MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe
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December 5, 2013

PROBLEM SET 10 (The Last!)

DUE DATE: Tuesday, December 10, 2013, at 5:00 pm.

READING ASSIGNMENT: None.

PROBLEM 1: THE MAGNETIC MONOPOLE PROBLEM *(20 points)*

In Lecture Notes 9, we learned that Grand Unified Theories (GUTs) imply the existence of magnetic monopoles, which form as "topological defects" (topologically stable knots) in the configuration of the Higgs fields that are responsible for breaking the grand unified symmetry to the SU(3)×SU(2)×U(1) symmetry of the standard model of particle physics. At very high temperatures the Higgs fields oscillate wildly, so the fields average to zero. As the temperature T falls, however, the system undergoes a phase transition. The phase transition occurs at a temperature T_c, called the critical temperature, where $kT_c \approx 10^{16}$ GeV. At this phase transition the Higgs fields acquire nonzero expectation values, and the grand unified symmetry is thereby spontaneously broken. The monopoles are twists in the Higgs field expectation values, so the monopoles form at the phase transition. Each monopole is expected to have a mass $M_M c^2 \approx 10^{18}$ GeV, where the subscript "M" stands for "monopole." According to an estimate first proposed by T.W.B. Kibble, the number density n_M of monopoles formed at the phase transition is of order

$$n_M \sim 1/\xi^3 \;, \qquad \text{(P1.1)}$$

where ξ is the correlation length of the field, defined roughly as the maximum distance over which the field at one point in space is correlated with the field at another point in space. The correlation length is certainly no larger than the physical horizon distance at the time of the phase transition, and it is believed to typically be comparable to this upper limit. Note that an upper limit on ξ is a lower limit on n_M — there must be at least of order one monopole per horizon-sized volume.

Assume that the particles of the grand unified theory form a thermal gas of blackbody radiation, as described by Eq. (6.48) of Lecture Notes 6,

$$u = g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} \;,$$

with $g_{\text{GUT}} \sim 200$. Further assume that the universe is flat and radiation-dominated from its beginning to the time of the GUT phase transition, t_{GUT}.

For each of the following questions, first write the answer in terms of physical constants and the parameters T_c, M_M, and g_{GUT}, and then evaluate the answers numerically.

(a) *(5 points)* Under the assumptions described above, at what time t_{GUT} does the phase transition occur? Express your answer first in terms of symbols, and then evaluate it in seconds.

(b) *(5 points)* Using Eq. (P1.1) and setting ξ equal to the horizon distance, estimate the number density n_M of magnetic monopoles just after the phase transition.

(c) *(5 points)* Calculate the ratio n_M/n_γ of the number of monopoles to the number of photons immediately after the phase transition. Refer to Lecture Notes 6 to remind yourself about the number density of photons.

(d) *(5 points)* For topological reasons monopoles cannot disappear, but they form with an equal number of monopoles and antimonopoles, where the antimonopoles correspond to twists in the Higgs field in the opposite sense. Monopoles and antimonopoles can annihilate each other, but estimates of the rate for this process show that it is negligible. Thus, in the context of the conventional (non-inflationary) hot big bang model, the ratio of monopoles to photons would be about the same today as it was just after the phase transition. Use this assumption to estimate the contribution that these monopoles would make to the value of Ω today.

PROBLEM 2: EXPONENTIAL EXPANSION OF THE INFLATIONARY UNIVERSE *(15 points)*

Recall that the evolution of a Robertson-Walker universe is described by the equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} .$$

Suppose that the mass density ρ is given by the constant mass density ρ_f of the false vacuum. For the case $k = 0$, the growing solution is given simply by

$$a(t) = \text{const } e^{\chi t},$$

where

$$\chi = \sqrt{\frac{8\pi}{3} G\rho_f}$$

and *const* is an arbitrary constant. Find the growing solution to this equation for an arbitrary value of k. Be sure to consider both possibilities for the sign of k. You may find the following integrals useful:

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \ .$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x \ .$$

Show that for large times one has

$$a(t) \propto e^{\chi t}$$

for all choices of k.

PROBLEM 3: THE HORIZON DISTANCE FOR THE PRESENT UNIVERSE *(25 points)*

We have not discussed horizon distances since the beginning of Lecture Notes 4, when we found that

$$\ell_{p,\mathrm{horizon}}(t) = a(t) \int_0^t \frac{c}{a(t')} dt' \ . \tag{P3.1}$$

This formula was derived before we discussed curved spacetimes, but the formula is valid for any Robertson-Walker universe, whether it is open, closed, or flat.

(a) Show that the formula above is valid for closed universes. Hint: write the closed universe metric as it was written in Eq. (7.27):

$$ds^2 = -c^2\,dt^2 + \tilde{a}^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ ,$$

where

$$\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{k}}$$

and ψ is related to the usual Robertson-Walker coordinate r by

$$\sin\psi \equiv \sqrt{k}\, r \ .$$

Use the fact that the physical speed of light is c, or equivalently the fact that $ds^2 = 0$ for any segment of the light ray's trajectory.

(b) The evaluation of the formula depends of course on the form of the function $a(t)$, which is governed by the Friedmann equations. For the Planck 2013 best fit to the parameters,

$$\begin{aligned} H_0 &= 67.3\ \mathrm{km}\cdot\mathrm{s}^{-1}\cdot\mathrm{Mpc}^{-1} \\ \Omega_{m,0} &= 0.315 \\ \Omega_{\mathrm{vac},0} &= 0.685 \\ \Omega_{r,0} &= 9.2\times10^{-5} \quad (T_{\gamma,0} = 2.725\,\mathrm{K}) \ , \end{aligned}$$

find the current horizon distance, expressed both in light-years and in Mpc. Hint: find an integral expression for the horizon distance, similar to Eq. (7.23a) for the age of the universe. Then do the integral numerically.

Note that the model for which you are calculating does not explicitly include inflation. If it did, the horizon distance would turn out to be vastly larger. By ignoring the inflationary era in calculating the integral of Eq. (P3.1), we are finding an effective horizon distance, defined as the present distance of the most distant objects that we can in principle observe by using only photons that have left their sources after the end of inflation. Photons that left their sources earlier than the end of inflation have undergone incredibly large redshifts, so it is reasonable to consider them to be completely unobservable in practice.

PROBLEM 4: THE INFLATIONARY SOLUTION TO THE HORIZON/HOMOGENEITY PROBLEM

(This problem is not required, but can be done for 20 points extra credit.)

In this problem we will calculate how much inflation is needed to explain the observed homogeneity of the universe. To make the calculation well-defined, we will adopt a simple description of how inflation works. Although we are trying to explain the homogeneity of the universe, to make the problem tractable we will need to assume that from the onset of inflation, at a time we call t_i, the universe was already very nearly homogeneous, so that we can approximate its evolution using simple equations. We will in fact assume that from time t_i onward the evolution equations can be approximated by those of a homogeneous, isotropic, and flat universe. We will assume that inflation is driven by a false vacuum with a fixed mass density ρ_f, which we will describe by relating it to a parameter E_f by

$$\rho_\mathrm{f} \equiv \frac{E_\mathrm{f}^4}{\hbar^3 c^5} \ , \tag{P4.1}$$

where E_f has the units of energy. To discuss inflation at the energy scale of grand unified theories, we will write E_f as

$$E_\mathrm{f} \equiv E_{16} \times 10^{16}\ \mathrm{GeV} \ , \tag{P4.2}$$

where E_{16} is a dimensionless number that will we will assume is of order 1. The Hubble parameter during inflation is then dictated by the Friedmann equation,

$$H_i^2 = \frac{8\pi}{3} G\rho_\mathrm{f} \ . \tag{P4.3}$$

While we are assuming enough homogenity to proceed with the calculation, we still want to assume that the high precision homogeneity of the observed universe (like the 1 part in 10^5 uniformity of the CMB) was not part of the initial conditions, but must be explained in terms of the evolution of the universe. The homogeneity is created first on short distance scales, and the length scale of homogeneity, denoted by $r_h(t)$, increases with time. At the onset of inflation we assume that normal thermal equilibrium processes have already smoothed the universe on scales smaller than the Hubble length, so we write

$$r_h(t_i) \approx \beta c H_i^{-1} \ , \tag{P4.4}$$

where β is a dimensionless constant with $\beta \lesssim 1$.

We assume that inflation continues long enough so that the universe expands by a factor Z, where we will be trying to calculate the minimum value of Z. We will assume for simplicity that inflation ends suddenly, at time t_e. Reheating is then assumed to occur instantly, with the mass density $\rho_{\rm f}$ of the false vacuum being converted to thermal equilibrium radiation, described as in Lecture Notes 6 by

$$\rho_{\rm RH} = g_{\rm RH} \frac{\pi^2}{30} \frac{(kT_{\rm RH})^4}{\hbar^3 c^5} \ , \tag{P4.5}$$

where $g_{\rm RH}$ reflects the total number of particles that are effectively massless at the energy scale of reheating. For a grand unified theory one might take $g_{\rm RH} \approx 300$, but fortunately the value of this highly uncertain number will not have much effect on the answer. The length scale of homogeneity is stretched by inflation to

$$r_h(t_e) = Z r_h(t_i) \ , \tag{P4.6}$$

and we will assume that $r_h(t)$ continues to evolve only by being stretched with the scale factor. The length scale today is then given by

$$r_h(t_0) = \frac{a(t_0)}{a(t_e)} Z r_h(t_i) \ . \tag{P4.7}$$

To evaluate $a(t_0)/a(t_e)$, you can use the conservation of entropy, $a^3 s = constant$, where s is the entropy density, which is very accurate from the end of inflation to the present. For the current entropy density, include photons and neutrinos, taking into account the temperature difference $T_\nu / T_\gamma = (4/11)^{1/3}$.

Problem: Find the minimum value of Z such that

$$r_h(t_0) > \ell_{p,\rm horizon}(t_0) \ , \tag{P4.8}$$

using the value of $\ell_{p,\rm horizon}(t_0)$ calculated in Problem 3. (If you did not do Problem 3, you could use instead $3ct_0$, the answer for a flat matter-dominated universe, with $t_0 \approx 13.8$ billion years.) Assume the parameters of the Planck 2013 best fit described in Problem 3, and write your answer for $Z_{\rm min}$ as a function of E_{16}, $g_{\rm RH}$, and β. Since inflation is an exponential process, it is useful to also express the numerical answer in terms of $N_{\rm min} \equiv \ln Z_{\rm min}$, which is the minimum number of e-foldings of inflation. (An "e-folding" refers to a period of one Hubble time, $\Delta t = H^{-1}$, so the scale factor expands by $e^{H\Delta t} = e^1 = e$.)

Total points for Problem Set 10: 65, plus an optional 20 points of extra credit.