MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 3, 2009
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PROBLEM SET 6 SOLUTIONS

PROBLEM 1: CIRCULAR ORBITS IN A SCHWARSZCHILD METRIC

(a) Along a perfectly circular orbit in the x-y plane, the expression for $d\tau^2$ simplifies greatly. Note that

$$\begin{aligned} r = \text{fixed} \quad &\Longrightarrow \quad dr = 0 \ ; \\ \theta = \pi/2 \quad &\Longrightarrow \quad d\theta = 0, \ \ \sin\theta = 1 \ ; \\ \phi = \omega t \quad &\Longrightarrow \quad d\phi = \omega dt \ . \end{aligned}$$

The expression for $d\tau^2$ then reduces to

$$\begin{aligned} ds^2 = -c^2 d\tau^2 &= -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + r^2\omega^2 dt^2 \\ &= -\left(1 - \frac{2GM}{rc^2} - \frac{r^2\omega^2}{c^2}\right) c^2 dt^2 \ . \end{aligned}$$

Therefore we find

$$\boxed{\frac{d\tau}{dt} = \sqrt{1 - \frac{2GM}{rc^2} - \frac{r^2\omega^2}{c^2}}}$$

as hoped.

(b) The geodesic equation (6.68) was written in the notes as

$$\boxed{\frac{d}{d\tau}\left[g_{\mu\nu}\frac{dx^\nu}{d\tau}\right] = \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau} \ ,} \qquad (6.68)$$

We will define $g_{\mu\nu}$ by

$$ds^2 = -c^2\, d\tau^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \ ,$$

but you should be aware that there are different conventions in use. Some textbooks would use $d\tau^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$, or $c^2 d\tau^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$. The geodesic equation above is valid for any of these definitions of $g_{\mu\nu}$, since one definition

is related to another by multiplying $g_{\mu\nu}$ by a constant factor. The geodesic equation is not changed if $g_{\mu\nu}$ is replaced by *constant* $\times\, g_{\mu\nu}$, since the constant would multiply both sides of the equation and would cancel out.

The nonzero components of $g_{\mu\nu}$ for this case are

$$g_{tt} = -\left(1 - \frac{2GM}{rc^2}\right)c^2 \ , \qquad g_{rr} = \left(1 - \frac{2GM}{rc^2}\right)^{-1} \ ,$$

$$g_{\theta\theta} = r^2 \ , \qquad g_{\phi\phi} = r^2 \ ,$$

where $\sin\theta = 1$ was used to simplify $g_{\phi\phi}$. For $\mu = r$ the left-hand side of the geodesic equation becomes

$$\frac{d}{d\tau}\left[g_{rr}\frac{dr}{d\tau}\right] \ ,$$

which is equal to zero for this problem, since $dr = 0$. The right-hand side of the geodesic equation is expanded by explicitly summing over λ and σ, recognizing that for this metric the only nonzero terms arise when $\lambda = \sigma$. The geodesic equation then becomes

$$0 = \frac{1}{2}\frac{\partial g_{tt}}{\partial r}\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{2}\frac{\partial g_{rr}}{\partial r}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\frac{\partial g_{\theta\theta}}{\partial r}\left(\frac{d\theta}{d\tau}\right)^2 + \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial r}\left(\frac{d\phi}{d\tau}\right)^2 \ .$$

Since $d\theta = dr = 0$ this reduces to

$$\boxed{0 = \frac{1}{2}\frac{\partial g_{tt}}{\partial r}\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial r}\left(\frac{d\phi}{d\tau}\right)^2 \ .}$$

(c) Take the derivatives

$$\frac{\partial}{\partial r} g_{tt} = -\frac{2GM}{r^2}$$

and

$$\frac{\partial}{\partial r} g_{\phi\phi} = 2r \ .$$

Substituting these into the result of part (b) gives

$$r\left(\frac{d\phi}{d\tau}\right)^2 = \frac{GM}{r^2}\left(\frac{dt}{d\tau}\right)^2 \ .$$

Use the chain rule to write

$$\frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau}$$

and divide both sides by $(dt/d\tau)^2$ to find

$$r\left(\frac{d\phi}{dt}\right)^2 = \frac{GM}{r^2} \quad .$$

Remember $d\phi/dt \equiv \omega$ so that

$$\boxed{r\omega^2 = \frac{GM}{r^2} \quad .}$$

(d) From part (a) of the solution we found that

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2GM}{rc^2} - \frac{r^2\omega^2}{c^2}} \,. \tag{1}$$

The quantity inside the square root must be positive and this will give us a constraint on the possible circular orbits. Using our final result from part (c) we have

$$\frac{r^2\omega^2}{c^2} = \frac{GM}{rc^2} \,,$$

so equation (1) becomes

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{3GM}{rc^2}} \,.$$

We must therefore require

$$1 - \frac{3GM}{rc^2} > 0 \quad \Longrightarrow \quad \boxed{r > \frac{3GM}{c^2} = \frac{3}{2}R_S \,,}$$

where we recalled that the Schwarzchild radius is $R_S = 2GM/c^2$. The smallest possible circular orbit in the Schwarzchild geometry has radius $\frac{3}{2}R_S$. At this limiting radius $d\tau/dt = 0$, which indicates that the orbital velocity is equal to the speed of light. Closer orbits would require a speed greater than that of light, which is not possible. Further analysis of orbits in this geometry shows that the smallest *stable* circular orbit occurs for $r = 3R_S$. Circular orbits are possible for $\frac{3}{2}R_S < r < 3R_S$, but they are not stable. A small inward nudge would cause the orbiting object to plunge inward, while a small outward nudge will allow the object to fly outward to infinity.