MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 5, 2013
Prof. Alan Guth

PROBLEM SET 4

DUE DATE: Friday, October 11, 2013

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes*, Chapter 4; Barbara Ryden, *Introduction to Cosmology*, Chapters 4 and 5 and Sec. 6.1. In Weinberg's Chapter 4 (and, later, Chapter 5) there are a lot of numbers mentioned. You certainly do not need to learn all these numbers, but you should be familiar with the orders of magnitude. In Ryden's Chapters 4 and 5 (and, later, Chapter 6), the material parallels what we either have done or will be doing in lecture. For these chapters you should consider Ryden's book as an aid to understanding the lecture material, and not as a source of new material. On the upcoming quizzes, there will be no questions based specifically on the material in these chapters.

PROBLEM 1: EVOLUTION OF A CLOSED, MATTER-DOMINATED UNIVERSE *(25 points)*

It was shown in Lecture Notes 4 that the evolution of a closed, matter-dominated universe can be described by introducing the time parameter θ, sometimes called the development angle, with

$$ct = \alpha(\theta - \sin\theta) ,$$

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) ,$$

where α is a constant with the units of length.

(a) *(6 points)* Use these expressions to find H, the Hubble expansion rate, as a function of α and θ. *(Hint: You can use the first of the equations above to calculate $d\theta/dt$.)*

(b) *(6 points)* Find ρ, the mass density, as a function of α and θ.

(c) *(6 points)* Find Ω, where $\Omega \equiv \rho/\rho_c$, as a function of α and θ. The relation is given in Lecture Notes 4 as Eq. (4.35), but you should show that you get the same answer by combining your answers from parts (a) and (b) of this question.

(d) *(7 points)* Although the evolution of a closed, matter-dominated universe seems complicated, it is nonetheless possible to carry out the integration needed to compute the horizon distance. The integral becomes simple if one changes the variable of integration so that one integrates over θ instead of integrating over t. Show that the physical horizon distance $\ell_{p,\text{horizon}}$ for the closed, matter-dominated universe is given by

$$\ell_{p,\text{horizon}} = \alpha\theta(1 - \cos\theta) .$$

PROBLEM 2: EVOLUTION OF AN OPEN, MATTER-DOMINATED UNIVERSE *(35 points)*

The following problem originated on Quiz 2 of 1992 (ancient history!), where it counted 30 points.

The equations describing the evolution of an open, matter-dominated universe were given in Lecture Notes 4 as

$$ct = \alpha\,(\sinh\theta - \theta)$$

and

$$\frac{a}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1) ,$$

where α is a constant with units of length. The following mathematical identities, which you should know, may also prove useful on parts (e) and (f):

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2} \quad , \quad \cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$$

$$e^\theta = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots .$$

a) *(5 points)* Find the Hubble expansion rate H as a function of α and θ.

b) *(5 points)* Find the mass density ρ as a function of α and θ.

c) *(5 points)* Find the mass density parameter Ω as a function of α and θ. As with part (c) of the previous problem, the answer to this part appears in Lecture Notes 4. However, you should show that you get the same answer by combining your answers to parts (a) and (b) of this question.

d) *(6 points)* Find the physical value of the horizon distance, $\ell_{p,\text{horizon}}$, as a function of α and θ.

e) *(7 points)* For very small values of t, it is possible to use the first nonzero term of a power-series expansion to express θ as a function of t, and then a as a function of t. Give the expression for $a(t)$ in this approximation. The approximation will be valid for $t \ll t^*$. Estimate the value of t^*.

f) *(7 points)* Even though these equations describe an open universe, one still finds that Ω approaches one for very early times. For $t \ll t^*$ (where t^* is defined in part (e)), the quantity $1 - \Omega$ behaves as a power of t. Find the expression for $1 - \Omega$ in this approximation.

PROBLEM 3: THE CRUNCH OF A CLOSED, MATTER-DOMINATED UNIVERSE *(25 points)*

This is Problem 6.5 from Barbara Ryden's ***Introduction to Cosmology****, with some paraphrasing to make it consistent with the language used in lecture.*

Consider a closed universe containing only nonrelativistic matter. This is the closed universe discussed in Lecture Notes 4, and it is also the "Big Crunch" model discussed in Ryden's section 6.1. At some time during the contracting phase (i.e., when $\theta > \pi$), an astronomer named Elbbuh Niwde discovers that nearby galaxies have blueshifts ($-1 \leq z < 0$) proportional to their distance. He then measures the present values of the Hubble expansion rate, H_0, and the mass density parameter, Ω_0. He finds, of course, that $H_0 < 0$ (because he is in the contracting phase) and $\Omega_0 > 1$ (because the universe is closed). In terms of H_0 and Ω_0, how long a time will elapse between Dr. Niwde's observation at $t = t_0$ and the final Big Crunch at $t = t_{\text{Crunch}} = 2\pi\alpha/c$? Assuming that Dr. Niwde is able to observe all objects within his horizon, what is the most blueshifted (i.e., most negative) value of z that Dr. Niwde is able to see? What is the lookback time to an object with this blueshift? (By lookback time, one means the difference between the time of observation t_0 and the time at which the light was emitted.)

PROBLEM 4: THE AGE OF A MATTER-DOMINATED UNIVERSE AS $\Omega \to 1$ *(15 points)*

The age t of a matter-dominated universe, for any value of Ω, was given in Lecture Notes 4 as

$$|H|\,t = \begin{cases} \dfrac{\Omega}{2(1-\Omega)^{3/2}}\left[\dfrac{2\sqrt{1-\Omega}}{\Omega} - \text{arcsinh}\left(\dfrac{2\sqrt{1-\Omega}}{\Omega}\right)\right] & \text{if } \Omega < 1 \\ 2/3 & \text{if } \Omega = 1 \\ \dfrac{\Omega}{2(\Omega-1)^{3/2}}\left[\arcsin\left(\pm\dfrac{2\sqrt{\Omega-1}}{\Omega}\right) \mp \dfrac{2\sqrt{\Omega-1}}{\Omega}\right] & \text{if } \Omega > 1 \end{cases} \qquad (4.46)$$

It was claimed that this formula is continuous at $\Omega = 1$. In this problem you are asked to show half of this statement. Specifically, you should show that as Ω approaches 1 from below, the expression for $|H|t$ approaches $2/3$. In doing this, you may find it useful to use the Taylor expansion for $\text{arcsinh}(x)$ about $x = 0$:

$$\text{arcsinh}(x) = x - \frac{(1)^2}{3!}x^3 + \frac{(3\cdot 1)^2}{5!}x^5 - \frac{(5\cdot 3\cdot 1)^2}{7!}x^7 + \ldots\ .$$

The proof of continuity as $\Omega \to 0$ from above is of course very similar, and you are not asked to show it.

PROBLEM 5: ISOTROPY ABOUT TWO POINTS IN EUCLIDEAN SPACES

(This problem is not required, but can be done for 15 points extra credit.)

In Steven Weinberg's *The First Three Minutes,* in Chapter 2 on page 24, he gives an argument to show that if a space is isotropic about two distinct points, then it is necessarily homogeneous. He is assuming Euclidean geometry, although he is not explicit about this point. (The statement is simply not true if one allows non-Euclidean spaces.) The statement is true for Euclidean spaces, but Weinberg's argument is not adequate. He constructs two circles, and then describes an argument based on the properties of the point C at which they intersect. The problem, however, is that two circles need not intersect. Thus Weinberg's proof is valid for some cases, but cannot be applied to all cases. For 15 points of extra credit, devise a proof that holds in all cases. We have not established axioms for Euclidean geometry, but you may use in your proof any well-known fact about Euclidean geometry.

Total points for Problem Set 4: 100, plus 15 points of extra credit.