Problem 1: The Magnetic Monopole Problem

Problem Set 10 (the last)
\[ \int dx \sqrt{1 - x^2} = \sin^{-1} x. \]

\[ \int dx \sqrt{x^2 - 1} = \cosh^{-1} x. \]

Show that for large times one has
\[ a(t) \propto e^{\chi t} \]
for all choices of \( k \).

**PROBLEM 3: THE HORIZON DISTANCE FOR THE PRESENT UNIVERSE**
(25 points)

We have not discussed horizon distances since the beginning of Lecture Notes 4, when we found that
\[ \ell_p,_{\text{horizon}}(t) = a(t) \int_0^t c \left( \frac{d}{dt'} a(t') \right) \, dt'. \]

(P3.1)

This formula was derived before we discussed curved spacetimes, but the formula is valid for any Robertson-Walker universe, whether it is open, closed, or flat.

(a) Show that the formula above is valid for closed universes. Hint: write the closed universe metric as it was written in Eq. (7.27):
\[ ds^2 = -c^2 dt^2 + \tilde{a}^2(t) \left\{ d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2) \right\}, \]
where \( \tilde{a}(t) \equiv a(t) \sqrt{k} \) and \( \psi \) is related to the usual Robertson-Walker coordinate \( r \) by
\[ \sin \psi \equiv \sqrt{k} r. \]
Use the fact that the physical speed of light is \( c \), or equivalently the fact that \( ds^2 = 0 \) for any segment of the light ray’s trajectory.

(b) The evaluation of the formula depends of course on the form of the function \( a(t) \), which is governed by the Friedmann equations. For the Planck 2013 best fit to the parameters,
\[ H_0 = 67.3 \, \text{km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}, \]
\[ \Omega_m = 0.315, \]
\[ \Omega_{\text{vac}} = 0.685, \]
\[ \Omega_r = 0.049, \]
\[ T_{\gamma, 0} = 2.725 \, \text{K}. \]

Find the current horizon distance, expressed both in light-years and in Mpc. Hint: find an integral expression for the horizon distance, similar to Eq. (7.23a) for the age of the universe. Then do the integral numerically.

Note that the model for which you are calculating does not explicitly include inflation. If it did, the horizon distance would turn out to be vastly larger. By ignoring the inflationary era in calculating the integral of Eq. (P3.1), we are finding an effective horizon distance, defined as the present distance of the most distant objects that we can in principle observe by using only photons that have left their sources after the end of inflation. Photons that left their sources earlier than the end of inflation have undergone incredibly large redshifts, so it is reasonable to consider them to be completely unobservable in practice.

**PROBLEM 4: THE INFLATIONARY SOLUTION TO THE HORIZON/HOMOGENEITY PROBLEM**
(This problem is not required, but can be done for 20 points extra credit.)

In this problem we will calculate how much inflation is needed to explain the observed homogeneity of the universe. To make the calculation well-defined, we will adopt a simple description of how inflation works. Although we are trying to explain the homogeneity of the universe, to make the problem tractable we will need to assume that from the onset of inflation, at a time we call \( t_i \), the universe was already very nearly homogeneous, so that we can approximate its evolution using simple equations. We will in fact assume that from time \( t_i \) onward the evolution equations can be approximated by those of a homogeneous, isotropic, and flat universe. We will describe inflation as being driven by the evolution of a universe, \( \Phi(t) \), which we will write as \( \Phi(t) = 1 + \sqrt{E_f} \). We will assume that this potential is driven by a false vacuum with a fixed mass density \( \rho_f \), which we will describe by relating it to a parameter \( E_f \) by
\[ \rho_f \equiv E_f^4 \bar{h}^3 c^5, \]
where \( E_f \) has the units of energy. To discuss inflation at the energy scale of grand unified theories, we will write
\[ E_f \equiv E_{16} \times 10^{16} \, \text{GeV}, \]
where \( E_{16} \) is a dimensionless number with a value of order 1.

We will use the function \( a(t) \) to describe inflation, and assume that the Hubble parameter during inflation is then described by the Friedmann equation,
\[ H_i^2 = \frac{8\pi G}{3} \rho_f. \]

(P4.3)

While we are assuming enough homogeneity to proceed with the calculation, we are assuming that the high precision homogeneity of the observed universe will still be destroyed by redshifts, so it is still true that the end of inflation has produced incredibly large redshifts, so it is better than the end of inflation have undergone incredibly large redshifts, so it is better than the end of inflation to describe inflation, and the end of inflation that after inflation, the universe has already become homogenous. Hence, at the end of inflation, the universe was not only homogeneous, but also isotropic. However, we will describe inflation as being driven by the evolution of a false vacuum with a fixed mass density, which we will describe by relating it to a parameter \( E_f \) by
\[ \rho_f \equiv E_f^4 \bar{h}^3 c^5, \]
where \( E_f \) has the units of energy. By ignoring the inflationary era in calculating the integral of Eq. (P3.1), we are finding an effective horizon distance, defined as the present distance of the most distant objects that we can in principle observe by using only photons that have left their sources after the end of inflation. Photons that left their sources earlier than the end of inflation have undergone incredibly large redshifts, so it is reasonable to consider them to be completely unobservable in practice.

**USEFUL VERSES**

**PROBLEM 3: THE HORIZON DISTANCE FOR THE PRESENT UNIVERSE**

For all choices of \( k \),
\[ x_1 \sin x + (k \pi)^{1/2} \int \frac{1 - x \sin x}{x} \, dx = x_1 \cos x \]
\[ x_1 \cos x + (k \pi)^{1/2} \int \frac{1 - x \sin x}{x} \, dx = x_1 \sin x. \]
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(like the 1 part in 10)

The uniformity of the CMB was not part of the initial conditions, but must be explained in terms of the evolution of the universe. The homogeneity is created first on short distance scales, and the length scale of homogeneity, denoted by $r_h(t)$, increases with time. At the onset of inflation we assume that normal thermal equilibrium processes have already smoothed the universe on scales smaller than the Hubble length, so we write

$$r_h(t) \approx \beta c H^{-1},$$

(P4.4)

where $\beta$ is a dimensionless constant with $\beta < 1\sim 1$.

We assume that inflation continues long enough so that the universe expands by a factor $Z$, where we will be trying to calculate the minimum value of $Z$. We will assume for simplicity that inflation ends suddenly, at time $t_e$. Reheating is then assumed to occur instantly, with the mass density $\rho_f$ of the false vacuum being converted to thermal equilibrium radiation, described as in Lecture Notes 6 by

$$\rho \approx \frac{g_{RH}}{30^2 \pi^2 (k T_{RH})^4 \bar{h}^3 c^5},$$

(P4.5)

where $g_{RH}$ reflects the total number of particles that are effectively massless at the energy scale of reheating. For a grand unified theory one might take $g_{RH} \approx 300$, but fortunately the value of this highly uncertain number will not have much effect on the answer. The length scale of homogeneity is stretched by inflation to

$$r_h(t) = Z r_h(t_i),$$

(P4.6)

and we will assume that $r_h(t_i)$ continues to evolve only by being stretched with the scale factor. The length scale today is then given by

$$r_h(0) = a(0)/a(t_e) Z r_h(t_i),$$

(P4.7)

To evaluate $a(0)/a(t_e)$, you can use the conservation of entropy,

$$s^3 = \text{constant},$$

where $s$ is the entropy density, which is very accurate from the end of inflation to the present. For the current entropy density, include photons and neutrinos, taking into account the temperature difference $T_\nu/T_\gamma = (4/11)^{1/3}$.

Problem: Find the minimum value of $Z$ such that

$$r_{horizon}(0) > \ell_p,$$

(P4.8)

where $r_{horizon}(0)$ is the horizon scale calculated in Problem 3. If you did not do Problem 3, you could use instead $3c t_0$, the answer for a flat matter-dominated universe, with $t_0 \approx 13.8$ billion years. Assume the parameters of the Planck 2013 best fit described in Problem 3, and write your answer for $Z_{min}$ as a function of $E_{16}$, $g_{RH}$, and $\beta$. Since inflation is an exponential process, it is useful to also express the numerical answer in terms of $N_{min} \equiv \ln Z_{min}$, which is the minimum number of e-foldings of inflation. (An "e-folding" refers to a period of one Hubble time, $\Delta t = H^{-1}$, so the scale factor expands by $e$ in a period of $H^{-1}$.)