MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 25, 2009

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# PROBLEM SET 9

**DUE DATE:** Tuesday, December 1, 2009

**READING ASSIGNMENT:** "Inflation and the New Era of High-Precision Cosmology," by Alan Guth, written for the MIT Physics Department annual newsletter, 2002. It is available at

http://web.mit.edu/physics/alumniandfriends/physicsjournal_fall_02_cosmology.pdf.

The data quoted in the article about the nonuniformities of the cosmic microwave background radiation has since been superceded by much better data, but the conclusions remain the same.

**UPCOMING QUIZ:** Thursday, December 3, 2009.

## PROBLEM 1: MASS DENSITY OF VACUUM FLUCTUATIONS *(10 points)*

*This problem was Problem 3 on Problem Set 8, but was held over.*

The energy density of vacuum fluctuations will be discussed qualitatively in lecture. In this problem we will calculate in detail the energy density associated with quantum fluctuations of the electromagnetic field. To keep the problem finite, we will not consider all of space at once, but instead we will consider the electromagnetic field inside a cube of side $L$, defined by coordinates

$$0 \leq x \leq L \; ,$$
$$0 \leq y \leq L \; ,$$
$$0 \leq z \leq L \; .$$

Our goal, however, will be to compute the energy density in the limit as the size of the box is taken to infinity.

(a) The electromagnetic waves inside the box can be decomposed into a Fourier sum of sinusoidal normal modes. Suppose we consider only modes that extend up to a maximum wave number $k_{\rm max}$, or equivalently modes that extend down to a minimum wavelength $\lambda_{\rm min}$, where

$$k_{\rm max} = \frac{2\pi}{\lambda_{\rm min}} \; .$$

How many such modes are there? I do not expect an exact answer, but your approximations should become arbitrarily accurate when $\lambda_{\rm min} \ll L$. (These



mode counting techniques are probably familiar to many of you, but in case they are not I have attached an extended hint after part (c).)

(b) When the electromagnetic field is described quantum mechanically, each normal mode behaves exactly as a harmonic oscillator: if the angular frequency of the mode is $\omega$, then the quantized energy levels have energies given by

$$E_n = \left(n + \tfrac{1}{2}\right) \hbar\omega \; ,$$

where $\hbar$ is Planck's original constant divided by $2\pi$, and $n$ is an integer. The integer $n$ is called the "occupation number," and is interpreted as the number of photons in the specified mode. The minimum energy is not zero, but instead is $\frac{1}{2}\hbar\omega$, which is the energy of the quantum fluctuations of the electromagnetic field. Assuming that the mode sum is cut off at $\lambda_{\rm min}$ equal to the Planck length (as defined in the Lecture Notes), what is the total mass density of these quantum fluctuations?

(c) How does the mass density of the quantum fluctuations of the electromagnetic field compare with the critical density of our universe?

*Extended Hint:*

The electromagnetic fields inside a closed box can be expanded as the sum of modes, each of which has a sinusoidally varying time dependence, but the precise form of these modes depends on the nature of the boundary conditions on the walls of the box. Physically reasonable boundary conditions, such as total reflection, are in fact difficult to use in calculations. However, it is known that in the limit of an infinite-sized box, the nature of the boundary conditions will not make any difference. We are therefore free to choose the simplest boundary conditions that we can imagine, and for this purpose we will choose periodic boundary conditions. That is, we will assume that the fields and their normal derivatives on each wall are fixed to precisely match the fields and their normal derivatives on the opposite wall.

To begin, we consider a wave in one dimension, moving at the speed of light. Such waves are most easily described in terms of complex exponentials. If $A(x,t)$ represents the amplitude of the wave, then a sinusoidal wave moving in the positive $x$-direction can be written as

$$A(x,t) = \mathrm{Re}\left[B e^{ik(x-ct)}\right] \; ,$$

where $B$ is a complex constant and $k$ is a real constant. Defining $\omega = c|k|$, waves in either direction can be written as

$$A(x,t) = \mathrm{Re}\left[B e^{i(kx-\omega t)}\right] \; ,$$

where the sign of $k$ determines the direction. To be periodic with period $L$, the parameter $k$ must satisfy

$$kL = 2\pi n \ ,$$

where $n$ is an integer. So the spacing between modes is $\Delta k = 2\pi/L$. The density of modes $dN/dk$ (i.e., the number of modes per interval of $k$) is then one divided by the spacing, or $1/\Delta k$, so

$$\frac{dN}{dk} = \frac{L}{2\pi} \quad \text{(one dimension)} \ .$$

In three dimensions, a sinusoidal wave can be written as

$$A(\vec{x}, t) = \mathrm{Re}\left[ B e^{i(\vec{k}\cdot\vec{x} - \omega t)} \right] \ ,$$

where $\omega = c|\vec{k}|$, and

$$k_x L = 2\pi n_x \ , \quad k_y L = 2\pi n_y \ , \quad k_z L = 2\pi n_z \ ,$$

where $n_x$, $n_y$, and $n_z$ are integers. Thus, in three-dimensional $\vec{k}$-space the allowed values of $\vec{k}$ lie on a cubical lattice, with spacing $2\pi/L$. In counting the modes, one should also remember that for photons there is an extra factor of 2 associated with the fact that electromagnetic waves have two possible polarizations for each allowed value of $\vec{k}$.

**PROBLEM 2: THE DEUTERIUM BOTTLENECK** *(10 points)*

The "deuterium bottleneck" plays a major role in the description of big bang nucleosynthesis: all of the nuclear reactions involved in nucleosynthesis depend on deuterium forming at the start, but deuterium does not become stable until the temperature reaches a rather low value. In this problem we will explore the statistical mechanics of the deuterium bottleneck.

An ideal gas of classical nonrelativistic particles of type $X$, in thermal equilibrium, has a number density given by

$$n_X = g_X \left( \frac{m_X kT}{2\pi\hbar^2} \right)^{3/2} \exp\left( -\frac{m_X c^2}{kT} \right) \exp\left( \frac{\mu_X}{kT} \right) \ . \qquad (1)$$

Here $g_X$ is the number of spin degrees of freedom associated with the particle (like the factor $g = 2$ that we encountered with photons), $m_X$ is the mass of the particle, $T$ is the temperature, and $\mu_X$ is the chemical potential of the particle. ($\hbar = h/2\pi$, $c$, and $k$ have their usual meanings: Planck's constant, the speed of light, and the Boltzmann constant.) You may or may not be familiar with chemical potential, but



it will suffice for you to know that it is a concept introduced to treat quantities that are conserved or at least effectively conserved over the time scales of interest. Such quantities can have any value in thermal equilibrium, since the value is determined by the initial conditions and cannot be changed. For each such conserved quantity $Q_i$ one introduces a chemical potential $\mu_i$. The chemical potential of particle $X$ is given by

$$\mu_X = \sum_i \mu_i q_i^X \ , \qquad (2)$$

where $q_i^X$ is the amount of quantity $Q_i$ contained in one particle of type $X$. The chemical potentials $\mu_i$ are then adjusted to produce the desired values for each of the conserved quantities $Q_i$. (In the grand canonical ensemble, which gives the probability distribution that leads to Eq. (1), each possible state for the system as a whole is assigned a probability proportional to $\exp(-E/kT)\exp(\sum_i \mu_i Q_i)$, where $E$ is the energy of the state and $Q_i$ is the amount of quantity $i$ in the state.) Note that Eq. (2) implies that for any allowed reaction, such as

$$A + B \longleftrightarrow C \ , \qquad (3)$$

we are guaranteed that

$$\mu_A + \mu_B = \mu_C \ , \qquad (4)$$

since the conserved quantities must balance on the two sides of the equation.

(a) I mentioned in lecture that our textbook writes Eq. (1) incorrectly, omitting the chemical potential factor. See for example Eqs. (10.11) and (10.12). The author does, however, have a footnote about this (p. 156), which concludes that "in most cosmological contexts, as it turns out, the chemical potential is small enough to be safely neglected." We can check this statement by using the author's formula to calculate the proton density at 3 minutes into the big bang, at the time of Steven Weinberg's Fifth Frame, from chapter 5 of *The First Three Minutes*. At that time the temperature was $T = 10^9$ K. To compare with the right answer, we make use of the fact that the ratio of the number density $n_b$ of baryons to the number density $n_\gamma$ of photons is estimated from WMAP data* as

$$\eta \equiv \frac{n_b}{n_\gamma} = (6.1 \pm 0.2) \times 10^{-10} \ . \qquad (5)$$

According to Weinberg, at that time 14% of the baryons were neutrons, with 86% protons. At the risk of appearing impertinent toward the author (but

* D.N. Spergel et al., "Wilkinson Microwave Anisotropy Probe (WMAP) three year results: implications for cosmology," *Astrophys. J. Suppl.* **170,** 377 (2007), also available at http://arxiv.org/abs/astro-ph/0603449. They actually write it as $6.116^{+0.197}_{-0.249} \times 10^{-10}$, but I don't think that we have any need for the extra digits.

physicists are known for their impertinence), I will phrase the question this way: By how many kilo-orders of magnitude is the author's formula for $n_p$ in error?† (Be prepared to have your calculators overflow — if they do, calculate the logarithm of the answer.)

(b) For deuterium production, the relevant reaction is

$$n + p \longleftrightarrow D \ , \tag{6}$$

so Eq. (4) tells us that $\mu_n + \mu_p = \mu_D$. This equality implies that if we form the ratio

$$\frac{n_D}{n_p\, n_n} \ , \tag{7}$$

expressing each number density as in Eq. (1), then the chemical potential factors will cancel out. (This is how the formula is normally used, and this is how Ryden uses it on p. 180. From here on her treatment is correct, but we will proceed with slightly more detail.) To describe the bookkeeping for the reaction of Eq. (6), we need to define our variables. I am using $n_n$, $n_p$, and $n_D$ to mean the number densities of free neutrons, free protons, and deuterium nuclei. $n_b$ denotes the total baryon number density, so

$$n_b = n_n + n_p + 2n_D \ . \tag{8}$$

Finally, I will use $n_n^{\rm TOT}$ and $n_p^{\rm TOT}$ to denote the total number densities of neutrons and protons respectively, whether free or bound inside deuterium. We assume that deuterium production happens fast enough so that there is no further change in the neutron-proton balance while deuterium if forming, so the ratio

$$f \equiv \frac{n_n^{\rm TOT}}{n_b} \tag{9}$$

is fixed. We will describe the extent to which the reaction has proceeded by specifying the fraction $x$ of neutrons that remain free,

$$x \equiv \frac{n_n}{n_n^{\rm TOT}} \ . \tag{10}$$

Using these definitions, write the equation that equates the ratio $n_D/(n_p\, n_n)$ to a function of temperature, using Eq. (1) for each of the number densities.

† I have exchanged email with Ryden about this, and she said she would fix it in the next edition.



The deuteron is spin-1, with $g = 3$, and the proton and neutron are each spin-$\frac{1}{2}$, with $g = 2$. You may approximate $m_n = m_p = m_D/2$. Manipulate this formula so that it has the form

$$F(\eta, f, x) = G(T) \ ,$$

where $F$ and $G$ are functions that you must determine. You will need the binding energy of deuterium,

$$B = (m_p + m_n - m_D)c^2 \approx 2.22 \text{ MeV}. \tag{11}$$

This formula determines $x$ as a function of $T$, or vice versa, but we will not try to write the function explicitly in either case.

(c) Using your result in part (b), and taking $f = 0.14$ from Weinberg's book, find the value of $x$, the fraction of neutrons that have been bound in deuterium, at the time of the Fifth Frame, when $T = 10^9$ K. You will probably want to solve the equation numerically. Two significant figures will be sufficient.

(d) Again using your result from part (b), and assuming that $f = 0.14$ is still accurate, find the temperature at which $x = \frac{1}{2}$, i.e., the temperature for which half of the neutrons have become combined into deuterium. Again you will presumably find the answer numerically, and 2 significant figures will be sufficient. What is the value of $kT$ at this temperature. Qualitatively, what feature of the calculation causes this number to be small compared to $B$.

**PROBLEM 3: THE HORIZON PROBLEM** *(8 points)*

The success of the big bang predictions for the abundances of the light elements suggests that the universe was already in thermal equilibrium at one second after the big bang. At this time, the region which later evolves to become the observed universe was many horizon distances across. Try to estimate how many. You may assume that the universe is flat.

**PROBLEM 4: THE FLATNESS PROBLEM** *(7 points)*

Although we now know that $\Omega_0 = 1$ to an accuracy of a few percent, let us pretend that the value of $\Omega$ today is 0.1. It nonetheless follows that at $10^{-37}$ second after the big bang (about the time of the grand unified theory phase transition), $\Omega$ must have been extraordinarily close to one. Writing $\Omega = 1 - \delta$ , estimate the value of $\delta$ at $t = 10^{-37}$ sec (using the standard cosmological model).

**Total points for Problem Set 9: 35.**