In this problem set we will consider a universe in which the scale factor is given by
\[ a(t) = bt^{2/3}, \]
where \( b \) is an arbitrary constant of proportionality which should not appear in the answers to any of the questions below. (We will see in Lecture Notes 3 that this is the behavior of a flat universe with a mass density that is dominated by nonrelativistic matter.) We will suppose that a distant galaxy is observed with a redshift \( z \).

As a concrete example we will consider the most distant known object with a well-determined redshift, the galaxy UDFy-38135539, which has a redshift \( z = 8.55 \).

PROBLEM 1: DISTANCE TO THE GALAXY

Let \( t_0 \) denote the present time, and let \( t_e \) denote the time at which the light that we are currently receiving was emitted by the galaxy. In terms of these quantities, find the present value of the physical distance \( \ell_p \) between this distant galaxy and us.

PROBLEM 2: TIME OF EMISSION

Express the redshift \( z \) in terms of \( t_0 \) and \( t_e \). Find the ratio \( t_e/t_0 \) for the \( z = 8.55 \) galaxy.

PROBLEM 3: DISTANCE TO THE GALAXY (9 points)

Let us denote the present value of the physical distance \( d \) between this distant galaxy and us as \( d = \ell_p \).

The redshift \( z \) is defined as the ratio of the present value of the physical distance \( d \) to the distance at which the light was emitted: \( z = d/d_e \), where \( d_e \) is the distance at which the light was emitted. The redshift \( z \) is a measure of the expansion of the universe, and it is given by the formula \( z = \frac{1}{2} \left( \frac{d}{d_e} - 1 \right) \).

In 1986 the highest measured redshift was only 3.78. It was 4.01 in 1988, 4.73 in 1992, 4.89 in 1994, and 4.92 in 1998, 6.28 in 2002, and 6.58 in 2003. In 2006 Iye et al. discovered a galaxy with a redshift of 6.96. In 2006 Richard McMahon compiled the graph on the right, which was published in a News article on p. 128 of the same issue of Nature as the Iye et al. discovery.

The search for high redshift objects continues to be an exciting area of research, as astronomers try to sort out the conditions in the universe when the first galaxies began to form.

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PROBLEM 3: DISTANCE IN TERMS OF REDSHIFT

Express the present value of the physical distance in terms of the present value of the Hubble expansion rate \( H_0 \) and the redshift \( z \).

Taking \( H_0 \approx 72 \text{ km-sec}^{-1}-\text{Mpc}^{-1} \), how far away is the galaxy? Express your answer both in light-years and in Mpc.

PROBLEM 4: SPEED OF RECESSION

Find the present rate at which the physical distance \( \ell_p \) between the distant galaxy and us is changing. Express your answer in terms of the redshift \( z \) and the speed of light \( c \), and evaluate it numerically for the case \( z = 8.55 \). Express your answer as a fraction of the speed of light. [If you get it right, this “fraction” is greater than one! Our expanding universe violates special relativity, but is consistent with general relativity.]

PROBLEM 5: APPARENT ANGULAR SIZES

Now suppose for simplicity that the galaxy is spherical, and that its physical diameter was \( w \) at the time it emitted the light. (The actual galaxy is seen as an unresolved point source, so we don’t know its actual size and shape.) Find the apparent angular size \( \theta \) (measured from one edge to the other) of the galaxy as it would be observed from Earth today. Express your answer in terms of \( w \), \( z \), \( H_0 \), and \( c \). You may assume that \( \theta \ll 1 \). Compare your answer to the apparent angular size of a circle of diameter \( w \) in a static Euclidean space, at a distance equal to the present value of the physical distance to the galaxy, as found in Problem 3. [Hint: draw diagrams which trace the light rays in the comoving coordinate system. If you have it right, you will find that \( \theta \) decreases as you increase the redshift \( z \). This phenomenon makes sense if you think about the distance to the galaxy at the time of emission. If the galaxy is very far away today, then the light that we now see must have left the galaxy very early, when it was rather close to us.]

PROBLEM 6: RECEIVED RADIATION FLUX

At the time of emission, the galaxy had a power output \( P \) (measured, say, in ergs/sec) which was radiated uniformly in all directions. This power was emitted as photons. What is the radiation energy flux \( J \) from this galaxy at the earth today? Energy flux (which might be measured in ergs-cm\(^{-2}\)-sec\(^{-1}\)) is defined as the energy per unit area per unit time striking a surface that is orthogonal to the direction of energy flow. The easiest way to solve this problem is to consider the trajectories of the photons, as viewed in comoving coordinates. You must calculate the number of photons, emitted in the direction of energy flow, that cross a surface of area \( A \) at a distance \( d \) from the source, and then use the fact that the energy of each photon is proportional to its frequency, and is therefore decreased by the redshift. You may find it useful to think of the detector as a small sphere of radius \( r \) centered on the source, as shown in the following diagram:

[Diagram showing the source galaxy, radiation, detector, and trajectories of photons in comoving coordinates.]