MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department
Physics 8.286: The Early Universe October 11, 2011
Prof. Alan Guth

PROBLEM SET 4

DUE DATE: Thursday, October 20, 2011

READING ASSIGNMENT: Steven Weinberg, The First Three Minutes, Chapters 4 and 5. Barbara Ryden, Introduction to Cosmology, Chapters 4 and 5. For these chapters there are no new problems, but the extra credit problems parallel the material discussed in these chapters. In Weinberg's Chapter 4 (and 5) there are a lot of numbers mentioned. You certainly do not need to learn all these numbers, but you should be familiar with the orders of magnitude.

NOTE ABOUT EXTRA CREDIT: This problem set contains 40 points of regular problems and 5 points extra credit, so it is probably worthwhile for me to clarify the operational definition of "extra credit." We keep track of the extra credit grades separately, and at the end of the term, and students whose average has been pushed down by single low grade, will be the ones most likely to be boosted.

The bottom line is that you should feel free to skip the extra credit problems, and you will still get an excellent grade in the course if you do well on the regular problems. However, if you are the kind of student who really wants to get the most out of the course, then I hope that you will find these extra credit problems challenging, interesting, and educational.

WARNING: Problem 6 would certainly be worth 10 points if it were not for extra credit. I am assigning fewer points for extra credit problems, because I would like students to attack these problems mainly for their intellectual interest, and not their grade-boosting potential.

PROBLEM 1: EVOLUTION OF A CLOSED, MATTER-DOMINATED UNIVERSE (10 points)

It was shown in Lecture Notes 4 that the evolution of a closed, matter-dominated universe can be described by introducing the time parameter θ, sometimes called the development angle, with

\[ \frac{d}{dt} = \alpha (\sin \theta - \theta) \]

and

\[ a = \alpha (1 - \cos \theta) \]

where \( \alpha \) is a constant with units of length.

(a) Use these expressions to find \( H \), the Hubble expansion rate, as a function of \( \alpha \) and \( \theta \).

(b) Find \( \rho \), the mass density, as a function of \( \alpha \) and \( \theta \).

(c) Find \( \Omega \equiv \rho/\rho_c \), where \( \rho_c \) is the critical density, as a function of \( \alpha \) and \( \theta \). The relation is given in Lecture Notes 4 as Eq. (4.34), but you should show that you get the same answer by combining your answers from parts (a) and (b).

(d) Although the evolution of a closed, matter-dominated universe seems complicated, it is nonetheless possible to carry out the integration needed to compute the horizon distance. The integral becomes simple if one changes the variable of integration so that one integrates over \( \theta \) instead of integrating over \( t \). Show that the physical horizon distance \( \ell_{\text{p,horizon}} \) for the closed, matter-dominated universe is given by

\[ \ell_{\text{p,horizon}} = \alpha \theta (1 - \cos \theta) \]

Problem 2: EVOLUTION OF AN OPEN, MATTER-DOMINATED UNIVERSE (10 points)

The following problem originated on Quiz 2 of 1992, where it counted 30 points. The equations describing the evolution of an open, matter-dominated universe were given in Lecture Notes 4 as

\[ ct = \alpha (\sinh \theta - \theta) \]

and

\[ a = \alpha (\cosh \theta - 1) \]

where \( \alpha \) is a constant with units of length.

(a) Use these expressions to find \( H \), the Hubble expansion rate, as a function of \( \alpha \) and \( \theta \). The relation is given in Lecture Notes 4 as Eq. (4.34), but you should show that you get the same answer by combining your answers from parts (a) and (b).

(b) Find \( \rho \), the mass density, as a function of \( \alpha \) and \( \theta \). You can use the expression above to calculate \( \rho/\rho_c \).

(c) Find \( \Omega \equiv \rho/\rho_c \), where \( \rho_c \) is the critical density, as a function of \( \alpha \) and \( \theta \). The relation is given in Lecture Notes 4 as Eq. (4.34), but you should show that you get the same answer by combining your answers from parts (a) and (b).

Note: All these numbers, but you should be familiar with the orders of magnitude.

Interestingly, there are a lot of numbers mentioned. You certainly do not need to learn all these numbers, but you should be familiar with the orders of magnitude.

Reading Assignment: Steven Weinberg, The First Three Minutes, Chapters 4 and 5. Barbara Ryden, Introduction to Cosmology, Chapters 4 and 5. For these chapters there are no new problems, but the extra credit problems parallel the material discussed in these chapters. In Weinberg's Chapter 4 (and 5) there are a lot of numbers mentioned. You certainly do not need to learn all these numbers, but you should be familiar with the orders of magnitude.
Consider a circle described by the following metric:

\[ ds^2 = c^2 dt^2 - \left( \frac{\rho}{\alpha} \right)^2 \left( d\phi^2 + \sin^2 \phi d\theta^2 \right) \]

(a) Find the Hubble expansion rate \( H \) as a function of \( \theta \) and \( \alpha \), assuming that Dr. Niwde is able to observe all objects within his horizon, what is the most blueshifted (i.e., most negative) value of \( z \)?

(b) Find the radius \( r \) of this circle. Note that \( r \) is the length of a line which runs from the origin to the circle (let \( \phi = \pi/2 \) and \( \pi/2 \leq \theta \leq \pi \)).

(c) Find the mass density \( \Omega \) as a function of \( \theta \) and \( \alpha \). Niwde is able to see? What is the lookback time to an object with this blueshift? (By lookback time, one means the difference between the time of observation and the time at which the light was emitted.)

(d) Find the expression for \( 1/t \) as a function of \( \alpha \) and \( \theta \). Estimate the value of \( \alpha \) and \( \theta \). Even though these equations describe an open universe, one still finds that \( \Omega \) behaves as a power of \( t \).

(e) For very small values of \( \theta \), it is possible to use the first nonzero term of a power-series expansion to express the metric of the previous problem, the answer to this part appears in Lecture Notes 4. However, you should show that you get the same answer by combining your answers to parts (a) and (b) of this question.
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Dr. , calculate the length of such a segment, and integrate. Consider the case of open and closed universes separately, and take \( k = \pm 1 \). (If you don't remember why we can take \( k = \pm 1 \), see the section called "Units" in Lecture Notes 3).

You will want the following integrals:

\[
\int dr \sqrt{1 - r^2} = \sin^{-1} r
\]

and

\[
\int dr \sqrt{1 + r^2} = \sinh^{-1} r.
\]

(c) Express the circumference \( S \) in terms of the radius \( \rho \). This result is independent of the coordinate system which was used for the calculation, since \( S \) and \( \rho \) are both measurable quantities. Since the space described by this metric is homogeneous and isotropic, the answer does not depend on where the circle is located or on how it is oriented. For the two cases of open and closed universes, state whether \( S \) is larger or smaller than the value it would have for a Euclidean circle of radius \( \rho \).

PROBLEM 5: VOLUME OF A CLOSED UNIVERSE

(5 points)

Calculate the total volume of a closed universe, as described by the metric of Eq. (5.14) of Lecture Notes 5:

\[
ds^2 = R^2 \left[ d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2) \right].
\]

Break the volume up into spherical shells of infinitesimal thickness, extending from \( \psi \) to \( \psi + d\psi \).

By comparing Eq. (5.14) with (5.8), the metric for the surface of a sphere of radius \( R \),

\[
\phi = 0 \to \phi \to \phi = \pi/2
\]

one can see that the volume is the same as that of a spherical surface of radius \( R \). Thus the area of the spherical surface can see that this is true for any value of \( \theta \) and the metric for the surface of a sphere is

\[
\sigma d^2 = \frac{R^2 d\rho \sin^2 \psi (d\phi d\theta + \sin^2 \theta d\phi^2)}{d\psi}
\]

and

\[
\sigma d^2 = \frac{R^2 d\rho \sin^2 \psi (d\phi d\theta + \sin^2 \theta d\phi^2)}{d\psi}
\]

You will want the following integrals:

\[
\left[ (\phi \rho \cos \psi \sin \phi + \phi \rho \sin \phi \sin \psi) \phi \cos \phi + \phi \rho \cos \psi \sin \phi \right] \rho = \phi
\]

PROBLEM 6: ISOTROPY ABOUT TWO POINTS IN EUCLIDEAN SPACES

(This problem is not required, but can be done for 5 points extra credit.)

In Steven Weinberg’s The First Three Minutes, on page 24, he gives an argument to show that if a space is isotropic about two distinct points, then it is necessarily Euclidean, and the metric is of the form

\[
ds^2 = \sum d\xi^2,
\]

where \( \xi \) is a coordinate system. Since the space described by this metric is homogeneous and isotropic, the answer does not depend on the coordinate system, since \( \xi \) and \( \sum d\xi^2 \) are both measurable quantities. Since the space described by this metric is homogeneous and isotropic, the answer does not depend on where the circle is located or on how it is oriented. For the two cases of open and closed universes, state whether \( S \) is larger or smaller than the value it would have for a Euclidean circle of radius \( \rho \).

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(c) Express the circumference \( S \) in terms of the radius \( \rho \). This result is independent of the coordinate system which was used for the calculation, since \( S \) and \( \rho \) are both measurable quantities. Since the space described by this metric is homogeneous and isotropic, the answer does not depend on where the circle is located or on how it is oriented. For the two cases of open and closed universes, state whether \( S \) is larger or smaller than the value it would have for a Euclidean circle of radius \( \rho \).