MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 3, 2009
Prof. Alan Guth

PROBLEM SET 10 (The Last!)

DUE DATE: Thursday, December 10, 2009

OPTIONAL READING ASSIGNMENT: Barbara Ryden, *Introduction to Cosmology*, Chapter 11 (*Inflation and the Very Early Universe)*.

PROBLEM 1: EXPONENTIAL EXPANSION OF THE INFLATIONARY UNIVERSE *(7 points)*

Recall that the evolution of a Robertson-Walker universe is described by the equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}.$$

Suppose that the mass density ρ is given by the constant mass density ρ_f of the false vacuum. For the case $k = 0$, the growing solution is given simply by

$$a(t) = \text{const } e^{\chi t},$$

where

$$\chi = \sqrt{\frac{8\pi}{3}G\rho_f}$$

and *const* is an arbitrary constant. Find the growing solution to this equation for an arbitrary value of k. Be sure to consider both possibilities for the sign of k. You may find the following integrals useful:

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \ .$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x \ .$$

Show that for large times one has

$$a(t) \propto e^{\chi t}$$

for all choices of k.

PROBLEM 2: THE HORIZON DISTANCE FOR THE PRESENT UNIVERSE *(10 points)*

We have not discussed horizon distances since the beginning of Lecture Notes 5, when we found that

$$\ell_{p,\text{horizon}}(t) = a(t)\int_0^t \frac{c}{a(t')}dt' \ . \tag{1}$$

This formula was derived before we discussed curved spacetimes, but the formula is valid for any Robertson-Walker universe, whether it is open, closed, or flat.

(a) Show that the formula above is valid for closed universes. Hint: write the closed universe metric as it was written in Eq. (8.29):

$$ds^2 = -c^2\,dt^2 + \tilde{a}^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ ,$$

where

$$\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{k}}$$

and ψ is related to the usual Robertson-Walker coordinate r by

$$\sin\psi \equiv \sqrt{k}\,r \ .$$

(b) The evaluation of the formula depends of course on the form of the function $a(t)$, which is governed by the Friedmann equations. For the WMAP 5-year best fit to the parameters,

$$\begin{aligned} H_0 &= 72 \text{ km}\cdot\text{s}^{-1}\cdot\text{Mpc}^{-1} \\ \Omega_m &= 0.26 \\ \Omega_{\text{vac}} &= 0.74 \\ \Omega_r &= 8.0\times10^{-5} \quad (T_{\gamma,0} = 2.725\,\text{K}) \ , \end{aligned}$$

find the current horizon distance, expressed both in light-years and in Mpc. Hint: find an integral expression for the horizon distance, similar to Eq. (8.23a) for the age of the universe. Then do the integral numerically.

Note that the model for which you are calculating does not explicitly include inflation. If it did, the horizon distance would turn out to be vastly larger. By ignoring the inflationary era in calculating the integral of Eq. (1), we are finding an effective horizon distance, defined as the present distance of the most distant objects that we can in principle observe by using only photons that have left their sources after the end of inflation. Photons that left their sources earlier than the end of inflation have undergone incredibly large redshifts, so it is reasonable to consider them to be completely unobservable in practice.

PROBLEM 3: THE INFLATIONARY SOLUTION TO THE HORIZON/HOMOGENEITY PROBLEM *(10 points)*

In this problem we will calculate how much inflation is needed to explain the observed homogeneity of the universe. To make the calculation well-defined, we will adopt a simple description of how inflation works. Although we are trying to explain the homogeneity of the universe, to make the problem tractable we will need to assume that from the onset of inflation, at a time we call t_i, the universe was already very nearly homogeneous, so that we can approximate its evolution using simple equations. We will in fact assume that from time t_i onward the evolution equations can be approximated by those of a homogeneous, isotropic, and flat universe. We will assume that inflation is driven by a false vacuum with a fixed mass density ρ_f, which we will describe by relating it to a parameter E_f by

$$\rho_\mathrm{f} \equiv \frac{E_\mathrm{f}^4}{\hbar^3 c^5} \;, \tag{1}$$

where E_f has the units of energy. To discuss inflation at the energy scale of grand unified theories, we will write E_f as

$$E_\mathrm{f} \equiv E_{16} \times 10^{16} \text{ GeV} \;, \tag{2}$$

where E_{16} is a dimensionless number that will we will assume is of order 1. The Hubble parameter during inflation is then dictated by the Friedmann equation,

$$H_i^2 = \frac{8\pi}{3}\, G\rho_\mathrm{f} \;. \tag{3}$$

While we are assuming enough homogenity to proceed with the calculation, we still want to assume that the high precision homogeneity of the observed universe (like the 1 part in 10^5 uniformity of the CMB) was not part of the initial conditions, but must be explained in terms of the evolution of the universe. The homogeneity is created first on short distance scales, and the length scale of homogeneity, denoted by $r_h(t)$, increases with time. At the onset of inflation we assume that normal thermal equilibrium processes have already smoothed the universe on scales smaller than the Hubble length, so we write

$$r_h(t_i) \approx \beta c H_i^{-1} \;, \tag{4}$$

where β is a dimensionless constant with $\beta \lesssim 1$.

We assume that inflation continues long enough so that the universe expands by a factor Z, where we will be trying to calculate the minimum value of Z. We will assume for simplicity that inflation ends suddenly, at time t_e. Reheating is

then assumed to occur instantly, with the mass density ρ_f of the false vacuum being converted to thermal equilibrium radiation, described as in Lecture Notes 7 by

$$\rho_\mathrm{RH} = g_\mathrm{RH} \frac{\pi^2}{30} \,\frac{(kT_\mathrm{RH})^4}{\hbar^3 c^5} \;, \tag{5}$$

where g_RH reflects the total number of particles that are effectively massless at the energy scale of reheating. For a grand unified theory one might take $g_\mathrm{RH} \approx 300$, but fortunately the value of this highly uncertain number will not have much effect on the answer. The length scale of homogeneity is stretched by inflation to

$$r_h(t_e) = Z r_h(t_i) \;, \tag{6}$$

and we will assume that $r_h(t)$ continues to evolve only by being stretched with the scale factor. The length scale today is then given by

$$r_h(t_0) = \frac{a(t_0)}{a(t_e)}\, Z r_h(t_i) \;. \tag{7}$$

To evaluate $a(t_0)/a(t_e)$, you can use the conservation of entropy, $a^3 s = \mathit{constant}$, where s is the entropy density, which is very accurate from the end of inflation to the present. For the current entropy density, include photons and neutrinos, taking into account the temperature difference $T_\nu/T_\gamma = (4/11)^{1/3}$.

Problem: Find the minimum value of Z such that

$$r_h(t_0) > \ell_{p,\mathrm{horizon}}(t_0) \;, \tag{8}$$

using the value of $\ell_{p,\mathrm{horizon}}(t_0)$ calculated in Problem 2. (If you did not do Problem 2, you could use instead $3ct_0$, the answer for a flat matter-dominated universe, with $t_0 \approx 13.7$ billion years.) Assume the parameters of the WMAP 3-year best fit described in Problem 2, and write your answer for Z_min as a function of E_{16}, g_RH, and β. Since inflation is an exponential process, it is useful to also express the numerical answer in terms of $N_\mathrm{min} \equiv \ln Z_\mathrm{min}$, which is the minimum number of e-foldings of inflation. (An “e-folding” refers to a period of one Hubble time, $\Delta t = H^{-1}$, so the scale factor expands by $e^{H\Delta t} = e^1 = e$.)

PROBLEM 4: HUBBLE CROSSINGS DURING INFLATION *(10 points)*

In the description of density fluctuations in inflationary models, the small amplitude of the fluctuations implies that they can be accurately described by linear perturbation theory. In this treatment the fluctuations can be expanded in modes of definite wavelength, and the interactions of one wavelength with another are ignored. Thus, we describe the perturbations one mode at a time. As the mode evolves the wavelength in comoving coordinates is fixed, so the physical wavelength of a mode grows as the universe expands.

The behavior of a mode changes qualitatively, depending on whether the physical wavelength is smaller or larger than the Hubble length, cH^{-1}. It is therefore important to keep track of when a given mode crosses the Hubble length, and therefore changes its behavior. A typical mode of observational interest has a wavelength today that is small compared to the Hubble length, but nonetheless such modes have spent a significant part of their life with a wavelength larger than the Hubble length.

During inflation the Hubble expansion rate is either constant or very slowly varying, so the Hubble length can be considered fixed. The physical wavelength of a given mode, however, grows with the scale factor, and hence grows exponentially. With this rapid growth, a typical mode with a wavelength shorter than the Hubble length will soon cross the Hubble length. This event is called the *first Hubble crossing*. Since the physical wavelength is growing exponentially, it rapidly becomes much larger than the Hubble length.

Since the physical wavelength of a mode grows monotonically as the universe expands, a little thought is required to understand how a mode with a wavelength larger than the Hubble length can at a later time have a wavelength which is smaller than the Hubble length. The key, of course, is that the Hubble length also changes with time. In a matter-dominated flat universe, for example, the physical wavelength of a specific mode grows with the scale factor, $a(t) \propto t^{2/3}$. The Hubble expansion rate $H = \dot{a}/a = 2/(3t)$, so the Hubble length is given by $\frac{3}{2}ct$, which grows faster than $a(t)$. Thus the Hubble length catches up with the wavelengths of modes that are outside the Hubble length, so each mode goes through a *second Hubble crossing*, at which the wavelength changes from outside to inside the Hubble length.

As will be discussed in lecture, the properties of the fluctuation mode are largely determined at first Hubble crossing, so it is important to know how to find when this occurs. In this problem we will calculate when wavelengths that are in the vicinity of 1 Mpc today underwent their first Hubble crossing. We will use the same model of instantaneous reheating that was used in the previous problem.

Let $t_{H1}(\lambda)$ denote the time of first Hubble crossing for a mode with physical wavelength today equal to λ. The quantity that we will actually calculate is

$$Z_{H1}(\lambda) \equiv \frac{a(t_e)}{a\big(t_{H1}(\lambda)\big)} \ , \qquad (9)$$

where $Z_{H1}(\lambda)$ can be described as the inflationary factor that occurs **after** the mode undergoes first Hubble crossing. Then

$$N_{H1}(\lambda) \equiv \ln Z_{H1}(\lambda) \qquad (10)$$

is the number of e-foldings of inflation that occur after first Hubble crossing.

Since we wish to express Z_{H1} in terms of the physical wavelength at the present time t_0, it is useful to rewrite Eq. (10) as the product of two factors:

$$Z_{H1}(\lambda) = \frac{a(t_e)}{a(t_0)} \frac{a(t_0)}{a\big(t_{H1}(\lambda)\big)} \ . \qquad (11)$$

Note that the first factor appeared in Eq. (7) of the previous problem, so you have already thought about it. You need to figure out how to evaluate the second factor.

Problem: Find $Z_{H1}(\lambda)$, using the same description of the inflationary model as in the previous problem. Define the dimensionless parameter

$$\tilde{\lambda} \equiv \frac{\lambda}{1 \text{ Mpc}} \ , \qquad (12)$$

and express your answer in terms of $\tilde{\lambda}$, E_{16}, and g_{RH}. Again it is useful to explicitly write the answer in terms of $N_{H1} \equiv \ln Z_{H1}(\tilde{\lambda})$, which can be described as the number of e-foldings of inflation that happen after the mode described by $\tilde{\lambda}$ has undergone first Hubble crossing.

Total points for Problem Set 10: 37