distant galaxy $G$, located at $\psi = \psi_G$. Write down an equation which determines the time $t_G$ at which the light pulse left the galaxy. (You may assume that the light pulse travels on a "null" trajectory, which means that $d\tau = 0$ for any segment of it. Since you don't know $a(t)$ you cannot solve this equation, so please do not try.)

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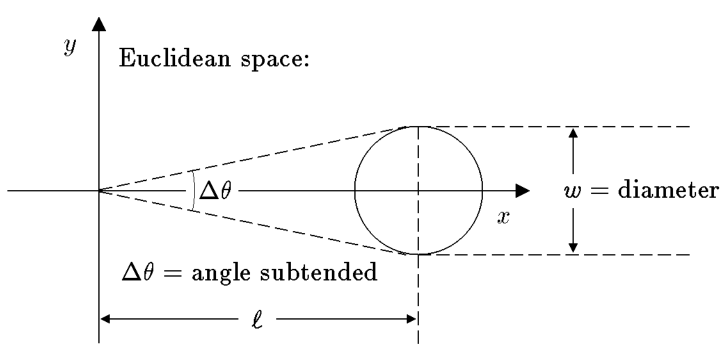
* *Definition of solid angle:* To define the solid angle subtended by the galaxy, imagine surrounding the observer by a sphere of arbitrary radius $r$. The sphere should be small compared to cosmological distances, so that Euclidean geometry is valid within the sphere. If a picture of the galaxy is painted on the surface of the sphere so that it just covers the real image, then the solid angle, in steradians, is the area of the picture on the sphere, divided by $r^2$.

b) What is the redshift $z_G$ of the light from galaxy $G$? (Your answer may depend on $t_G$, as well as $\psi_G$ or any property of the function $a(t)$.)

c) To estimate the number of galaxies that one expects to see in a given range of redshifts, it is necessary to know the volume of the region of space that corresponds to this range. Write an expression for the present value of the volume that corresponds to redshifts smaller than that of galaxy $G$. (You may leave your answer in the form of a definite integral, which may be expressed in terms of $\psi_G$, $t_G$, $z_G$, or the function $a(t)$.)

d) There are a number of different ways of defining distances in cosmology, and generally they are not equal to each other. One choice is called ***proper distance***, which corresponds to the distance that one could in principle measure with rulers. The proper distance is defined as the total length of a network of rulers that are laid end to end from here to the distant galaxy. The rulers have different velocities, because each is at rest with respect to the matter in its own vicinity. They are arranged so that, at the present instant of time, each ruler just touches its neighbors on either side. Write down an expression for the proper distance $\ell_{\rm prop}$ of galaxy $G$.

e) Another common definition of distance is ***angular size distance,*** determined by measuring the apparent size of an object of known physical size. In a static, Euclidean space, a small sphere of diameter $w$ at a distance $\ell$ will subtend an angle $\Delta\theta = w/\ell$:

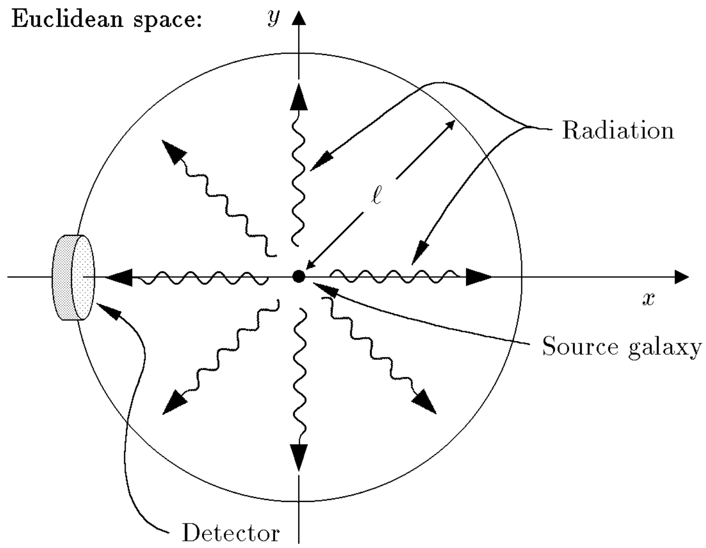



Motivated by this relation, cosmologists define the angular size distance $\ell_{\rm ang}$ of an object by

$$\ell_{\rm ang} \equiv \frac{w}{\Delta\theta} \ .$$

What is the angular size distance $\ell_{\rm ang}$ of galaxy $G$?

f) A third common definition of distance is called ***luminosity distance***, which is determined by measuring the apparent brightness of an object for which the actual total power output is known. In a static, Euclidean space, the energy flux $J$ received from a source of power $P$ at a distance $\ell$ is given by $J = P/(4\pi\ell^2)$:




Cosmologists therefore define the luminosity distance by

$$\ell_{\text{lum}} \equiv \sqrt{\frac{P}{4\pi J}} \quad .$$

Find the luminosity distance $\ell_{\text{lum}}$ of galaxy $G$. (Hint: the Robertson-Walker coordinates can be shifted so that the galaxy $G$ is at the origin.)

## PROBLEM 3: GEODESICS IN A FLAT UNIVERSE *(10 points)*

According to general relativity, in the absence of any non-gravitational forces a particle will travel along a spacetime geodesic. In this sense, gravity is reduced to a distortion in spacetime.

Consider the case of a flat (*i.e.*, $k = 0$) Robertson–Walker metric, which has the simple form

$$ds^2 = -c^2 dt^2 + a^2(t) \left[ dx^2 + dy^2 + dz^2 \right] \quad .$$

Since the spatial metric is flat, we have the option of writing it in terms of Cartesian rather than polar coordinates. Now consider a particle which moves along the $x$-axis. (Note that the galaxies are on the average at rest in this system, but one can still discuss the trajectory of a particle which moves through the model universe.)

(a) Use the geodesic equation to show that the coordinate velocity computed with respect to proper time (*i.e.*, $dx/d\tau$) falls off as $1/a^2(t)$.

(b) Use the expression for the spacetime metric to relate $dx/dt$ to $dx/d\tau$.

(c) The physical velocity of the particle relative to the galaxies that it is passing is given by

$$v = a(t)\frac{dx}{dt} \quad .$$

Show that the momentum of the particle, defined relativistically by

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

falls off as $1/a(t)$. (This implies, by the way, that if the particle were described as a quantum mechanical wave with wavelength $\lambda = h/|\vec{p}\,|$, then its wavelength would stretch with the expansion of the universe, in the same way that the wavelength of light is redshifted.)

**PROBLEM 4: METRIC OF A STATIC GRAVITATIONAL FIELD** *(10 points)*

In this problem we will consider the metric

$$ds^2 = -\left[c^2 + 2\phi(\vec{x})\right]\, dt^2 + \sum_{i=1}^{3} \left(dx^i\right)^2 \quad ,$$

which describes a static gravitational field. Here $i$ runs from 1 to 3, with the identifications $x^1 \equiv x$, $x^2 \equiv y$, and $x^3 \equiv z$. The function $\phi(\vec{x})$ depends only on the spatial variables $\vec{x} \equiv (x^1, x^2, x^3)$, and not on the time coordinate $t$.

(a) Suppose that a radio transmitter, located at $\vec{x}_e$, emits a series of evenly spaced pulses. The pulses are separated by a proper time interval $\Delta T_e$, as measured by a clock at the same location. What is the coordinate time interval $\Delta t_e$ between the emission of the pulses? (I.e., $\Delta t_e$ is the difference between the time coordinate $t$ at the emission of one pulse and the time coordinate $t$ at the emission of the next pulse.)

(b) The pulses are received by an observer at $\vec{x}_r$, who measures the time of arrival of each pulse. What is the ***coordinate*** time interval $\Delta t_r$ between the reception of successive pulses?

(c) The observer uses his own clocks to measure the proper time interval $\Delta T_r$ between the reception of successive pulses. Find this time interval, and also the redshift $z$, defined by

$$1 + z = \frac{\Delta T_r}{\Delta T_e} \quad .$$

First compute an exact expression for $z$, and then expand the answer to lowest order in $\phi(\vec{x})$ to obtain a weak-field approximation. (This weak-field approximation is in fact highly accurate in all terrestrial and solar system applications.)

(d) A freely falling particle travels on a spacetime geodesic $x^\mu(\tau)$, where $\tau$ is the proper time. (I.e., $\tau$ is the time that would be measured by a clock moving with the particle.) The trajectory is described by the geodesic equation

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau} \quad ,$$

where the Greek indices ($\mu, \nu, \lambda, \sigma$, etc.) run from 0 to 3, and are summed over when repeated. Calculate an explicit expression for

$$\frac{d^2x^i}{d\tau^2} \; ,$$

valid for $i = 1, 2$, or 3. (It is acceptable to leave quantities such as $dt/d\tau$ or $dx^i/d\tau$ in the answer.)

(e) In the weak-field nonrelativistic-velocity approximation, the answer to the previous part reduces to

$$\frac{d^2x^i}{dt^2} = -\partial_i\phi \; ,$$

so $\phi(\vec{x})$ can be identified as the Newtonian gravitational potential. Use this fact to estimate the gravitational redshift $z$ of a photon that rises from the floor of this room to the ceiling (say 4 meters). (One significant figure will be sufficient.)

**PROBLEM 5: THE KLEIN DESCRIPTION OF THE G-B-L GEOMETRY**

*(This problem is not required, but can be done for 5 points extra credit.)*

I stated in Lecture Notes 6 that the space invented by Klein, described by the distance relation

$$\cosh\left[\frac{d(1,2)}{a}\right] = \frac{1 - x_1x_2 - y_1y_2}{\sqrt{1-x_1^2-y_1^2}\,\sqrt{1-x_2^2-y_2^2}} \quad ,$$

where

$$x^2 + y^2 < 1 \quad ,$$

is a two-dimensional space of constant negative curvature. In other words, this is just a two-dimensional Robertson–Walker metric, as would be described by a two-dimensional version of Eq. (6.27), with $k = -1$:

$$ds^2 = a^2\left\{\frac{dr^2}{1+r^2} + r^2d\theta^2\right\} \quad .$$

The problem is to prove the equivalence.

(a) As a first step, show that if $x$ and $y$ are replaced by the polar coordinates defined by

$$x = u\cos\theta$$

$$y = u\sin\theta \quad ,$$

then the distance equation can be rewritten as

$$\cosh\left[\frac{d(1,2)}{a}\right] = \frac{1 - u_1 u_2\cos(\theta_1 - \theta_2)}{\sqrt{1-u_1^2}\,\sqrt{1-u_2^2}} \quad .$$

(b) The next step is to derive the metric from the distance function above. Let

$$u_1 = u \qquad \theta_1 = \theta \quad ,$$

$$u_2 = u + du \qquad \theta_2 = \theta + d\theta \quad ,$$

and

$$d(1,2) = ds \quad .$$

Insert these expressions into the distance function, expand everything to second order in the infinitesimal quantities, and show that

$$ds^2 = a^2\left\{\frac{du^2}{(1-u^2)^2} + \frac{u^2 d\theta^2}{1-u^2}\right\} \quad .$$

(This part is rather messy, but you should be able to do it.)

(c) Now find the relationship between $r$ and $u$ and show that the two metric functions are identical. Hint: The coefficients of $d\theta^2$ must be the same in the two cases. Can you now see why Klein had to impose the condition $x^2 + y^2 < 1$?

**Total points for Problem Set 5: 45, plus up to 5 points extra credit.**