MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 31, 2009
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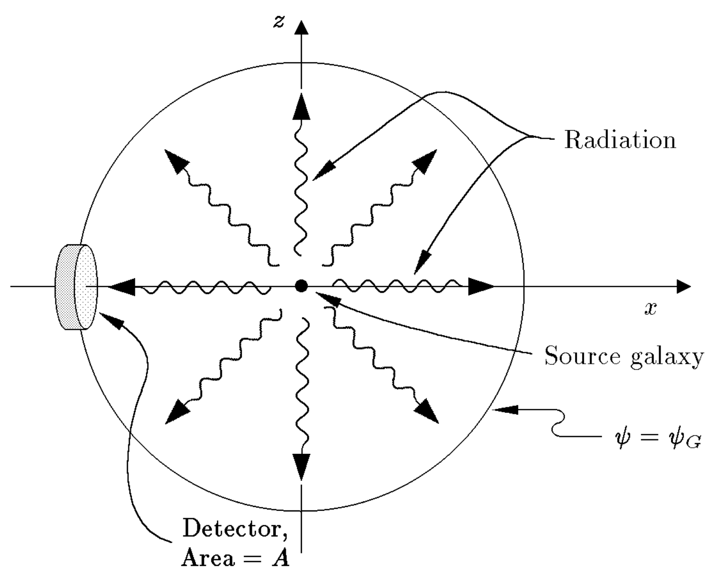
PROBLEM SET 5 SOLUTIONS

PROBLEM 1: SURFACE BRIGHTNESS IN A CLOSED UNIVERSE*
(10 points)

In this problem we use the form of the metric

$$ds^2 = -c^2 dt^2 + a^2(t)\left\{d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)\right\}$$

(a) Following the hint, we draw Robertson-Walker coordinates with the galaxy G in the center. The radial coordinate of the detector on Earth will be ψ_G. The diagram also shows a sphere at the same radial coordinate ψ_G:



Since the speed of light is independent of angle, all the photons that left the galaxy G at time t_G are arriving at the $\psi = \psi_G$ sphere at the present time t_0. To calculate the power received by the detector we need to know what fraction of those photons hit the detector. The fraction is simply the area A of the detector divided by the area $A_s(t_0)$ of the sphere at time t_0. The area of the sphere can be calculated by restricting the metric to the case $dt = d\psi = 0$, $t = t_0$, $\psi = \psi_G$:

$$ds^2 = a^2(t_0)\sin^2\psi_G\,(d\theta^2 + \sin^2\theta\, d\phi^2)\ .$$

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* Solution by Barton Zwiebach, based on a prior version by Alan Guth.

This expression is identical to the metric of the surface of a sphere of radius $r = a(t_0)\sin\psi_G$. The area is therefore $A_s(t_0) = 4\pi r^2 = 4\pi a^2(t_0)\sin^2\psi_G$. So,

$$\text{fraction} = \frac{\text{area of detector}}{\text{area of sphere}} = \frac{A}{4\pi a^2(t_0)\sin^2\psi_G} \ .$$

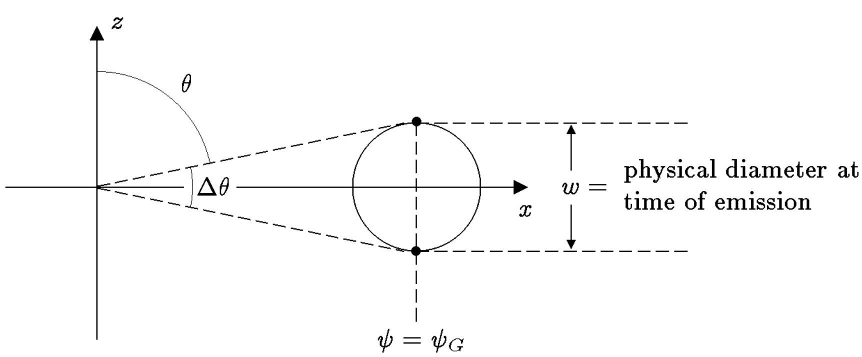
The power hitting the detector is further reduced by one factor of $(1+z) = a(t_0)/a(t_G)$ because the frequency, and hence the energy, of each photon is reduced by this factor. In addition, the power is reduced by another factor of $(1+z)$ because the rate of arrival of photons is reduced by this factor. Thus, if P is the power that the galaxy was emitting at time t_G, then the power received by the detector today is

$$\begin{aligned} P_{\text{received}} &= P\frac{A}{4\pi a^2(t_0)\sin^2\psi_G}\left[\frac{a(t_G)}{a(t_0)}\right]^2 \\ &= P\frac{Aa^2(t_G)}{4\pi a^4(t_0)\sin^2\psi_G} \ . \end{aligned}$$

The flux is given by

$$\boxed{J = \frac{P_{\text{received}}}{A} = P\frac{a^2(t_G)}{4\pi a^4(t_0)\sin^2\psi_G} \ .}$$

(b) If we choose the axis shown vertically in the diagram to be the z-axis, then the angle labeled $\Delta\theta$ will represent an increment of the Robertson-Walker coordinate θ, as the label $\Delta\theta$ suggests:



At time t_G the distance between the two edges of the galaxy is given, according to the metric, by

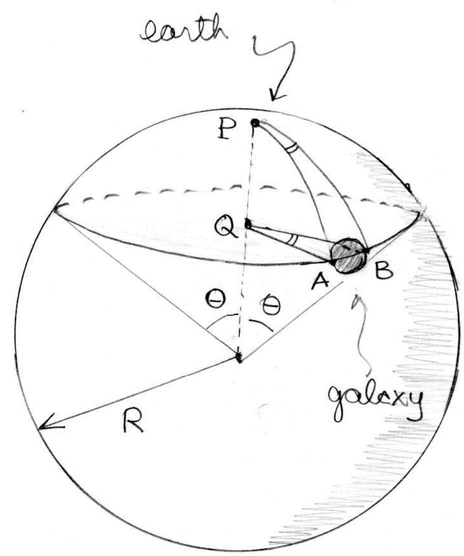
$$ds = a(t_G)\sin\psi_G\,\Delta\theta \ ,$$

where I have assumed that $\Delta\theta \ll 1$. But the problem tells us that this distance is w, so

$$w = a(t_G)\sin\psi_G\,\Delta\theta \quad \Longrightarrow \quad \boxed{\Delta\theta = \frac{w}{a(t_G)\sin\psi_G}}\ .$$

Remarks. In solving the problem this way, we used the diagram above only to label the coordinates, but we needed the metric to determine the angle. It would have been incorrect to assume that $\Delta\theta$ is calculated by dividing a comoving distance $w/a(t_G)$ by the radial distance ψ_G. The problem is that in curved space the angle between two geodesics cannot be calculated by dividing an arc length over a radial distance. Think, for example, of two geodesics starting at the north pole of planet Earth at an angle of 90°. By the time they reach the equator, the distance between them, along the equator, is equal to the radial distance from the north pole to the equator. A naive calculation of the angle would then give 1 radian, smaller than the correct angle of 90°.

By taking advantage of the fact that a closed universe can be viewed as a sphere embedded in a Euclidean space, the two dimensional analog of the problem can be visualized nicely. As shown in the figure to the right, we assume planet Earth is at the north pole P and a galaxy, shown as a small disk, is on a latitude circle with some value of θ. We also show two geodesics (or light ray trajectories) that leave the two sides A and B of the galaxy and reach P. Suppose the distance between the sides A and B of the galaxy is w, what is the angle $\Delta\phi$ at P? The angle is *not* w divided by the distance $a\theta$ from P to the galaxy. To find the correct value we must first show that the angle $\Delta\phi$ or, equivalently, the angle APB is equal to the angle AQB.



One way to see this is to note that the vertical projection of the line PA to the plane through the latitude circle gives the line QA and the vertical projection of the line PB gives the line QB. In a very small neighborhood of P the surface of the sphere is approximately flat and locally parallel to the plane through the latitude circle. The projection then does pretty much nothing, showing that the angles are the same. Note that this means that the angle at P can be calculated by dividing arc length by radial distance *only* in the limit as the radial distance goes to zero.

To calculate $\Delta\phi$ we use the disk through the latitude circle. On this circle the galaxy spans a distance w and the radial distance is equal to the radius $a\sin\theta$. Since this is a circle on flat space, we find

$$\Delta\phi = \frac{w}{a\sin\theta} .$$

By our argument above this is the correct value for the angle seen at P. The more operational way to obtain this answer uses the metric on the sphere:

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2) .$$

The galaxy corresponds to $d\theta = 0$ and $d\phi = \Delta\phi$. We then have

$$w = ds = a\sin\theta\Delta\phi,$$

which coincides with the answer obtained before.

c) To evaluate the solid angle subtended by the galaxy, imagine surrounding the observer by a small sphere of arbitrary radius r. The galaxy would appear on this sphere as a disk with a angular radius $\Delta\theta/2$, which implies a radius of $r\,\Delta\theta/2$, and an area $A = \pi r^2\,\Delta\theta^2/4$. The solid angle is given by

$$\Delta\Omega \equiv \frac{A}{r^2} = \frac{\pi\,\Delta\theta^2}{4} .$$

Using the answers from the previous two parts, the surface brightness is given by

$$\sigma = \frac{J}{\Delta\Omega} = \frac{4J}{\pi\,\Delta\theta^2} = \frac{4Ja^2(t_G)\sin^2\psi_G}{\pi w^2}$$

$$= \frac{Pa^4(t_G)}{\pi^2 w^2 a^4(t_0)} = \boxed{\frac{P}{\pi^2 w^2}\frac{1}{(1+z)^4}} .$$

While we derived this formula for a closed universe, we would have found the same result in an open or flat universe.

Note that this result implies that for $z \ll 1$, the surface brightness is independent of distance. This result is consistent with Euclidean geometry, which says that both the energy flux and the solid angle are inversely proportional to the square of the distance, so the surface brightness is independent of distance.

Recall that in Problem Set 2 (2009) we discussed the most distant galaxy with a well-determined redshift, with $z = 6.96$. Note that for this galaxy the surface brightness is suppressed by a whopping factor of $(1+z)^4 \approx 4,015$, which indicates why such high redshift objects are difficult to see!

PROBLEM 2: TRAJECTORIES AND DISTANCES IN AN OPEN UNIVERSE

a) The geodesic is along a radial line, so $\mathrm{d}\theta = \mathrm{d}\phi = 0$. Then $\mathrm{d}\tau = 0$, which is always true for a light pulse traveling in a vacuum, implies that

$$-c^2\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\psi^2 = 0\ , \tag{1}$$

or

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\frac{c}{a(t)}\ .$$

Note that Eq. (1) has two roots, $\mathrm{d}\psi/\mathrm{d}t = \pm c/a(t)$, but the negative sign is right for this problem because the value of ψ for the light pulse starts at ψ_G (which is always positive) and ***decreases*** to 0. Integrating,

$$\mathrm{d}\psi = -\frac{c}{a(t)}\,\mathrm{d}t$$

$$\int_{\psi_G}^{0} \mathrm{d}\psi = -\int_{t_G}^{t_0} \frac{c}{a(t)}\,\mathrm{d}t$$

$$\boxed{\psi_G = \int_{t_G}^{t_0} \frac{c}{a(t)}\,\mathrm{d}t}$$

which, since ψ_G is known, determines t_G in terms of $a(t)$.

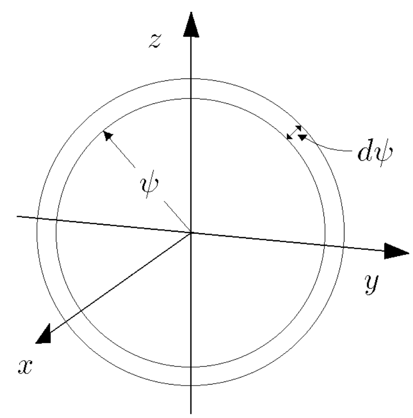
b) The cosmological red shift is given by

$$1 + z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_{\text{observed}})}{a(t_{\text{emitted}})}\ .$$

Since $t_{\text{observed}} = t_0$ and $t_{\text{emitted}} = t_G$, it follows that

$$\boxed{z_G = \frac{a(t_0)}{a(t_G)} - 1\ .}$$

c) To find the volume of space with redshifts smaller than that of galaxy G, the first step is to recognize that the redshift increases monotonically with ψ_G. (If you doubt this statement, note that the answer to (a) implies that t_G decreases monotonically with ψ_G. Assuming that $a(t)$ is monotonically increasing, the answer to (b) then implies that z_G increases monotonically with ψ_G.) Thus, the region with z smaller than that of galaxy G is the region with $0 < \psi < \psi_G$. In other words, we need to find the physical volume of a sphere of radius ψ_G, so conceptually this will be very similar to Problem 2. To integrate the volume of this region, we again divide space into concentric shells, with radial coordinate ψ and coordinate thickness $d\psi$, as shown at the right.



The area of the spherical shell is determined by the metric on the surface, which can be obtained from the full metric by treating t and ψ as fixed:

$$ds^2 = a^2(t)\sinh^2\psi\,(d\theta^2 + \sin^2\theta\,d\phi^2)\ .$$

This expression is identical to the metric of the surface of a sphere of radius $r = a(t)\sinh\psi$. The area is therefore $A = 4\pi r^2 = 4\pi a^2(t)\sinh^2\psi$. Looking again at the metric, one sees that the physical thickness of the shell is $ds = a(t)\,d\psi$. The volume of the shell is then

$$dV = A\,a(t)\,d\psi = 4\pi a^3(t)\sinh^2\psi\,d\psi\ ,$$

and the total volume is found by integration:

$$\boxed{V = 4\pi a^3(t)\int_0^{\psi_G}\sinh^2\psi\,d\psi\ .}$$

Extension: You were not asked to evaluate the integral, but it can be done as follows:

$$\begin{aligned}\int_0^{\psi_G}\sinh^2\psi\,d\psi &= \int_0^{\psi_G}\left[\frac{e^{\psi}-e^{-\psi}}{2}\right]^2 d\psi\\ &= \frac{1}{4}\int_0^{\psi_G}\left[e^{2\psi}+e^{-2\psi}-2\right] d\psi\\ &= \frac{1}{4}\left[\frac{1}{2}e^{2\psi}-\frac{1}{2}e^{-2\psi}-2\psi\right]\Bigg|_0^{\psi_G}\\ &= \frac{1}{4}\left[\frac{1}{2}\left(e^{2\psi_G}-e^{-2\psi_G}\right)-2\psi_G\right]\\ &= \frac{1}{4}\left[\sinh(2\psi_G)-2\psi_G\right]\ .\end{aligned}$$

The volume is then

$$\boxed{V = \pi a^3(t)\left[\sinh(2\psi_G) - 2\psi_G\right] \ .}$$

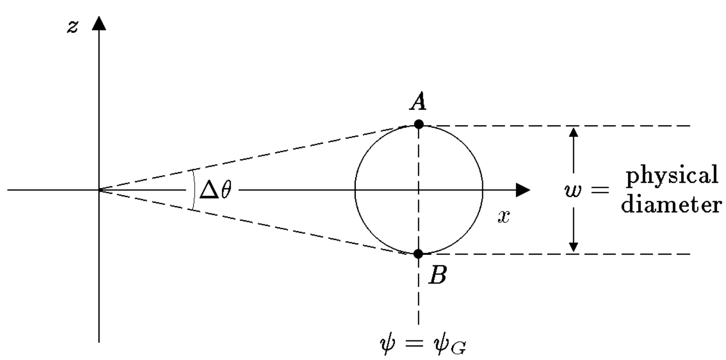
d) The proper distance between (ψ, θ, ϕ) and $(\psi + \mathrm{d}\psi, \theta, \phi)$ is what is measured by a ruler at rest in this coordinate system, but that is exactly the meaning of the ds that appears in the expression for the metric. Since t, θ, and ϕ are constant along the radial line between Earth and the galaxy G, the metric at $t = t_0$ reduces to

$$\mathrm{d}s = a(t_0)\,\mathrm{d}\psi \ .$$

Integrating,

$$\boxed{\ell_{\rm prop} = \int \mathrm{d}s = a(t_0)\,\psi_G \ .}$$

e) The calculation of the angular size distance is similar to the angular size calculation in Problem 5 of Problem Set 2 (2009), but it is not quite identical. There we were talking about a flat universe, but this time we are interested in a curved universe. The basic method is the same, however, so long as we remember that all distances have to be determined via the metric. Placing the galaxy for convenience along the x axis ($\theta = 0$), we draw it at the time of emission, t_G:



We draw the picture at the time of emission, because the photons that we receive today arrive on trajectories that were determined solely by the position of the galaxy at that time. Using the metric, we can express the physical diameter of the galaxy at the time of emission. The only coordinate that changes between the points A and B is θ, so

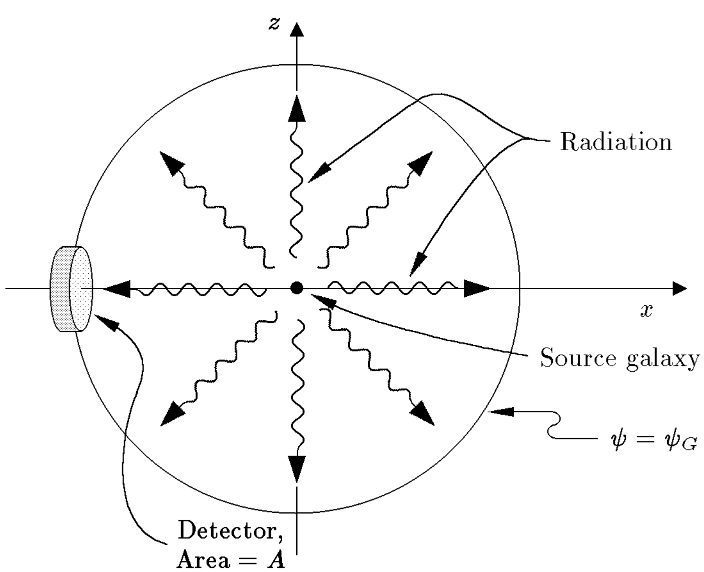
$$w = \mathrm{d}s = a(t_G)\,\sinh\psi_G\,\Delta\theta \ .$$

(Note that we have assumed in the above equation that $\Delta\theta \ll 1$, so that we do not need to distinguish between the angle between A to B, which are at opposite ends of the diameter of the sphere, and the angle of visibility, which is bounded by lines which are tangent to the sphere.) The angular size distance is then

$$\boxed{\ell_{\rm ang} \equiv \frac{w}{\Delta\theta} = a(t_G)\,\sinh\psi_G\ .}$$

Subtlety: To be sure that the above solution is correct, one must know that the coordinate separation $\Delta\theta$ between the points A and B, at the time of emission, is equal to the angular size that we observe today. This equality can be justified by using the fact that we are located at the origin of this coordinate system, and therefore the photons that we detect arrive along radial lines. (One can verify that trajectories that move along radial lines at the speed of light are geodesics, but I will not try to do that here.) The photons that left point A, at $\theta = \frac{1}{2}\Delta\theta$, will arrive today along the radial line at $\theta = \frac{1}{2}\Delta\theta$. Similarly, the photons that left point B, at $\theta = -\frac{1}{2}\Delta\theta$, will arrive today along the radial line at $\theta = -\frac{1}{2}\Delta\theta$. Thus, the angular size that we observe, the angular separation between these two radial lines, is $\Delta\theta$.

f) Following the hint, we draw Robertson-Walker coordinates with the galaxy G in the center. The radial coordinate of the detector, on Earth, will be ψ_G. The diagram also shows a sphere at the same radial coordinate, ψ_G:



Since the speed of light is independent of angle, all the photons that left the galaxy G at time t_G are arriving at the $\psi = \psi_G$ sphere at the present time,

t_0. To calculate the power received by the detector, we need to know what fraction of those photons hit the detector. The fraction is simply the area of the detector divided by the area of the sphere, or

$$\text{fraction} = \frac{\text{area of detector}}{\text{area of sphere}} = \frac{A}{4\pi a^2(t_0)\sinh^2\psi_G} \; .$$

(The formula for the area was discussed in the answer to (c).) The power hitting the detector is further reduced by one factor of $(1+z) = a(t_0)/a(t_G)$ because the frequency, and hence the energy, of each photon is reduced by this factor. In addition, the power is reduced by another factor of $(1+z)$ because the rate of arrival of photons is reduced by this factor. Thus, if P is the power that the galaxy was emitting at time t_G, then the power received by the detector today is

$$\begin{aligned} P_{\text{received}} &= P\frac{A}{4\pi a^2(t_0)\sinh^2\psi_G}\left[\frac{a(t_G)}{a(t_0)}\right]^2 \\ &= P\frac{Aa^2(t_G)}{4\pi a^4(t_0)\sinh^2\psi_G} \; . \end{aligned}$$

The flux is given by

$$J = \frac{P_{\text{received}}}{A} = P\frac{a^2(t_G)}{4\pi a^4(t_0)\sinh^2\psi_G} \; .$$

From the definition of luminosity distance,

$$\boxed{\ell_{\text{lum}} \equiv \sqrt{\frac{P}{4\pi J}} = \frac{a^2(t_0)\sinh\psi_G}{a(t_G)} \; .}$$

Note, by the way, that the luminosity distance and the angular size distance have a simple relationship to each other:

$$\ell_{\text{lum}} = \left(\frac{a(t_0)}{a(t_G)}\right)^2 \ell_{\text{ang}} = (1+z)^2\,\ell_{\text{ang}} \; .$$

While we derived this relation for open universes, the same relation would apply in a flat or closed universe. From the answer to part (d), we can see that the proper distance is related in a slightly more complicated way:

$$\begin{aligned} \ell_{\text{prop}} &= (1+z)\,\frac{\psi_G}{\sinh\psi_G}\,\ell_{\text{ang}} \\ &= \frac{1}{(1+z)}\,\frac{\psi_G}{\sinh\psi_G}\,\ell_{\text{lum}} \; . \end{aligned}$$

The factor $\psi_G/\sinh\psi_G$ is a geometric factor, independent of the expansion of the universe, which is less than 1 for the open universe case that we are considering. In a closed universe the analogous factor would have been $\psi_G/\sin\psi_G$, which is greater than 1, and in a flat universe the corresponding factor would be 1.

PROBLEM 3: GEODESICS IN A FLAT UNIVERSE

(a) The geodesic equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \ .$$

If we take the free index μ to be x , using $g_{xx} = a^2(t)$ and $\partial g_{\lambda\sigma}/\partial x = 0$ gives us:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[a^2(t)\frac{\mathrm{d}x}{\mathrm{d}\tau}\right] = 0$$

$$\Longrightarrow \quad \boxed{\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\text{const.}}{a^2(t)} \ .}$$

(b) Specializing the metric to the case of motion along the x-axis,

$$\begin{aligned} -c^2\mathrm{d}\tau^2 &= -c^2\mathrm{d}t^2 + a^2(t)\mathrm{d}x^2 \\ &= -c^2\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2\mathrm{d}t^2 \ . \end{aligned}$$

Then

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{1 - \frac{1}{c^2}a^2(t)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2} \ ,$$

and

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\mathrm{d}x/\mathrm{d}t}{\sqrt{1 - \frac{1}{c^2}a^2(t)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}} \ .}$$

(c) Using the physical velocity of the particle in the formula for its relativistic momentum yields

$$\begin{aligned} p &= \frac{mv}{\sqrt{1 - v^2/c^2}} = \frac{ma(t)\frac{\mathrm{d}x}{\mathrm{d}t}}{\sqrt{1 - \frac{1}{c^2}a^2(t)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}} \\ &= ma(t)\frac{\mathrm{d}x}{\mathrm{d}\tau} \end{aligned}$$

As we found previously,

$$a^2(t)\frac{\mathrm{d}x}{\mathrm{d}\tau} = \text{const.} \Longrightarrow$$

$$\boxed{p = ma(t)\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\text{const.}}{a(t)} \quad .}$$

PROBLEM 4: METRIC OF A STATIC GRAVITATIONAL FIELD

(a) $\mathrm{d}s^2$ is the invariant separation between the event at (x^i, t) and the event at $(x^i + \mathrm{d}x^i, t + \mathrm{d}t)$. Here x^i and t are arbitrary coordinates that are connected to measurements only through the metric. As discussed in Lecture Notes 6 ("The Generalization from Space to Spacetime"), when $\mathrm{d}s^2$ is negative one uses

$$\mathrm{d}s^2 \equiv -c^2\,\mathrm{d}\tau^2 \ ,$$

where $\mathrm{d}\tau$ is interpreted as the proper time separation between the events, which is the time interval that would be measured on a clock carried by a free-falling observer who sees the two events happen at the same location. In this case the radio transmitter sees the emission of two successive pulses as occurring at the same location, so the time ΔT_e that it measures is the proper time:*

$$\mathrm{d}s^2 = -c^2\,\Delta T_e^2 \ .$$

To connect with the metric, note that the successive emissions have a separation in the time coordinate of Δt_e, and a separation of space coordinates $\mathrm{d}x^i = 0$. So

$$\mathrm{d}s^2 = -c^2\,\Delta T_e^2 = -[c^2 + 2\phi(\vec{x}_e)]\,\Delta t_e^2 \quad \Longrightarrow$$

$$\boxed{\Delta t_e = \frac{\Delta T_e}{\sqrt{1 + \frac{2\phi(\vec{x}_e)}{c^2}}} \ .}$$

* The transmitter is not really a freely falling observer, but is presumably held at rest in this coordinate system. Thus gravity is acting on the clock, and could in principle affect its speed. It is standard, however, to assume that such effects are negligible. That is, one assumes that the clock is *ideal*, meaning that it ticks at the same rate as a freely falling clock that is instantaneously moving with the same velocity. Note that this property of ideal timekeeping is an assumption about the physical makeup of the clock, and is not a consequence of general relativity. There is nothing in general relativity that prevents one from building a very delicate clock that will stop completely if subjected to an acceleration of 1 m/s^2.

(b) Since the metric is independent of t, each pulse follows a trajectory identical to the previous pulse, but delayed in t. Thus each pulse requires the same time interval Δt to travel from emitter to receiver, so the pulses arrive with the same t-separation as they have at emission:

$$\boxed{\Delta t_r = \Delta t_e\ .}$$

(c) This is similar to part (a), but in this case we consider the two events corresponding to the reception of two successive pulses. $\mathrm{d}s^2$ is related to the physical measurement ΔT_r by

$$\mathrm{d}s^2 = -c^2\,\Delta T_r^2\ .$$

It is connected to the coordinate separation Δt_r through the metric, where again we use the fact that the two events have zero separation in their space coordinates— i.e., $\mathrm{d}x^i = 0$. So

$$\mathrm{d}s^2 = -c^2\,\Delta T_r^2 = -[c^2 + 2\phi(\vec{x}_r)]\,\Delta t_e^2 \quad \Longrightarrow$$

$$\Delta T_r = \sqrt{1 + \frac{2\phi(\vec{x}_r)}{c^2}}\,\Delta t_e\ .$$

We can cast this into a more useful form for the problem by using the solution for Δt_e found in part (a). This gives

$$\Delta T_r = \left[\frac{\sqrt{1 + \frac{2\phi(\vec{x}_r)}{c^2}}}{\sqrt{1 + \frac{2\phi(\vec{x}_e)}{c^2}}}\right]\Delta T_e\ .$$

Substitute this result for ΔT_r directly into the definition for z to obtain the exact expression for the redshift,

$$\boxed{1 + z = \frac{\sqrt{1 + \frac{2\phi(\vec{x}_r)}{c^2}}}{\sqrt{1 + \frac{2\phi(\vec{x}_e)}{c^2}}}\ .}$$

Remember that $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ for small x. For weak fields, that is, for small values of $\phi(\vec{x})$, we can expand our result to lowest order in $\phi(\vec{x})$. Expanding the numerator we have

$$\sqrt{1 + \frac{2\phi(\vec{x}_r)}{c^2}} \approx 1 + \frac{\phi(\vec{x}_r)}{c^2}\ .$$

Similarly we find for

$$\frac{1}{\sqrt{1+\frac{2\phi(\vec{x}_e)}{c^2}}} \approx 1 - \frac{\phi(\vec{x}_e)}{c^2} \; .$$

Putting these approximations into our exact expression for $1+z$ we obtain

$$1+z \approx \left(1+\frac{\phi(\vec{x}_r)}{c^2}\right)\left(1-\frac{\phi(\vec{x}_e)}{c^2}\right) \approx 1 + \frac{\phi(\vec{x}_r)}{c^2} - \frac{\phi(\vec{x}_e)}{c^2} \; ,$$

where we dropped terms in $\phi(\vec{x}_e)\phi(\vec{x}_r)$. Finally,

$$\boxed{z \approx \frac{\phi(\vec{x}_r)-\phi(\vec{x}_e)}{c^2} \; .}$$

(d) For the metric at hand we know $g_{00} = -[c^2 + 2\phi(\vec{x})]$, $g_{k0} = g_{0k} = 0$ and $g_{ik} = g_{ki} = \delta_{ik}$. It is useful to notice that only g_{00} depends on $\vec{x}$ and thus $\partial_i g_{km} = 0$. The geodesic equation corresponding to $\mu = i$, where i runs from 1 to 3, is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{ik}\frac{\mathrm{d}x^k}{\mathrm{d}\tau}\right) = \frac{1}{2}(\partial_i g_{\lambda\sigma})\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \quad \Longrightarrow$$

$$\delta_{ik}\frac{\mathrm{d}^2x^k}{\mathrm{d}\tau^2} = \frac{1}{2}(\partial_i g_{00})\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \; .$$

Using $x^0 \equiv t$, $\delta_{ik}y^k = y^i$ and

$$\partial_i g_{00} = -\partial_i(c^2 + 2\phi(\vec{x})) = -2\partial_i\phi(\vec{x})$$

we find

$$\boxed{\frac{\mathrm{d}^2x^i}{\mathrm{d}^2\tau} = -\partial_i\phi(\vec{x})\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \; .}$$

[Pedagogical Note: You might prefer to use the notation $x^0 \equiv ct$, which is also a very common choice. In that case the metric is rewritten as

$$\mathrm{d}s^2 = -\left[1+\frac{2\phi(\vec{x})}{c^2}\right]\left(\mathrm{d}x^0\right)^2 + \sum_{i=1}^{3}\left(\mathrm{d}x^i\right)^2 \quad ,$$

so one takes $g_{00} = -\left[1 + (2\phi(\vec{x})/c^2)\right]$. In the end one finds the same answer as the boxed equation above.

Note also that when ϕ is small and velocities are nonrelativistic, then $\mathrm{d}t/\mathrm{d}\tau \approx 1$. Thus one has $\mathrm{d}^2x^i/\mathrm{d}^2t \approx -\partial_i\phi(\vec{x})$, so $\phi(\vec{x})$ can be identified with the Newtonian gravitational potential. In the context of general relativity, Newtonian gravity is a distortion of the metric in the time-direction.]

(e) As shown in part (c), the gravitational redshift z is given in the weak field limit as

$$z \approx \frac{\Delta\phi}{c^2} \ ,$$

where $\Delta\phi$ is the difference in gravitational potential. Near the surface of the Earth the gravitational potential can be written, up to an arbitrary additive constant, as

$$\phi(z) \approx gz \ ,$$

where z is the vertical coordinate, and $g \approx 9.8\ \mathrm{m/s}^2$. Thus, if the height of the room is $h \approx 4$ m, the redshift is given by

$$z \approx \frac{gh}{c^2} \approx \frac{9.8\ \text{m-s}^{-2} \cdot 4\ \text{m}}{(3.0 \times 10^8\ \text{m-s}^{-1})^2} \approx \boxed{4.4 \times 10^{-16}\ .}$$

PROBLEM 5: THE KLEIN DESCRIPTION OF THE G-B-L GEOMETRY (optional)

(a) The Klein formula for distance is given by

$$\cosh\left[\frac{d(1,2)}{a}\right] = \frac{1 - x_1x_2 - y_1y_2}{\sqrt{1-x_1^2-y_1^2}\,\sqrt{1-x_2^2-y_2^2}} \quad .$$

Defining

$$x = u\cos\theta$$

$$y = u\sin\theta$$

for both (x_1, y_1) and (x_2, y_2), one has

$$\begin{aligned} x_1x_2 + y_1y_2 &= u_1u_2\cos\theta_1\cos\theta_2 + u_1u_2\sin\theta_1\sin\theta_2 \\ &= u_1u_2\left[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right] \\ &= u_1u_2\cos\left(\theta_1 - \theta_2\right) \quad . \end{aligned}$$

$$\sqrt{1-x_1^2-y_1^2} = \sqrt{1-u_1^2\cos^2\theta_1 - u_1^2\sin^2\theta_1} = \sqrt{1-u_1^2} \quad .$$

So

$$\boxed{\cos\left[\frac{d(1,2)}{a}\right] = \frac{1 - u_1 u_2 \cos(\theta_1 - \theta_2)}{\sqrt{1-u_1^2}\,\sqrt{1-u_2^2}}}$$

(b) As a useful prelude, let us expand the function

$$f(x) = \frac{1}{\sqrt{b-x}}$$

in a power series. Note that

$$f(x) = (b-x)^{-1/2} \qquad f(0) = b^{-1/2}$$

$$f'(x) = \frac{1}{2}(b-x)^{-3/2} \qquad f'(0) = \frac{1}{2}b^{-3/2}$$

$$f''(x) = \frac{3}{4}(b-x)^{-5/2} \qquad f''(0) = \frac{3}{4}b^{-5/2}$$

so

$$f(x) = f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''(x)x^2 + \ldots$$

$$\frac{1}{\sqrt{b-x}} = \frac{1}{\sqrt{b}}\left\{1 + \frac{1}{2}\,\frac{x}{b} + \frac{3}{8}\,\frac{x^2}{b^2} + \ldots\right\}$$

Using

$$u_1 = u \qquad \theta_1 = \theta$$

$$u_2 = u + \mathrm{d}u \qquad \theta_2 = \theta + \mathrm{d}\theta$$

$$d(1,2) \equiv \mathrm{d}s \quad ,$$

One has

$$\begin{aligned}
\cosh\left[\frac{\mathrm{d}s}{a}\right] &= 1 + \frac{\mathrm{d}s^2}{2!a^2} + \ldots \\
&= \frac{1 - u(u+\mathrm{d}u)\cos(\mathrm{d}\theta)}{\sqrt{1-u^2}\,\sqrt{1-u^2-2u\mathrm{d}u-\mathrm{d}u^2}} \quad . \\
&= \frac{1}{1-u^2}\left\{\left[1 - (u^2 + u\mathrm{d}u)\left(1 - \frac{1}{2}\mathrm{d}\theta^2 + \ldots\right)\right]\right. \\
&\quad \left.\times\left[1 + \frac{1}{2}\frac{2u\mathrm{d}u + \mathrm{d}u^2}{1-u^2} + \frac{3}{8}\frac{\left(2u\mathrm{d}u + \mathrm{d}u^2\right)^2}{\left(1-u^2\right)^2} + \ldots\right]\right\}
\end{aligned}$$

$$= \frac{1}{1-u^2}\left\{\left[1 - u^2 - u\mathrm{d}u + \frac{1}{2}u^2\mathrm{d}\theta^2 + \ldots\right]\right.$$

$$\left.\times\left[1 + \frac{u\mathrm{d}u}{1-u^2} + \frac{1}{2}\frac{\mathrm{d}u^2}{1-u^2} + \frac{3}{2}\frac{u^2\mathrm{d}u^2}{(1-u^2)^2} + \ldots\right]\right\}$$

$$= \frac{1}{(1-u^2)^2}\left\{\left[1 - u^2 - u\mathrm{d}u + \frac{1}{2}u^2\mathrm{d}\theta^2 + \ldots\right]\right.$$

$$\left.\times\left[1 - u^2 + u\mathrm{d}u + \frac{1}{2}\mathrm{d}u^2 + \frac{3}{2}\frac{u^2\mathrm{d}u^2}{1-u^2} + \ldots\right]\right\}$$

$$= \frac{1}{(1-u^2)^2}\left\{(1-u^2)^2 + (1-u^2)u\mathrm{d}u + \frac{1}{2}(1-u^2)\mathrm{d}u^2\right.$$

$$\left.+ \frac{3}{2}u^2\mathrm{d}u^2 - u\mathrm{d}u(1-u^2) - u^2\mathrm{d}u^2 + \frac{1}{2}u^2\mathrm{d}\theta^2(1-u^2) + \ldots\right\}$$

$$= \frac{1}{(1-u^2)^2}\left\{(1-u^2)^2 + \frac{1}{2}\mathrm{d}u^2 + \frac{1}{2}u^2(1-u^2)\mathrm{d}\theta^2 + \ldots\right\}$$

$$= 1 + \frac{1}{2}\frac{\mathrm{d}u^2}{(1-u^2)^2} + \frac{1}{2}\frac{u^2\mathrm{d}\theta^2}{(1-u^2)} \quad .$$

At each stage one can drop all terms higher than second order in the infinitesimals $\mathrm{d}u$ and $\mathrm{d}\theta$. So

$$\boxed{\mathrm{d}s^2 = a^2\left\{\frac{\mathrm{d}u^2}{(1-u^2)^2} + \frac{u^2\mathrm{d}\theta^2}{(1-u^2)}\right\} \quad .}$$

(c) The above metric must be compared with the Robertson–Walker form

$$\mathrm{d}s^2 = a^2\left\{\frac{\mathrm{d}r^2}{1+r^2} + r^2\mathrm{d}\theta^2\right\} \quad .$$

Since the coefficients of $\mathrm{d}\theta^2$ must match, one must have

$$r^2 = \frac{u^2}{1-u^2} \quad , \quad \text{or} \quad \boxed{r = \frac{u}{\sqrt{1-u^2}} \quad .}$$

We must now check to see if the first terms match.

$$\begin{aligned} \mathrm{d}r &= \frac{\mathrm{d}u}{\sqrt{1-u^2}} + \frac{1}{2}\,\frac{u}{\left(1-u^2\right)^{3/2}} 2u\mathrm{d}u \\ &= \mathrm{d}u\left\{\frac{1}{\sqrt{1-u^2}} + \frac{u^2}{\left(1-u^2\right)^{3/2}}\right\} \\ &= \mathrm{d}u\left\{\frac{1}{\left(1-u^2\right)^{3/2}}\left[1-u^2+u^2\right]\right\} = \frac{\mathrm{d}u}{\left(1-u^2\right)^{3/2}} \\ 1+r^2 &= 1+\frac{u^2}{1-u^2} = \frac{1}{1-u^2} \end{aligned}$$

So,

$$\frac{\mathrm{d}r^2}{1+r^2} = \frac{\mathrm{d}u^2}{\left(1-u^2\right)^2} \qquad \text{— It agrees!}$$

Note that $r \to \infty$ as $u \to 1$, so the range of u is restricted.