MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 10, 2009
Prof. Alan Guth

PROBLEM SET 7

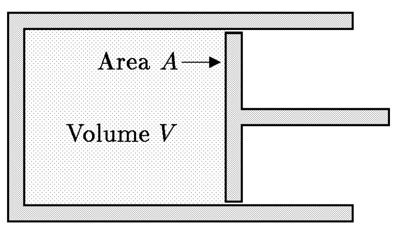
DUE DATE: Tuesday, November 17, 2009

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes*, Chapter 8 and Afterword. Barbara Ryden, *Introduction to Cosmology*, Chapter 8.

UPCOMING QUIZ: Thursday, December 3, 2009.

PROBLEM 1: GAS PRESSURE AND ENERGY CONSERVATION *(10 points)*

In this problem we will pursue the implications of the conservation of energy. Consider first a gas contained in a chamber with a movable piston, as shown below:



Let U denote the total energy of the gas, and let p denote the pressure. Suppose that the piston is moved a distance dx to the right. (We suppose that the motion is slow, so that the gas particles have time to respond and to maintain a uniform pressure throughout the volume.) The gas exerts a force pA on the piston, so the gas does work $dW = pAdx$ as the piston is moved. Note that the volume increases by an amount $dV = Adx$, so $dW = pdV$. The energy of the gas decreases by this amount, so

$$dU = -pdV \ . \tag{1}$$

It turns out that this relation is valid whenever the volume of a gas is changed, regardless of the shape of the volume.

Now consider a homogeneous, isotropic, expanding universe, described by a scale factor $a(t)$. Let u denote the energy density of the gas that fills it. (Remember that $u = \rho c^2$, where ρ is the mass density of the gas.) We will consider a fixed coordinate volume V_{coord}, so the physical volume will vary as

$$V_{\rm phys}(t) = a^3(t) V_{\rm coord} \ . \tag{2}$$

The energy of the gas in this region is then given by

$$U = V_{\rm phys} u \ . \tag{3}$$

(a) Using these relations, show that

$$\frac{d}{dt}\left(a^3 \rho c^2\right) = -p \frac{d}{dt}(a^3) \ , \tag{4}$$

and then that

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) \ , \tag{5}$$

where the dot denotes differentiation with respect to t.

(b) The scale factor evolves according to the relation

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \ . \tag{6}$$

Using (5) and (6), show that

$$\ddot a = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) a \ . \tag{7}$$

This equation describes directly the deceleration of the cosmic expansion. Note that there are contributions from the mass density ρ, but also from the pressure p.

(c) So far our equations have been valid for any sort of a gas, but let us now specialize to the case of black-body radiation. For this case we know that $\rho = bT^4$, where b is a constant and T is the temperature. We also know that as the universe expands, aT remains constant. Using these facts and Eq. (5), find an expression for p in terms of ρ.

PROBLEM 2: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION *(10 points)*

A radiation-dominated universe behaves differently from a matter-dominated universe because the pressure of the radiation is significant. In this problem we explore the role of pressure for several fictitious forms of matter.

(a) For the first fictitious form of matter, the mass density ρ decreases as the scale factor $a(t)$ grows, with the relation

$$\rho(t) \propto \frac{1}{a^6(t)} \ .$$

What is the pressure of this form of matter? *[Hint: the answer is proportional to the mass density.]*

(b) Find the behavior of the scale factor $a(t)$ for a flat universe dominated by the form of matter described in part (a). You should be able to determine the function $a(t)$ up to a constant factor.

(c) Now consider a universe dominated by a different form of fictitious matter, with a pressure given by

$$p = \frac{1}{2}\rho c^2 .$$

As the universe expands, the mass density of this form of matter behaves as

$$\rho(t) \propto \frac{1}{a^n(t)} .$$

Find the power n.

PROBLEM 3: TIME EVOLUTION OF A UNIVERSE WITH MYSTERIOUS STUFF *(5 points)*

Suppose that a model universe is filled with a peculiar form of matter for which

$$\rho \propto \frac{1}{a^5(t)} .$$

Assuming that the model universe is flat, calculate

(a) The behavior of the scale factor, $a(t)$. You should be able to find $a(t)$ up to an arbitrary constant of proportionality.

(b) The value of the Hubble parameter $H(t)$, as a function of t.

(c) The physical horizon distance, $\ell_{p,\text{horizon}}(t)$.

(d) The mass density $\rho(t)$.

PROBLEM 4: EFFECT OF AN EXTRA NEUTRINO SPECIES *(5 points)*

According to the standard assumptions (which were used in the lecture notes), there are three species of effectively massless neutrinos. In the temperature range of 1 MeV $< kT <$ 100 MeV, the mass density of the universe is believed to have been dominated by the black-body radiation of photons, electron-positron pairs, and these neutrinos, all of which were in thermal equilibrium.

(a) Under these assumptions, how long did it take (starting from the instant of the big bang) for the temperature to fall to the value such that $kT = 1$ MeV?

(b) How much time would it have taken if there were one other species of massless neutrino, in addition to the three which we are currently assuming?

(c) What would be the mass density of the universe when $kT = 1$ MeV under the standard assumptions, and what would it be if there were one other species of massless neutrino?

PROBLEM 5: ENTROPY AND THE BACKGROUND NEUTRINO TEMPERATURE *(5 points)*

The formula for the entropy density of black-body radiation is given in Lecture Notes 7. The derivation of this formula has been left to the statistical mechanics course that you either have taken or hopefully will take. For our purposes, the important point is that the early universe remains very close to thermal equilibrium, and therefore entropy is conserved. The conservation of entropy applies even during periods when particles, such as electron-positron pairs, are "freezing out" of the thermal equilibrium mix. Since total entropy is conserved, the entropy density falls off as $1/a^3(t)$.

When the electron-positron pairs disappear from the thermal equilibrium mixture as kT falls below $m_e c^2 = 0.511$ MeV, the weak interactions have such low cross sections that the neutrinos have essentially decoupled. To a good approximation, all of the energy and entropy released by the annihilation of electrons and positrons is added to the photon gas, and the neutrinos are unaffected. Use these facts to show that as electron-positron pair annihilation takes place, aT_γ increases by a factor of $(11/4)^{1/3}$, while aT_ν remains constant. It follows that after the disappearance of the electron-positron pairs, $T_\nu/T_\gamma = (4/11)^{1/3}$. As far as we know, nothing happens that significantly effects this ratio right up to the present day. So we expect today a background of thermal neutrinos which are slightly colder than the 2.7°K background of photons.

PROBLEM 6: FREEZE-OUT OF MUONS *(10 points)*

A particle called the muon seems to be essentially identical to the electron, except that it is heavier— the mass/energy of a muon is 106 MeV, compared to 0.511 MeV for the electron. The muon (μ^-) has the same charge as an electron, denoted by $-e$. There is also an antimuon (μ^+), analogous to the positron, with charge $+e$. The muon and antimuon have the same spin as the electron. There is no known particle with a mass between that of an electron and that of a muon.

(a) The formula for the energy density of black-body radiation,

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} ,$$

is written in terms of a normalization constant g. What is the value of g for the muons, taking μ^+ and μ^- together?

(b) When kT is just above 106 MeV as the universe cools, what particles besides the muons are contained in the thermal radiation that fills the universe? What is the contribution to g from each of these particles?

(c) As kT falls below 106 MeV, the muons disappear from the thermal equilibrium radiation. At these temperatures all of the other particles in the black-body radiation are interacting fast enough to maintain equilibrium, so the heat given off from the muons is shared among all the other particles. Letting a denote the Robertson-Walker scale factor, by what factor does the quantity aT increase when the muons disappear?

Total points for Problem Set 7: 45.