MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 29, 2009
Prof. Alan Guth

PROBLEM SET 6

DUE DATE: Tuesday, November 3, 2009

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes*, Chapter 7. Barbara Ryden, *Introduction to Cosmology*, Chapter 10. We have skipped Chapters 7–9 for now, but we will come back to at least some of them. Chapter 10, about *Nucleosynthesis and the Early Universe,* makes good parallel reading to Weinberg's book, and really has no dependence on the chapters that we are skipping.

UPCOMING QUIZZES: Thursday, November 5, and Thursday, December 3, 2009.

PROBLEM 1: CIRCULAR ORBITS IN A SCHWARZSCHILD METRIC *(10 points plus 3 extra credit points)*

The Schwarzschild metric, which describes the external gravitational field of any spherically symmetric distribution of mass, is given by

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \ ,$$

where M is the total mass of the object, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$, and $\phi = 2\pi$ is identified with $\phi = 0$. We will be concerned only with motion outside the Schwarzschild horizon $R_S = 2GM/c^2$, so we can take $r > R_S$. (This restriction allows us to avoid the complications of understanding the effects of the singularity at $r = R_S$.) In this problem we will use the geodesic equation to calculate the behavior of circular orbits in this metric. We will assume a perfectly circular orbit in the x-y plane: the radial coordinate r is fixed, $\theta = 90°$, and $\phi = \omega t$, for some angular velocity ω.

(a) Use the metric to find the proper time interval $d\tau$ for a segment of the path corresponding to a coordinate time interval dt. Note that $d\tau$ represents the time that would actually be measured by a clock moving with the orbiting body. Your result should show that

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2GM}{rc^2} - \frac{r^2\omega^2}{c^2}} \ .$$

Note that for $M = 0$ this reduces to the special relativistic relation $d\tau/dt = \sqrt{1 - v^2/c^2}$, but the extra term proportional to M describes an effect that is new with general relativity— the gravitational field causes clocks to slow down, just as motion does.

(b) Show that the geodesic equation of motion (Eq. (6.68)) for one of the coordinates takes the form

$$0 = \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial r}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{1}{2}\frac{\partial g_{tt}}{\partial r}\left(\frac{dt}{d\tau}\right)^2 \ .$$

(c) Show that the above equation implies

$$r\left(\frac{d\phi}{d\tau}\right)^2 = \frac{GM}{r^2}\left(\frac{dt}{d\tau}\right)^2 \ ,$$

which in turn implies that

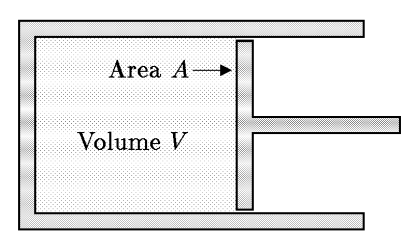
$$r\omega^2 = \frac{GM}{r^2} \ .$$

Thus, the relation between r and ω is exactly the same as in Newtonian mechanics. *[Note, however, that this does not really mean that general relativity has no effect. First, ω has been defined by $d\phi/dt$, where t is a time coordinate which is not the same as the proper time τ that would be measured by a clock on the orbiting body. Second, r does not really have the same meaning as in the Newtonian calculation, since it is not the measured distance from the center of motion. Measured distances, you will recall, are calculated by integrating the metric, as for example in Problem 4 of Problem Set 4, A Circle in a Non-Euclidean Geometry. Since the angular ($d\theta^2$ and $d\phi^2$) terms in the Schwarzschild metric are unaffected by the mass, however, it can be seen that the circumference of the circle is equal to $2\pi r$, as in the Newtonian calculation.]*

(d) *(For 3 points extra credit)* Show that circular orbits around a black hole have a minimum value of the radial coordinate r, which is larger than R_S. What is it?

PROBLEM 2: GAS PRESSURE AND ENERGY CONSERVATION *(10 points)*

In this problem we will pursue the implications of the conservation of energy. Consider first a gas contained in a chamber with a movable piston, as shown below:



Let U denote the total energy of the gas, and let p denote the pressure. Suppose that the piston is moved a distance dx to the right. (We suppose that the motion is slow, so that the gas particles have time to respond and to maintain a uniform pressure throughout the volume.) The gas exerts a force pA on the piston, so the gas does work $dW = pAdx$ as the piston is moved. Note that the volume increases by an amount $dV = Adx$, so $dW = pdV$. The energy of the gas decreases by this amount, so

$$dU = -pdV \ . \tag{1}$$

It turns out that this relation is valid whenever the volume of a gas is changed, regardless of the shape of the volume.

Now consider a homogeneous, isotropic, expanding universe, described by a scale factor $a(t)$. Let u denote the energy density of the gas that fills it. (Remember that $u = \rho c^2$, where ρ is the mass density of the gas.) We will consider a fixed coordinate volume V_{coord}, so the physical volume will vary as

$$V_{\rm phys}(t) = a^3(t) V_{\rm coord} \ . \tag{2}$$

The energy of the gas in this region is then given by

$$U = V_{\rm phys} u \ . \tag{3}$$

(a) Using these relations, show that

$$\frac{d}{dt}\left(a^3 \rho c^2\right) = -p \frac{d}{dt}(a^3) \ , \tag{4}$$

and then that

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) \ , \tag{5}$$

where the dot denotes differentiation with respect to t.

(b) The scale factor evolves according to the relation

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \ . \tag{6}$$

Using (5) and (6), show that

$$\ddot a = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) a \ . \tag{7}$$

This equation describes directly the deceleration of the cosmic expansion. Note that there are contributions from the mass density ρ, but also from the pressure p.

(c) So far our equations have been valid for any sort of a gas, but let us now specialize to the case of black-body radiation. For this case we know that $\rho = bT^4$, where b is a constant and T is the temperature. We also know that as the universe expands, aT remains constant. Using these facts and Eq. (5), find an expression for p in terms of ρ.

PROBLEM 3: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION *(10 points)*

A radiation-dominated universe behaves differently from a matter-dominated universe because the pressure of the radiation is significant. In this problem we explore the role of pressure for several fictitious forms of matter.

(a) For the first fictitious form of matter, the mass density ρ decreases as the scale factor $a(t)$ grows, with the relation

$$\rho(t) \propto \frac{1}{a^6(t)} \ .$$

What is the pressure of this form of matter? *[Hint: the answer is proportional to the mass density.]*

(b) Find the behavior of the scale factor $R(t)$ for a flat universe dominated by the form of matter described in part (a). You should be able to determine the function $a(t)$ up to a constant factor.

(c) Now consider a universe dominated by a different form of fictitious matter, with a pressure given by

$$p = \frac{1}{2}\rho c^2 \ .$$

As the universe expands, the mass density of this form of matter behaves as

$$\rho(t) \propto \frac{1}{a^n(t)} \ .$$

Find the power n.

PROBLEM 4: TIME EVOLUTION OF A UNIVERSE WITH MYSTERIOUS STUFF *(5 points)*

Suppose that a model universe is filled with a peculiar form of matter for which

$$\rho \propto \frac{1}{a^5(t)} \ .$$

Assuming that the model universe is flat, calculate

(a) The behavior of the scale factor, $a(t)$. You should be able to find $a(t)$ up to an arbitrary constant of proportionality.

(b) The value of the Hubble parameter $H(t)$, as a function of t.

(c) The physical horizon distance, $\ell_{p,\rm horizon}(t)$.

(d) The mass density $\rho(t)$.

Total points for Problem Set 6: 35, plus up to 3 points extra credit.