MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe October 25, 2009
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PROBLEM SET 4 SOLUTIONS

PROBLEM 1: EVOLUTION OF A CLOSED, MATTER-DOMINATED UNIVERSE

(a) Using the chain rule, the standard formula for the Hubble constant can be rewritten as

$$H(\theta) = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} .$$

By differentiating the parametric equations for a and t, one finds

$$\frac{\mathrm{d}a}{\mathrm{d}\theta} = \alpha\sqrt{k}\sin\theta ,$$

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{\alpha}{c}(1-\cos\theta) = \frac{1}{\mathrm{d}\theta/\mathrm{d}t} .$$

Then

$$H(\theta) = \left[\frac{1}{\sqrt{k}\alpha(1-\cos\theta)}\right]\left[\alpha\sqrt{k}\sin\theta\right]\left[\frac{c}{\alpha(1-\cos\theta)}\right]$$

$$= \boxed{\frac{c\sin\theta}{\alpha(1-\cos\theta)^2}} .$$

(b) The evolution equation for a homogeneous isotropic universe can be written as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} .$$

Then, solving for ρ gives,

$$\rho = \frac{3}{8\pi G}\left(H^2 + \frac{kc^2}{a^2}\right) .$$

Using the answer for $H(\alpha,\theta)$ from (a), and the parametric expression for $a/\sqrt{k}$, one has

$$\rho = \frac{3}{8\pi G}\left[\frac{c^2\sin^2\theta}{\alpha^2(1-\cos\theta)^4} + \frac{c^2}{\alpha^2(1-\cos\theta)^2}\right]$$

$$= \frac{3c^2}{8\pi G\alpha^2(1-\cos\theta)^2}\left[\frac{\sin^2\theta}{(1-\cos\theta)^2} + 1\right]$$

This expression is greatly simplified by using the following trigonometric identity:

$$\sin^2\theta = 1 - \cos^2\theta = (1-\cos\theta)(1+\cos\theta) \ .$$

Using this in our expression for ρ we have

$$\begin{aligned}
\rho &= \frac{3c^2}{8\pi G\alpha^2(1-\cos\theta)^2}\left[\frac{(1+\cos\theta)(1-\cos\theta)}{(1-\cos\theta)(1-\cos\theta)}+1\right] \\
&= \frac{3c^2}{8\pi G\alpha^2(1-\cos\theta)^2}\left[\frac{1+\cos\theta}{1-\cos\theta}+1\right] \\
&= \frac{3c^2}{8\pi G\alpha^2(1-\cos\theta)^2}\left[\frac{2}{1-\cos\theta}\right] \\
&= \boxed{\frac{3c^2}{4\pi G\alpha^2(1-\cos\theta)^3} \ .}
\end{aligned}$$

Alternatively, one can use Eq. (5.17) from Lecture Notes 5,

$$\alpha \equiv \frac{4\pi}{3}\frac{G\rho\tilde{a}^3}{c^2} \ ,$$

which can be solved for ρ. Then if $a/\sqrt{k}$ is replaced by the expression given in the problem statement, one obtains the same formula for ρ that appears in the box above.

(c) Using the answer for $H(\alpha,\theta)$ from (a) and the expression for ρ_c in terms of H, one has

$$\rho_c = \frac{3H^2}{8\pi G} = \frac{3c^2\sin^2\theta}{8\pi G\alpha^2(1-\cos\theta)^4} \ .$$

Then, using the answer to part (b) for ρ,

$$\Omega \equiv \rho/\rho_c = \frac{2(1-\cos\theta)}{\sin^2\theta} = \boxed{\frac{2}{1+\cos\theta} \ .}$$

PROBLEM 2: EVOLUTION OF AN OPEN, MATTER-DOMINATED UNIVERSE

(a) As in the previous problem, we use the chain rule to write the definition of the Hubble parameter in terms of derivatives with respect to θ:

$$H(\theta) = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \ .$$

The parametric equations for a and t for an open, matter-dominated universe are given by

$$ct = \alpha\,(\sinh\theta - \theta)$$
$$\frac{a}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1) \ .$$

Recall that the hyperbolic trigonometric functions are defined by

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} \ ,$$
$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} \ ,$$

and they are differentiated as

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\sinh\theta = \cosh\theta \ ,$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta}\cosh\theta = \sinh\theta \ .$$

So, differentiating the parametric equations,

$$\frac{\mathrm{d}a}{\mathrm{d}\theta} = \alpha\sqrt{k}\sinh\theta \ ,$$
$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{\alpha}{c}(\cosh\theta - 1) = \frac{1}{\mathrm{d}\theta/\mathrm{d}t} \ .$$

Then

$$H(\theta) = \left[\frac{1}{\sqrt{\kappa}\alpha(\cosh\theta - 1)}\right]\left[\alpha\sqrt{\kappa}\sinh\theta\right]\left[\frac{c}{\alpha(\cosh\theta - 1)}\right]$$
$$= \boxed{\frac{c\sinh\theta}{\alpha(\cosh\theta - 1)^2}} \ .$$

(b) This problem can be attacked by at least three different methods. While you were expected to use only one, we will show all three.

(i) One way to find ρ is to use

$$H^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \ .$$

This is usually the safest method to find ρ for a cosmological model, since the above equation is one of the general Friedmann equations. The equation requires that the universe be homogeneous and isotropic, but it is valid for any form of matter. By contrast, the two other methods that will be shown below are valid only for "matter-dominated" universes (i.e., universes that are dominated by nonrelativistic matter, for which the pressure is always negligible). One can rewrite this equation as

$$\frac{8\pi}{3}G\rho = H^2 + \frac{kc^2}{a^2} \ .$$

Recalling that we described open universes by using $\kappa \equiv -k$, this can be rewritten as

$$\frac{8\pi}{3}G\rho = H^2 - \frac{\kappa c^2}{a^2} \ .$$

Replacing H by the answer in part (a) and a by its parametric equation, one finds

$$\begin{aligned} \frac{8\pi}{3}G\rho &= \frac{c^2 \sinh^2\theta}{\alpha^2(\cosh\theta - 1)^4} - \frac{\kappa c^2}{\alpha^2\kappa(\cosh\theta-1)^2} \\ &= \frac{c^2}{\alpha^2(\cosh\theta-1)^4}\left[\sinh^2\theta - (\cosh\theta-1)^2\right] \ . \end{aligned}$$

Now make use of the hypertrigonometric identity

$$\cosh^2\theta - \sinh^2\theta = 1$$

to simplify:

$$\begin{aligned} \sinh^2\theta - (\cosh\theta-1)^2 &= \sinh^2\theta - \cosh^2\theta + 2\cosh\theta - 1 \\ &= 2(\cosh\theta - 1) \ , \end{aligned}$$

so

$$\frac{8\pi}{3}G\rho = \frac{2c^2}{\alpha^2(\cosh\theta-1)^3} \ .$$

Dividing both sides of the equation by $(8\pi/3)G$, one finds

$$\boxed{\rho = \frac{3c^2}{4\pi G\alpha^2(\cosh\theta - 1)^3}\ .}$$

(ii) Using Eq. (5.17) of Lecture Notes 5, with Eq. (5.38),

$$\alpha = \frac{4\pi}{3}\frac{G\rho a^3}{\kappa^{3/2}c^2}\ ,$$

can be solved for ρ to give

$$\rho = \frac{3}{4\pi}\frac{\alpha\kappa^{3/2}c^2}{Ga^3}\ .$$

Then substitute the parametric equation for $a(\theta)$:

$$\begin{aligned}\rho &= \frac{3}{4\pi}\frac{\alpha\kappa^{3/2}c^2}{G}\frac{1}{\alpha^3\kappa^{3/2}(\cosh\theta - 1)^3} \\ &= \frac{3c^2}{4\pi G\alpha^2(\cosh\theta - 1)^3}\ .\end{aligned}$$

(iii) ρ can also be found from $\ddot{a} = -(4\pi/3)G\rho a$, as long as we know that the universe is matter-dominated. (Be careful, however, about applying this formula in other situations: if the pressure cannot be neglected, then this equation has to be modified.) To evaluate $\ddot{a}$, again use the chain rule. Starting with $\dot{a}$,

$$\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \alpha\sqrt{\kappa}\sinh\theta\frac{c}{\alpha(\cosh\theta - 1)} = \frac{c\sqrt{\kappa}\sinh\theta}{\cosh\theta - 1}\ .$$

Then

$$\begin{aligned}\ddot{a} &= \frac{\mathrm{d}\dot{a}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{c\sqrt{\kappa}\sinh\theta}{\cosh\theta - 1}\right]\frac{c}{\alpha(\cosh\theta - 1)} \\ &= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)}\left[\frac{\cosh\theta}{\cosh\theta - 1} - \frac{\sinh^2\theta}{(\cosh\theta - 1)^2}\right] \\ &= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^3}\left[\cosh\theta(\cosh\theta - 1) - \sinh^2\theta\right] \\ &= \frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^3}(1 - \cosh\theta) = -\frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^2}\ .\end{aligned}$$

So

$$\ddot{a} = -\frac{4\pi}{3}G\rho a \quad \Longrightarrow \quad -\frac{c^2\sqrt{\kappa}}{\alpha(\cosh\theta - 1)^2} = -\frac{4\pi}{3}G\rho\alpha\sqrt{\kappa}(\cosh\theta - 1) \ ,$$

and

$$\rho = \frac{3c^2}{4\pi G\alpha^2(\cosh\theta - 1)^3} \ .$$

(c) The critical mass density satisfies the cosmological evolution equations for $k = 0$, so

$$H^2 = \frac{8\pi}{3}G\rho_c \ .$$

Then

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2} \ .$$

Now replace H by the answer to part (a), and ρ by the answer to part (b):

$$\begin{aligned}\Omega &= \frac{8\pi G}{3}\left[\frac{3}{4\pi}\frac{c^2}{G\alpha^2(\cosh\theta-1)^3}\right]\left[\frac{\alpha^2(\cosh\theta-1)^4}{c^2\sinh^2\theta}\right] \\ &= 2\frac{\cosh\theta-1}{\sinh^2\theta} = 2\frac{\cosh\theta-1}{\cosh^2\theta-1} \\ &= 2\frac{\cosh\theta-1}{(\cosh\theta+1)(\cosh\theta-1)} = \boxed{\frac{2}{\cosh\theta+1}} \ .\end{aligned}$$

The answer can be written even more compactly, if one wishes, by using a further hypertrigonometric identity:

$$\Omega = \frac{2}{\cosh\theta + 1} = \frac{1}{\cosh^2\frac{1}{2}\theta} = \mathrm{sech}^2\,\frac{1}{2}\theta \ .$$

(d) The basic formula that determines the physical value of the horizon distance is given by Eq. (5.7) of the lecture notes:

$$\ell_{p,\mathrm{horizon}}(t) = a(t)\int_0^t \frac{c}{a(t')}dt' \ .$$

The complication here is that a is given as a function of θ, rather than t. The problem is handled, however, by a simple change of integration variables. One can change the integral over t' to an integral over θ', provided that one replaces

$$\mathrm{d}t' \to \frac{dt'}{\mathrm{d}\theta'}\mathrm{d}\theta' = \frac{\alpha}{c}(\cosh\theta' - 1)\mathrm{d}\theta' \ .$$

One must also re-express the limits of integration in terms of θ. So

$$\begin{aligned}\ell_{p,\text{horizon}}(\theta) &= a(\theta)\int_0^\theta \frac{c}{a(\theta')}\frac{dt'}{\mathrm{d}\theta'}\mathrm{d}\theta' \\ &= \alpha\sqrt{\kappa}(\cosh\theta-1)\int_0^\theta \frac{c}{\alpha\sqrt{\kappa}(\cosh\theta'-1)}\frac{\alpha}{c}(\cosh\theta'-1)\,\mathrm{d}\theta' \; . \\ &= \alpha(\cosh\theta-1)\int_0^\theta \mathrm{d}\theta' = \boxed{\alpha\,\theta(\cosh\theta-1)\; .}\end{aligned}$$

(e) The key to this problem is the use of power series expansions. When this problem appeared as a quiz problem in 1992, I was rather surprised to find that many of the students seemed very inexperienced in this technique. It is a very useful method of approximation, so I strongly urge you to learn it if you don't know it already. In general, any sufficiently smooth function $f(x)$ can be expanded about the point x_0 by the series

$$f(x) = f(x_0) + \frac{1}{1!}f'(x_0)(x-x_0) + \frac{1}{2!}f''(x_0)(x-x_0)^2 + \frac{1}{3!}f'''(x_0)(x-x_0)^3 + \ldots \; ,$$

where the prime is used to denote a derivative. In particular, the exponential, sinh, and cosh functions can be expanded about $\theta = 0$ by the formulas

$$\begin{aligned} e^\theta &= 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots \\ \sinh\theta &= \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^5}{7!}\ldots \\ \cosh\theta &= 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \ldots \; . \end{aligned}$$

For this problem, we expand the parametric equations for $a(\theta)$ and $t(\theta)$, keeping the first nonvanishing term in the power series expansions:

$$\begin{aligned} t &= \frac{\alpha}{c}(\sinh\theta-\theta) = \frac{\alpha}{c}\left(\frac{\theta^3}{3!}+\ldots\right) \\ a &= \alpha\sqrt{\kappa}(\cosh\theta-1) = \alpha\sqrt{\kappa}\left(\frac{\theta^2}{2!}+\ldots\right) \; . \end{aligned}$$

The first expression can be solved for θ, giving

$$\theta \approx \left(\frac{6ct}{\alpha}\right)^{1/3} ,$$

which can be substituted into the second expression to give

$$\boxed{a \approx \frac{1}{2}\alpha\sqrt{\kappa}\left(\frac{6ct}{\alpha}\right)^{2/3} .}$$

The power series expansions for the sinh and cosh are valid whenever the terms left out are much smaller than the last term kept, which happens when $\theta \ll 1$. Given the above relation between θ and t, this condition is equivalent to

$$t \ll \frac{\alpha}{6c} .$$

Thus,

$$\boxed{t^* \approx \frac{\alpha}{6c}, \text{ or } t^* \approx \frac{\alpha}{c} .}$$

Since there is no precise meaning to the statement that an approximation is valid, there is no precise value for t^*. It is possible to be more precise by placing criteria on the size of the first omitted term in the series, and using these criteria to derive a more precise value for t^*. These expressions for t^* are always in the form of a dimensionless constant times α/c. This approach is very good, but it was not required to get full credit for this problem.

(f) From part (c), the expression for Ω is given by

$$\Omega = \frac{2}{\cosh\theta + 1} .$$

So,

$$1 - \Omega = 1 - \frac{2}{\cosh\theta + 1} = \frac{\cosh\theta - 1}{\cosh\theta + 1} .$$

Expanding numerator and denominator in power series,

$$1 - \Omega \approx \frac{\frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots}{2 + \frac{\theta^2}{2!} + \dots} .$$

Keeping only the leading terms,

$$1 - \Omega \approx \frac{\frac{\theta^2}{2}}{2} = \frac{1}{4}\theta^2 \; ,$$

so

$$\boxed{1 - \Omega \approx \frac{1}{4}\left(\frac{6ct}{\alpha}\right)^{2/3} \; .}$$

This result shows that the deviation of Ω from 1 is amplified with time. This fact leads to a conundrum called the "flatness problem", which will be discussed later in the course.

A common mistake (very minor) was to keep extra terms, especially in the denominator. Keeping extra terms allows a higher degree of accuracy, so there is nothing wrong with it. However, one should always be sure to keep ***all*** terms of a given order, since keeping only a subset of terms may or may not increase the accuracy. In this case, an extra term in the denominator can be rewritten as a term in the numerator:

$$\begin{aligned} \frac{\frac{\theta^2}{2!}}{2 + \frac{\theta^2}{2!}} = \frac{1}{4}\frac{\theta^2}{1 + \frac{\theta^2}{4}} &= \frac{1}{4}\theta^2\left(1 - \frac{\theta^2}{4} + \ldots\right) \\ &= \frac{1}{4}\theta^2 - \frac{1}{16}\theta^4 + \ldots \; , \end{aligned}$$

where I used the expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \ldots \; .$$

Thus, the extra term in the denominator is equivalent to a term in the numerator of order θ^4, but other terms proportional to θ^4 have been dropped. So, it is not worthwhile to keep the 2nd term in the expansion of the denominator.

PROBLEM 3: THE CRUNCH OF A CLOSED, MATTER-DOMINATED UNIVERSE *(10 points)**†

* Solution written by Leo Stein and Alan Guth.

† The Problem Set claimed that this problem was taken from Ryden's Problem 6.14, while most of you probably found that it was Problem 6.5. While it is conceivable that I may have miscopied the number, in fact it was Problem 6.14 in the earliest printing of the book. You might wonder, however, if there were so many problems in the earlier printing, why the current printings have only 8 problems at the end of chapter 6. The answer is that the early printing also had only 8 problems at the end of chapter 6. Due to some software bug, they were numbered 6.10 though 6.17. Apparently some counter variable was not reset after Chapter 5, which has 9 problems.

Dr. Niwde measures the two quantities Ω_0 (in the range of $(1, \infty)$) and $H_0 < 0$. All of the quantities of interest (the time until the end of the universe t_{left}, the minimum z, and the lookback time $t_{\text{lb,bluest}}$) must all be stated in terms of the two physical observables Ω_0 and H_0. The parametric form of the evolution of the closed universe is parameterized by the development angle θ, which needs to be determined from the two physical observables, and the constant α which is a measure of the mass density of the universe. From class (or Eq. (5.32) in the Lecture Notes), we found

$$\alpha = \frac{c}{2|H|}\frac{\Omega}{(\Omega - 1)^{3/2}} \ . \tag{3.1}$$

The value is a constant over the course of the universe, so it can be evaluated at any time (except $\theta = \pi$); therefore insert the values of H_0 and Ω_0. We also have the relation (Eq. (5.34))

$$\cos\theta = \frac{2 - \Omega}{\Omega} \ . \tag{3.2}$$

This needs to be inverted for θ_0 when evaluated with Ω_0 on the right hand side. The function $\cos\theta$ is not one to one, so the inverse is not unique. We could write

$$\theta = \arccos\left(\frac{2 - \Omega}{\Omega}\right) \ , \tag{3.3}$$

adding the words that θ is to be chosen in the interval $\theta \in [\pi, 2\pi]$. Such an answer should get full credit, but it is hard to use, since calculators are not capable of responding to such verbal instructions. Calculators normally return the "principal branch" of the $\arccos(x)$ function, which maps $x \in [-1, 1]$ to $\theta \in [0, \pi]$. Similar principal branches are conventional for other inverse trigonometric functions; for example, the principal branch of $\arcsin(x)$ maps $x \in [-1, 1]$ to $\theta \in [-\pi/2, \pi/2]$. Since the principal branch of $\arccos(x)$ covers more of the range of the development angle θ than the principal branch of $\arcsin(x)$ does, $\cos\theta$ will be more convenient to invert than $\sin\theta$; but one also needs the value of $\sin\theta$ for some expressions. To find θ_0 with the principal branch of $\arccos(x)$, note that $\cos\theta = \cos(2\pi - \theta)$; thus values of $\theta \in [\pi, 2\pi]$ can be evaluated in terms of values in $[0, \pi]$, which is the principal branch of $\arccos(x)$. Using this, we can write

$$\theta_0 = 2\pi - \arccos\left(\frac{2 - \Omega_0}{\Omega_0}\right) \ , \tag{3.4}$$

where $\arccos(x)$ is evaluated using the principal branch. That is, θ_0 defined by Eq. (3.4) satisfies Eq. (3.2), and it lies in the range of π to 2π. For values of $\sin\theta_0$, one uses the identity $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$ (Eq. (5.36)). Since one knows that $\theta_0 \in [\pi, 2\pi]$, and $\sin\theta$ is negative on this interval, one takes the negative root:

$$\sin\theta_0 = -\frac{2\sqrt{\Omega_0 - 1}}{\Omega_0} \ . \tag{3.5}$$

Thus, the value of t_0, when Dr. Niwde makes his measurements, is given by

$$\begin{aligned} t_0 &= \frac{\alpha}{c}(\theta_0 - \sin\theta_0) \\ &= \frac{1}{2|H|}\frac{\Omega}{(\Omega-1)^{3/2}}\left[2\pi - \arccos\left(\frac{2-\Omega_0}{\Omega_0}\right) + \frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right] . \end{aligned} \tag{3.6}$$

One is now ready to find $t_{\text{left}} = t_{\text{Crunch}} - t_0$, using $ct_{\text{Crunch}} = 2\pi\alpha$. Evaluating this, one finds

$$\boxed{t_{\text{left}} = \frac{\Omega_0}{2|H_0|(\Omega_0-1)^{3/2}}\left[\arccos\left(\frac{2-\Omega_0}{\Omega_0}\right) - \frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right] .} \tag{3.7}$$

(Alternatively, one could have taken t_0 directly from Eq. (5.37) of Lecture Notes 5, using the choices described in the table following the equation. Rewriting Eq. (5.37) explicitly for the contracting phase,

$$t_0 = \frac{1}{2|H_0|}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}\left\{\arcsin\left(-\frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right) + \frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right\} , \tag{3.8}$$

where $\arcsin(x)$ is chosen between π and $\frac{3}{2}\pi$ if $\infty \geq \Omega_0 \geq 2$, and between $\frac{3}{2}\pi$ and 2π if $2 \geq \Omega_0 \geq 1$. In terms of the principal branch of the sine function, this can be written

$$t_0 = \frac{1}{2|H_0|}\frac{\Omega_0}{(\Omega_0-1)^{3/2}} \times \begin{cases} \left[\pi + \arcsin\left(\frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right) + \frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right] & \text{if } \infty \geq \Omega_0 \geq 2, \\ \left[2\pi - \arcsin\left(\frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right) + \frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right] & \text{if } 2 \geq \Omega_0 \geq 1 . \end{cases} \tag{3.9}$$

Finally, $t_{\text{left}} = 2\pi\alpha/c - t_0$ implies that

$$\boxed{t_{\text{left}} = \frac{1}{2|H_0|}\frac{\Omega_0}{(\Omega_0-1)^{3/2}} \times \begin{cases} \left[\pi - \arcsin\left(\frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right) - \frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right] & \text{if } \infty \geq \Omega_0 \geq 2, \\ \left[\arcsin\left(\frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right) - \frac{2\sqrt{\Omega_0-1}}{\Omega_0}\right] & \text{if } 2 \geq \Omega_0 \geq 1 , \end{cases}} \tag{3.10}$$

where $\arcsin(x)$ is evaluated using the principal branch. Note that the complexity of the if-construction above is avoided by using the arccos function, as in Eq. (3.7))

Continuing, we are next asked to determine the bluest blueshift that Dr. Niwde can observe. Assume that the density of galaxies is high enough so that all possible distances (within the horizon distance) are well represented. Then there is always a galaxy whose light is just arriving at Dr. Niwde's observatory at t_0 for **any** t_e in the range $0 < t_e < t_0$. We let $\theta_e \equiv \theta(t_e)$ and $a_e \equiv a(t_e)$ denote respectively the development angle and scale factor at time t_e. The bluest blueshift is then found by minimizing $1+z = \frac{a_0}{a_e}$ over all the values of a_e that are in the past of Dr. Niwde.

(As an aside, one may be concerned about whether some given value of t_e might correspond to a distance beyond the horizon. This, however, can never happen. Recall that the horizon is defined as the distance beyond which light has not yet had time to reach us. So if light emitted at time t_e from some location is reaching us today, then the location is not beyond the horizon.)

Returning to the question of minimization, z is minimized when a_e is maximized, which happens at $\theta_e = \pi$. Using $a/\sqrt{k} = \alpha(1 - \cos\theta)$, the value of $z_{\rm min}$ is found to be

$$1 + z_{\rm min} = \frac{a_0}{a_e} = \frac{1 - \cos\theta_0}{1 - \cos\theta_e} = \frac{1 - \cos\theta_0}{2} \ . \tag{3.11}$$

Using the value of $\cos\theta_0$ from Eq. (3.2), one finds

$$\boxed{z_{\rm min} = -\frac{1}{\Omega_0} \ .} \tag{3.12}$$

Finally, the lookback time is simply $t_{\rm lb} = t_0 - t_e$, where $t_e = t(\theta = \pi) = \pi\alpha/c$. Using Eq. (3.6) for t_0, this gives

$$\boxed{t_{\rm lb} = \frac{\Omega_0}{2|H_0|(\Omega_0 - 1)^{3/2}}\left[\pi - \arccos\left(\frac{2 - \Omega_0}{\Omega_0}\right) + \frac{2\sqrt{\Omega_0 - 1}}{\Omega_0}\right] .} \tag{3.13}$$

Or, using Eq. (3.9), one can write

$$\boxed{t_{\rm lb} = \frac{1}{2|H_0|}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \times \begin{cases} \left[\arcsin\left(\frac{2\sqrt{\Omega_0 - 1}}{\Omega_0}\right) + \frac{2\sqrt{\Omega_0 - 1}}{\Omega_0}\right] & \text{if } \infty \geq \Omega_0 \geq 2, \\ \left[\pi - \arcsin\left(\frac{2\sqrt{\Omega_0 - 1}}{\Omega_0}\right) + \frac{2\sqrt{\Omega_0 - 1}}{\Omega_0}\right] & \text{if } 2 \geq \Omega_0 \geq 1 \ . \end{cases}} \tag{3.14}$$

It was not asked in the problem, but one may want to know the distance to a galaxy which is most blueshifted. The physical distance integral becomes simple when written in terms of θ, giving

$$\ell_{p,\rm bluest} = \alpha(1 - \cos\theta_0)(\theta_0 - \theta_e) \ . \tag{3.15}$$

Inserting α, θ_0, and $\theta_e = \pi$, this is

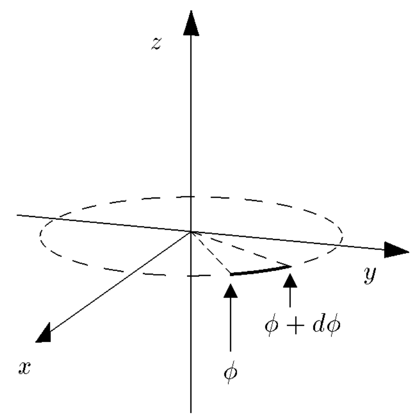
$$\ell_{p,\text{bluest}} = \frac{c}{|H_0|\sqrt{\Omega_0 - 1}}\left[\pi - \arccos\left(\frac{2-\Omega_0}{\Omega_0}\right)\right] \quad . \tag{3.16}$$

PROBLEM 4: A CIRCLE IN A NON-EUCLIDEAN GEOMETRY

(a) The metric is given by

$$\mathrm{d}s^2 = a^2\left\{\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} .$$

The metric allows us to find the proper, or physical, distance associated with a given coordinate displacement. For our problem, we are interested in the physical distance that results from motion in the ϕ direction, with the other coordinates held fixed, as shown at the right. For an infinitesimal segment of the circle, $r = r_0$ and $\theta = \pi/2$ are held constant; thus, $\mathrm{d}r = \mathrm{d}\theta = 0$, while $\mathrm{d}\phi$ is arbitrary. Plugging these values into the metric, we find that the physical arc length for an infinitesimal piece $\mathrm{d}\phi$ of the circumference is given by



$$\mathrm{d}s^2 = a^2\left\{r_0^2\,\mathrm{d}\phi^2\right\}$$

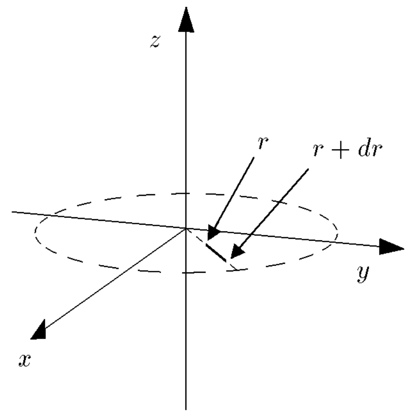
and therefore

$$\mathrm{d}s = ar_0\,\mathrm{d}\phi \quad .$$

To find the total circumference, we must integrate ϕ from 0 to 2π, so

$$S = \int \mathrm{d}s = ar_0\int_0^{2\pi}\mathrm{d}\phi = \boxed{2\pi a r_0} \quad .$$

(b) We now want to find the radius of the circle, so we must find the physical path length that corresponds to an infinitesimal displacement of the radial coordinate, with all angles held fixed. With $\theta = \pi/2$ and $\phi = \text{const.}$, $\mathrm{d}\theta = \mathrm{d}\phi = 0$, the metric becomes



$$ds^2 = a^2 \left\{ \frac{dr^2}{1 - kr^2} \right\} ,$$

and therefore

$$ds = \frac{a dr}{\sqrt{1 - kr^2}} .$$

To find the radius, we simply integrate the infinitesimal displacement,

$$\rho = \int ds = \int_0^{r_0} \frac{a dr}{\sqrt{1 - kr^2}} .$$

For a closed universe, $k = 1$ and

$$\rho = \int_0^{r_0} \frac{a dr}{\sqrt{1 - r^2}} = a \sin^{-1} r \Big]_0^{r_0} = \boxed{a \sin^{-1} r_0 .}$$

For an open universe, $k = -1$ and

$$\rho = \int_0^{r_0} \frac{a dr}{\sqrt{1 + r^2}} = a \sinh^{-1} r \Big]_0^{r_0} = \boxed{a \sinh^{-1} r_0 .}$$

(c) For $k = 1$, $S = 2\pi a r_0$ and $\rho = a \sin^{-1} r_0$

$$\Longrightarrow \boxed{S = 2\pi a \sin\left(\frac{\rho}{a}\right)}$$

Note that for $\rho \ll a$, $\sin\left(\frac{\rho}{a}\right) \approx \frac{\rho}{a}$ and so $S \approx 2\pi\rho$, in agreement with Euclidean geometry. For $\rho > 0$, $\sin(\rho/a) < \rho/a$, so $\boxed{S < 2\pi\rho}$.

For $k = -1$, $S = 2\pi a r_0$ and $\rho = a \sinh^{-1} r_0$

$$\Longrightarrow \boxed{S = 2\pi a \sinh\left(\frac{\rho}{a}\right) .}$$

Again, if $\rho \ll a$, $\sinh(\rho/a) \approx \rho/a$ and $S \approx 2\pi\rho$. For $\rho > 0$, $\sinh(\rho/a) > \rho/a$, so $\boxed{S > 2\pi\rho}$.

PROBLEM 5: VOLUME OF A CLOSED UNIVERSE

Note on Equation Numbers: Unfortunately I did not realize soon enough that the equation numbers mentioned in the statement of the problem referred to Lecture Notes 6 for 2005. Correcting for this year's numbering, the problem should have read:

> . . . It will be easiest to use the metric in the form of Eq. (6.14):
>
> $$ds^2 = a^2 \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \quad .$$
>
> . . . By comparing Eqs. (6.9) and (6.14), one can see that as long as ψ is held fixed, the metric for varying θ and ϕ is the same as that for a spherical surface of radius $a \sin\psi$, and thus the area of the spherical surface is $4\pi a^2 \sin^2\psi$.

In these solutions all equation numbers have been updated to correspond to the 2007 notes.

The metric for the closed universe can be written as

$$ds^2 = a^2 \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \quad ,$$

which is Eq. (6.14), with a^2 replaced by a^2 by using Eq. (6.21) with $k = 1$. For comparison, the metric for the surface of a sphere of radius a is given by Eq. (6.9),

$$ds^2 = a^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad .$$

By comparing these two, one sees that the set of points described by $\psi =$ constant (varying θ and ϕ) has the same metric as a sphere of radius $a = a \sin\psi$. We can save ourselves some trouble in calculating by remembering that the area of such a spherical surface is $4\pi a^2 = 4\pi a^2 \sin^2\psi$.

The volume of the spherical shell shown in the problem is just the area times the thickness. The thickness is not $d\psi$, since ψ is only a coordinate — remember that in curved space a coordinate and a distance are two different things. The distance is given by the metric. Consider in this case a radial line extending from ψ to $\psi + d\psi$, at constant θ and ϕ. Then

$$ds^2 = a^2 d\psi^2 \quad ,$$

and so the length of the line segment is $ds = a d\psi$.

The volume of the spherical shell is then given by

$$dV = \left[4\pi a^2 \sin^2\psi\right] a d\psi \quad .$$

We must now integrate over the range of ψ. The variable ψ was introduced with Eq. (6.12) and the accompanying diagram, and it was defined as the angle between the radial line and the positive w-axis. This angle ranges from 0, when the two lines are parallel, to π when they point in opposite directions. So

$$V = 4\pi a^3 \int_0^\pi \sin^2\psi \, \mathrm{d}\psi \quad .$$

Integrating,

$$\begin{aligned} \int_0^\pi \sin^2\psi \, \mathrm{d}\psi &= \int_0^\pi \frac{1-\cos 2\psi}{2} \mathrm{d}\psi \\ &= \frac{1}{2}\left\{ \psi \Big]_0^\pi - \frac{1}{2}\sin 2\psi \Big]_0^\pi \right\} \\ &= \frac{\pi}{2} \quad . \end{aligned}$$

So

$$\boxed{V = 2\pi^2 a^3 \quad .}$$

Alternatively, there is a famous (and very useful) mathematical "trick" for integrating $\sin^2\psi$ over any interval which is a multiple of $\pi/2$. Over such an interval $\sin^2\psi$ varies over its full range, and $\cos^2\psi$ would do the same (although out of phase). Using the fact that $\cos^2\psi + \sin^2\psi = 1$, it follows that $\cos^2\psi$ and $\sin^2\psi$ must each average to $1/2$ when integrated over any interval which is a multiple of $\pi/2$. The integral

$$\int_0^\pi \sin^2\psi \mathrm{d}\psi = \int_0^\pi \mathrm{d}\psi \times \left[\text{average of} \sin^2\theta\right] = \frac{\pi}{2} \quad .$$