MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 3, 2009
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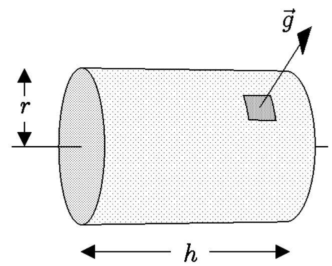
PROBLEM SET 3 SOLUTIONS

PROBLEM 1: A CYLINDRICAL UNIVERSE

a) Gauss's law of gravity states that

$$\oint \vec{g} \cdot d\vec{s} = -4\pi G M \ ,$$

where $\vec{g}$ is the acceleration of gravity, G is Newton's constant, and M is the total mass enclosed inside the volume. Apply this relation to the following slice of the infinite cylinder:



By symmetry $\vec{g}$ points radially outward, so the dot product $\vec{g} \cdot d\vec{s}$ vanishes for the disks that bound the cylindrical slice on the left and right. The only contribution comes from the curved surface of the cylinder, for which the cosine of the dot product is 1. Thus,

$$\oint \vec{g} \cdot d\vec{s} = 2\pi r h g_r \ ,$$

where g_r is the radial component of $\vec{g}$. The mass enclosed, M, is the length times the mass per length, or $h\mu$. (Recall that μ denotes the total mass within the radius r, so it is really a function of r and the mass density ρ.) Thus, Gauss's law gives

$$2\pi r h g_r = -4\pi G h \mu \ ,$$

so

$$g_r = -\frac{2G\mu}{r} \ .$$

Since the other components of $\vec{g}$ vanish by symmetry,

$$\boxed{\vec{g} = -\frac{2G\mu}{r}\,\hat{r} \ ,}$$

and we can read off the constant A,

$$\boxed{A = 2G} \ .$$

b) $\ddot{r}$ is the acceleration of r. The only force in the problem is gravity, so

$$\ddot{r} = g_r = -\frac{2G\mu}{r} \ ,$$

where I used the gravitational acceleration that was calculated in the previous part of the problem.

The evaluation of μ involves logic that is essentially identical to the spherical case discussed in Lecture Notes 4. If we label each particle by its initial radius r_i, then all the particles for a given r_i will form a cylindrical shell. As the motion proceeds it is conceivable that shells might cross each other, and then the evaluation of μ would become very complicated. However, the Hubble expansion incorporated into the initial conditions implies that initially any two shells are moving apart from each other. Since there are no infinite accelerations, we can count on the fact that there will be at least some nonzero time interval before any crossings can take place. Until such crossings take place, it is easy to find μ: the total mass contained within any shell of particles is independent of time, since the particles that lie at a smaller radius than any given shell at time t are exactly the same particles as those that were at smaller radius at time t_i. Then, even if shells start to cross at some time, the equations that we derive under the no-shell-crossing assumption will be valid until the time that shells cross. Thus, shells can cross only if our equations show the possibility that two shells starting at different values of r_i can come together. However, we will find below that the no-shell-crossing assumption leads to uniform expansion described by an overall scale factor $a(t)$. Since two shells never come together under this uniform expansion, we can conclude that no shell crossings will ever take place.

With no shell crossings, we can evaluate μ for a given shell r_i at the initial time. For definiteness we can consider a length h of the cylinder, so the volume of the cylinder of radius r_i is $\pi r_i^2 h$. The mass per length is then

$$\mu(r_i) = \frac{\pi r_i^2 h \rho_i}{h} = \pi r_i^2 \rho_i \ .$$

Thus,

$$\boxed{\ddot{r} = -\frac{2\pi G r_i^2 \rho_i}{r} \ .}$$

c) The function $u(r_i, t)$ is determined by the differential equation that it obeys, combined with the initial conditions. Using the answer from part (b), the differential equation for u is

$$\ddot{u} = \frac{\ddot{r}}{r_i} = -\frac{2\pi G r_i \rho_i}{r} ,$$

so

$$\ddot{u} = -\frac{2\pi G \rho_i}{u} ,$$

which does not depend on r_i. We still need to check that the initial conditions for u and $\dot{u}$ are independent of r_i. (There are two initial conditions, because the differential equation for u is second order.) Since $r(r_i, t_i) \equiv r_i$, the initial value of u is given by

$$u(r_i, t_i) = 1 .$$

Finally, since the initial velocities are set to agree with Hubble's law,

$$\dot{r}(r_i, t_i) = H_i r_i ,$$

it follows that

$$\dot{u}(r_i, t_i) = \frac{\dot{r}(r_i, t_i)}{r_i} = H_i .$$

Thus, neither the differential equation for $u(r_i, t)$ nor the initial conditions depend on r_i, so the solution will not depend on r_i. Thus, we can define $a(t) \equiv u(r_i, t)$.

d) For clarity, we can consider a finite length h of the cylinder. The mass contained inside a cylinder of radius r_i at the initial time t_i is then

$$M(r_i, h) = \pi r_i^2 h \rho_i .$$

At time t, this same mass will be uniformly spread in a cylinder of radius $r(r_i, t)$ and length h. The density is therefore

$$\rho(t) = \frac{M(r_i, h)}{\pi r^2 h} = \boxed{\frac{\rho_i}{a^2(t)} .}$$

Using this result to replace ρ_i in the differential equation found in (c), we find

$$\boxed{\ddot{a} = -2\pi G \rho a .}$$

e) Multiplying the differential equation by $\dot{a}$,

$$\dot{a}\left[\ddot{a}+\frac{2\pi G\rho_i}{a}\right]=0\ .$$

Note that I wrote the differential equation in terms of ρ_i rather than ρ, so that $a(t)$ is the only time-dependent quantity in the equation. This expression can be rewritten as

$$\frac{d}{dt}\left[\frac{1}{2}\dot{a}^2+2\pi G\rho_i\ln a\right]\equiv\frac{d}{dt}E=0\ .$$

In other words, the quantity in square brackets, which we have labeled E, must be constant:

$$E=\frac{1}{2}\dot{a}^2+2\pi G\rho_i\ln a\ \ =\text{const.}$$

We can now identify the rescaled potential energy as

$$\boxed{V(a)=2\pi G\rho_i\ln a\ .}$$

The potential energy term $V(a)$ grows as $\ln a$ and is hence unbounded. No matter how large the initial value of $\dot{a}^2$, there can never be enough energy to allow the universe to grow to arbitrarily large a. Eventually the $V(a)$ term will grow to be as large as E, at which point $\dot{a}$ will vanish and then change sign. This universe necessarily recollapses.

Note added: Since the mass distribution described here is uniform in the z-direction, one might worry about the validity of our assumption that there is no motion in the z-direction. After all, we learned in Lecture Notes 4 that a uniform distribution of matter in 3 dimensions cannot be static, but will necessarily collapse. Thus we have good reason to ask whether the mass distribution described here will collapse in the z-direction. The answer, however, is that it will not. The situation here differs in at least two crucial ways from the case of a uniform mass distribution in 3 dimensions:

1) A 3-dimensional static mass distribution is inconsistent with Gauss' law, while in the solution to this problem we constructed a configuration which is static in the z-direction, but completely consistent with Gauss' law. For the cylindrical problem the flux lines can point radially outward, while for a 3-dimensional static mass distribution there is no place for the flux lines to go.

2) For the 3-dimensional problem, the absence of a static solution was intimately linked to the statement that Newton's force law could not be integrated to determine the force on a particle, since the integral was only

conditionally convergent. If the force law could be integrated, the symmetry of the problem is sufficient to guarantee that it would integrate to zero. For the cylindrical universe, on the other hand, the force law can be integrated. To see this, note that if we want to calculate the force on some particle P located for example at $z = 0$, the question of convergence depends only of the force due to particles that are at arbitrarily large values of $|z|$. For such distant particles the thickness of the cylinder can be ignored, so the distance can be approximated as $|z|$. The force due to particles at large z is then given by

$$\vec{F} = \hat{k} \int \frac{G\mu}{z^2} dz \ ,$$

which converges at large z. Thus for the cylinder problem the total force on each particle is given by a convergent integral, which by symmetry can have no component in the z-direction.

PROBLEM 2: A FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION

(a) The calculation of the Hubble expansion rate is a straightforward application of Eq. (3.7), $H = \dot{a}/a$:

$$H(t) = \frac{\dot{a}}{a} = \frac{\frac{3}{4}bt^{-1/4}}{bt^{3/4}} = \boxed{\frac{3}{4t}} \ .$$

(b) For a general (homogeneous isotropic) universe the time evolution of the scale factor is governed by the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \ .$$

As stated in the problem, this equation is valid for arbitrary forms of matter (although we have so far derived it only for nonrelativistic matter). For a flat universe one has $k = 0$, so the mass density is simply related to the Hubble expansion rate. Since $H = \dot{a}/a$, the above equation can be rewritten as

$$H^2 = \frac{8\pi}{3}G\rho \ ,$$

or equivalently

$$\rho(t) = \frac{3H^2}{8\pi G} = \rho_c.$$

Taking the Hubble expansion rate from part (a), one has

$$\boxed{\rho = \frac{27}{128\pi G t^2}\ .}$$

Note that the mass density ρ might conceivably be inferred from Eq. (4.23),

$$\ddot{a} = -\frac{4\pi}{3} G\rho(t) a\ ,$$

but if you try it you will find that the answer does not agree with the answer boxed above. We will learn later that whenever the pressure is significant, as it is in this model universe, the above equation requires modification, while the equation for $(\dot{a}/a)$ that we used to solve the problem does not.

PROBLEM 3: ENERGY AND THE FRIEDMANN EQUATION

(a) We are given the conserved quantity E

$$E = \frac{1}{2}\dot{a}^2 - \frac{4\pi}{3}\frac{G\rho_i}{a}\,, \tag{1}$$

associated with the Friedmann equation and asked to show that $E_{\rm phys}$, defined by

$$E_{\rm phys} \equiv m r_i^2 E\,, \tag{2}$$

coincides with the effective energy $E_{\rm eff}$ of a particle of mass m in a certain analogue problem. The effective energy is defined by

$$E_{\rm eff} = \frac{1}{2} m v^2 + V_{\rm eff}(r)\ , \tag{3}$$

where

$$V_{\rm eff}(r) = -\frac{GmM(r_i)}{r}\ . \tag{4}$$

Using (1) and (2) we write

$$E_{\rm phys} = \frac{1}{2} m \big[\dot{a}(t)\, r_i\big]^2 - \frac{G\big(\frac{4\pi}{3}\rho_i r_i^3\big)m}{a(t)\, r_i}\ . \tag{5}$$

If the particle had initial radius r_i, at time t its radius is $r(t) = a(t)\, r_i$ and its speed is $v = \dot{r} = \dot{a}(t)\, r_i$. Moreover the mass enclosed by the sphere whose radius is $r(t)$ is independent of time and equal to $M(r_i) = \frac{4\pi}{3}\rho_i r_i^3$. It follows that we can write Eq. (5) as

$$E_{\rm phys} = \frac{1}{2}mv^2 - \frac{GM(r_i)m}{r} = \frac{1}{2}mv^2 + V_{\rm eff}(r) = E_{\rm eff} \ , \tag{6}$$

which is what we wanted to show.

(b) At time t, the sphere has radius

$$R_{\rm max}(t) = a(t) R_{{\rm max},i} \ , \tag{7}$$

and density

$$\rho(t) = \frac{\rho_i}{a^3(t)} \ , \tag{8}$$

and each particle moves with a speed

$$v(t) = H(t)\, r \ , \tag{9}$$

where r is the radius of the particle (measured from the center of the sphere), and $H(t) = \dot{a}(t)/a(t)$. For a shell of particles of radius r and thickness dr, the volume is

$$dV = 4\pi r^2 \, dr \ ,^* \tag{10}$$

so the mass is

$$dm = \rho \, dV = 4\pi\rho r^2 \, dr \ . \tag{11}$$

The kinetic energy of the shell is then

$$dK = \frac{1}{2}(dm)v^2 = 2\pi\rho H^2 r^4 \ \ dr \ . \tag{12}$$

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* If this formula is not obvious, think of the thin shell as a thin spherical sheet of rubber. If the sheet is very thin, then the curvature will not affect the volume, so the volume is the same as a flat sheet of area $4\pi r^2$ and thickness dr. If you want to make sure that these approximations are valid, you can start by writing an exact formula for the volume of the shell, which would be valid even if dr were not small:

$$dV = \frac{4\pi}{3}(r + dr)^3 - \frac{4\pi}{3} r^3 \ .$$

That is, the volume of the outer shell is equal to the volume of the full sphere, minus the volume of the smaller sphere that does not include the outer shell. If you expand this formula and drop terms that have more than one power of the infinitesimal quantity dr, you will find Eq. (10).

To find the total kinetic energy in the sphere we integrate r from 0 to its maximum value, $R_{\rm max}$:

$$K = 2\pi\rho H^2 \int_0^{R_{\rm max}} r^4\, dr = \frac{2}{5}\pi\rho H^2 R_{\rm max}^5 = \frac{2}{5}\pi\rho_i \dot{a}^2 R_{{\rm max},i}^5 \ , \tag{13}$$

where in the last step we expressed ρ and $R_{\rm max}$ in terms of their initial values, using Eqs. (7) and (8). Since the total mass of the sphere is given by

$$M = \frac{4\pi}{3}\rho_i R_{{\rm max},i}^3 \ , \tag{14}$$

the result can be rewritten as

$$\boxed{K = \frac{3}{5} M R_{{\rm max},i}^2 \left\{\frac{1}{2}\dot{a}^2\right\} \ ,} \tag{15}$$

which is the formula that we were asked to show, with

$$\boxed{c_K = \frac{3}{5} \ .} \tag{16}$$

(c) To assemble the sphere shell by shell, we consider some fixed time t, so the sphere has radius $R_{\rm max} = R_{\rm max}(t)$ and density $\rho = \rho(t)$, where $R_{\rm max}(t)$ and $\rho(t)$ are given by Eqs. (7) and (8) respectively. Let us suppose that the sphere has already been built up to radius r, and therefore has mass

$$M(r) = \frac{4\pi}{3}\rho r^3 \ . \tag{17}$$

We now add a shell of radius r and thickness dr, so the mass dm of the shell is given by Eq. (11), and the change in the potential energy is

$$\begin{aligned} dU &= \int_\infty^r \frac{GM(r)\, dm}{r'^2}\, dr' = -\frac{GM(r)\, dm}{r} \\ &= -\frac{G\left[\frac{4\pi}{3}\rho r^3\right]\left[4\pi\rho r^2\, dr\right]}{r} = -\frac{16\pi^2}{3} G\rho^2 r^4\, dr \ . \end{aligned} \tag{18}$$

To find the total potential energy, we integrate over r from 0 to $R_{\rm max}$:

$$U = -\frac{16\pi^2}{3} G\rho^2 \int_0^{R_{\rm max}} r^4\, dr = -\frac{16\pi^2}{15} G\rho^2\, R_{\rm max}^5 = -\frac{16\pi^2}{15}\frac{G\rho_i^2\, R_{{\rm max},i}^5}{a(t)} \ , \tag{19}$$

where in the last step we used Eqs. (7) and (8) to express $R_{\rm max}$ and ρ in terms of their initial values. Finally, using Eq. (14) for the total mass M,

$$U = -\frac{4\pi}{5}\frac{GM\rho_i\,R^2_{{\rm max},i}}{a(t)} = \boxed{\frac{3}{5}MR^2_{{\rm max},i}\left\{-\frac{4\pi}{3}G\frac{\rho_i}{a}\right\}\ ,} \tag{20}$$

which is what we wanted to show, with

$$\boxed{c_U = \frac{3}{5} = c_K\ .} \tag{21}$$

Thus,

$$K+U = \frac{3}{5}M\,R^2_{\rm max,i}\left\{\frac{1}{2}\dot{a}^2 - \frac{4\pi}{3}\frac{G\rho_i}{a}\right\} = \frac{3}{5}M\,R^2_{\rm max,i}\,E\ . \tag{22}$$

The total energy of the expanding sphere is indeed equal to a constant times the conserved quantity E.