MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 19, 2013
Prof. Alan Guth

PROBLEM SET 5

DUE DATE: Friday, October 25, 2013

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes,* Chapters 5 and 6, and also Barbara Ryden, *Introduction to Cosmology,* Chapter 10. We are skipping Chapters 7-9 of Ryden for now, but we will come back to them. Chapter 10, about *Nucleosynthesis and the Early Universe,* makes good parallel reading to Weinberg's book, and really has no dependence on the chapters that we are skipping.

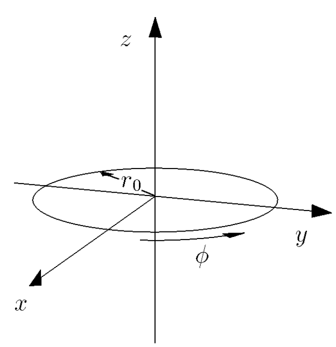
PROBLEM 1: A CIRCLE IN A NON-EUCLIDEAN GEOMETRY *(15 points)*

Consider a three-dimensional space described by the following metric:

$$ds^2 = R^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right\} .$$

Here R and k are constants, where k will always have one of the values 1, -1, or 0. θ and ϕ are angular coordinates with the usual properties: $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$, where $\phi = 2\pi$ and $\phi = 0$ are identified. r is a radial coordinate, which runs from 0 to 1 if $k = 1$, and otherwise from 0 to ∞. (This is the Robertson-Walker metric of Eq. (5.27) of Lecture Notes 5, evaluated at some particular time t, with $R \equiv a(t)$. You should be able to work this problem, however, whether or not you have gotten that far. The problem requires only that you understand what a metric means.) Consider a circle described by the equations

$$z = 0$$
$$x^2 + y^2 = r_0^2 \ ,$$



or equivalently by the angular coordinates

$$r = r_0$$
$$\theta = \pi/2 \ .$$

(a) *(5 points)* Find the circumference S of this circle. Hint: break the circle into infinitesimal segments of angular size $d\phi$, calculate the arc length of such a segment, and integrate.

(b) *(5 points)* Find the radius ρ of this circle. Note that ρ is the length of a line which runs from the origin to the circle ($r = r_0$), along a trajectory of $\theta = \pi/2$ and $\phi =$ constant. Hint: Break the line into infinitesimal segments of coordinate length dr, calculate the length of such a segment, and integrate. Consider the case of open and closed universes separately, and take $k = \pm 1$. (If you don't remember why we can take $k = \pm 1$, see the section called "Units" in Lecture Notes 3,). You will want the following integrals:

$$\int \frac{dr}{\sqrt{1-r^2}} = \sin^{-1} r$$

and

$$\int \frac{dr}{\sqrt{1+r^2}} = \sinh^{-1} r \ .$$

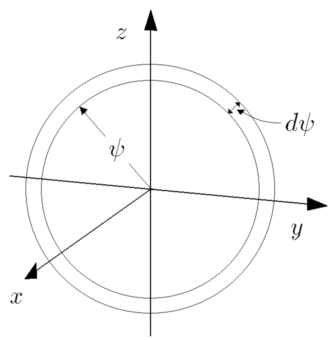
(c) *(5 points)* Express the circumference S in terms of the radius ρ. This result is independent of the coordinate system which was used for the calculation, since S and ρ are both measurable quantities. Since the space described by this metric is homogeneous and isotropic, the answer does not depend on where the circle is located or on how it is oriented. For the two cases of open and closed universes, state whether S is larger or smaller than the value it would have for a Euclidean circle of radius ρ.

PROBLEM 2: VOLUME OF A CLOSED UNIVERSE *(15 points)*

Calculate the total volume of a closed universe, as described by the metric of Eq. (5.14) of Lecture Notes 5:

$$ds^2 = R^2 \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] \ .$$

Break the volume up into spherical shells of infinitesimal thickness, extending from ψ to $\psi + d\psi$:



By comparing Eq. (5.14) with Eq. (5.8),

$$\mathrm{d}s^2 = R^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) ,$$

the metric for the surface of a sphere, one can see that as long as ψ is held fixed, the metric for varying θ and ϕ is the same as that for a spherical surface of radius $R\sin\psi$. Thus the area of the spherical surface is $4\pi R^2\sin^2\psi$. To find the volume, multiply this area by the thickness of the shell (which you can read off from the metric), and then integrate over the full range of ψ, from 0 to π.

PROBLEM 3: SURFACE BRIGHTNESS IN A CLOSED UNIVERSE
(25 points)

The spacetime metric for a homogeneous, isotropic, closed universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} ,$$

where I have taken $k = 1$. To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate ψ, related to r by

$$r = \sin\psi .$$

Then

$$\frac{dr}{\sqrt{1-r^2}} = d\psi ,$$

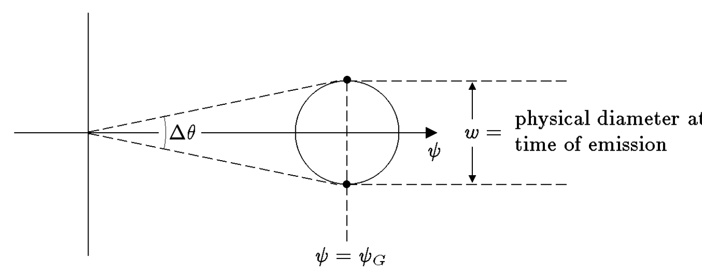
so the metric simplifies to

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} .$$

The form of $a(t)$ depends on the nature of the matter in the universe, but for this problem you should consider $a(t)$ to be an arbitrary function. You should simplify your answers as far as it is possible without knowing the function $a(t)$.

(a) *(10 points)* Suppose that the Earth is at the center of these coordinates, and that we observe a spherical galaxy that is located at $\psi = \psi_G$. The light that we see was emitted from the galaxy at time t_G, and is being received today, at a time that we call t_0. At the time of emission, the galaxy had a power output P (which could be measured, for example, in watts, where 1 watt = 1 joule/sec). The power was radiated uniformly in all directions, in the form of photons. What is the radiation energy flux J from this galaxy at the Earth today? Energy flux (which might be measured in joule-m^{-2}-sec^{-1}) is defined as the energy per unit area per unit time striking a surface that is orthogonal to the direction of the energy flow. [Hint: it is easiest to use a comoving coordinate system with the radiating galaxy at the origin.]

(b) *(10 points)* Suppose that the physical diameter of the galaxy at time t_G was w. Find the apparent angular size $\Delta\theta$ (measured from one edge to the other) of the galaxy as it would be observed from Earth today.



(c) *(5 points)* The *surface brightness* σ of the distant galaxy is defined to be the energy flux J per solid angle subtended by the galaxy.* Calculate the surface brightness σ of the galaxy described in parts (a) and (b). [Hint: if you have the right answer, it can be written in terms of P, w, and the redshift z, without any reference to ψ_G. The rapid decrease in σ with z means that high-z galaxies are difficult to distinguish from the night sky.]

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* *Definition of solid angle:* To define the solid angle subtended by the galaxy, imagine surrounding the observer by a sphere of arbitrary radius r. The sphere should be small compared to cosmological distances, so that Euclidean geometry is valid within the sphere. If a picture of the galaxy is painted on the surface of the sphere so that it just covers the real image, then the solid angle, in steradians, is the area of the picture on the sphere, divided by r^2.

PROBLEM 4: TRAJECTORIES AND DISTANCES IN AN OPEN UNIVERSE *(30 points)*

The spacetime metric for a homogeneous, isotropic, open universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + a^2(t)\left\{\frac{dr^2}{1+r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\} ,$$

where I have taken $k = -1$. As in Problem 1, for the discussion of radial motion it is convenient to introduce an alternative radial coordinate ψ, which in this case is related to r by

$$r = \sinh\psi \ .$$

Then

$$\frac{dr}{\sqrt{1+r^2}} = d\psi \ ,$$

so the metric simplifies to

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + a^2(t)\left\{d\psi^2 + \sinh^2\psi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\} \ .$$

You should treat the function $a(t)$ as a given function. You should simplify your answers as far as it is possible without knowing explicitly the function $a(t)$.

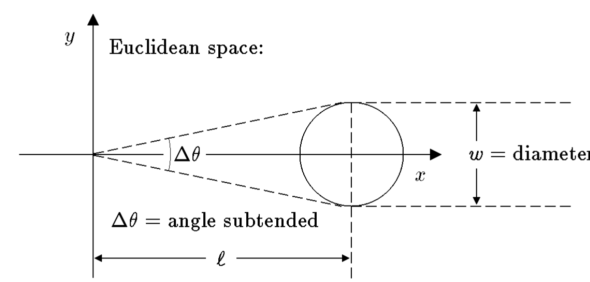
(a) *(5 points)* Suppose that the Earth is at the origin of the coordinate system ($\psi = 0$), and that at the present time, t_0, we receive a light pulse from a distant galaxy G, located at $\psi = \psi_G$. Write down an equation which determines the time t_G at which the light pulse left the galaxy. (You may assume that the light pulse travels on a "null" trajectory, which means that $d\tau = 0$ for any segment of it. Since you don't know $a(t)$ you cannot solve this equation, so please do not try.)

(b) *(5 points)* What is the redshift z_G of the light from galaxy G? (Your answer may depend on t_G, as well as ψ_G, t_0, or any property of the function $a(t)$.)

(c) *(5 points)* To estimate the number of galaxies that one expects to see in a given range of redshifts, it is necessary to know the volume of the region of space that corresponds to this range. Write an expression for the present value of the volume that corresponds to redshifts smaller than that of galaxy G. (You may leave your answer in the form of a definite integral, which may be expressed in terms of ψ_G, t_G, t_0, z_G, or the function $a(t)$.)

(d) *(5 points)* There are a number of different ways of defining distances in cosmology, and generally they are not equal to each other. One choice is called ***proper distance***, which corresponds to the distance that one could in principle measure with rulers. The proper distance is defined as the total length of a network of rulers that are laid end to end from here to the distant galaxy. The rulers have different velocities, because each is at rest with respect to the matter in its own vicinity. They are arranged so that, at the present instant of time, each ruler just touches its neighbors on either side. Write down an expression for the proper distance ℓ_{prop} of galaxy G.

(e) *(5 points)* Another common definition of distance is ***angular size distance,*** determined by measuring the apparent size of an object of known physical size. In a static, Euclidean space, a small sphere of diameter w at a distance ℓ will subtend an angle $\Delta\theta = w/\ell$:



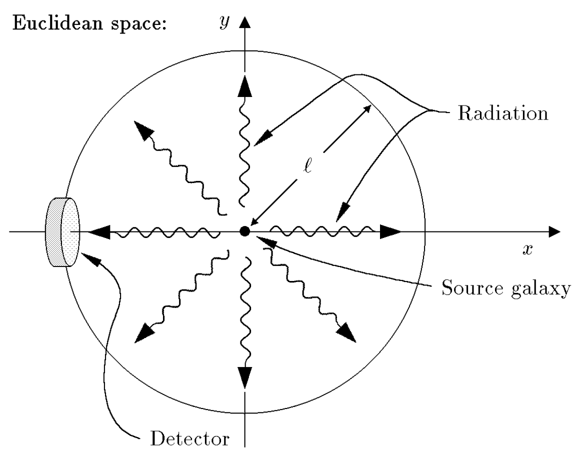
Motivated by this relation, cosmologists define the angular size distance ℓ_{ang} of an object by

$$\ell_{\text{ang}} \equiv \frac{w}{\Delta\theta} \ .$$

What is the angular size distance ℓ_{ang} of galaxy G?

(f) *(5 points)* A third common definition of distance is called ***luminosity distance***, which is determined by measuring the apparent brightness of an object for which the actual total power output is known. In a static, Euclidean space, the energy flux J received from a source of power P at a distance ℓ is given by

$J = P/(4\pi\ell^2)$:



Cosmologists therefore define the luminosity distance by

$$\ell_{\text{lum}} \equiv \sqrt{\frac{P}{4\pi J}} \quad .$$

Find the luminosity distance ℓ_{lum} of galaxy G. (Hint: the Robertson-Walker coordinates can be shifted so that the galaxy G is at the origin.)

PROBLEM 5: THE KLEIN DESCRIPTION OF THE G-B-L GEOMETRY

(This problem is not required, but can be done for 15 points extra credit.)

I stated in Lecture Notes 5 that the space invented by Klein, described by the distance relation

$$\cosh\left[\frac{d(1,2)}{a}\right] = \frac{1 - x_1 x_2 - y_1 y_2}{\sqrt{1 - x_1^2 - y_1^2}\,\sqrt{1 - x_2^2 - y_2^2}} \quad ,$$

where

$$x^2 + y^2 < 1 \quad ,$$

is a two-dimensional space of constant negative curvature. In other words, this is just a two-dimensional Robertson–Walker metric, as would be described by a two-dimensional version of Eq. (5.27), with $k = -1$:

$$ds^2 = a^2 \left\{ \frac{dr^2}{1 + r^2} + r^2 d\theta^2 \right\} \quad .$$

The problem is to prove the equivalence.

(a) *(5 points)* As a first step, show that if x and y are replaced by the polar coordinates defined by

$$x = u\cos\theta$$

$$y = u\sin\theta \quad ,$$

then the distance equation can be rewritten as

$$\cosh\left[\frac{d(1,2)}{a}\right] = \frac{1 - u_1 u_2 \cos(\theta_1 - \theta_2)}{\sqrt{1 - u_1^2}\,\sqrt{1 - u_2^2}} \quad .$$

(b) *(5 points)* The next step is to derive the metric from the distance function above. Let

$$u_1 = u \qquad \theta_1 = \theta \quad ,$$

$$u_2 = u + du \qquad \theta_2 = \theta + d\theta \quad ,$$

and

$$d(1,2) = ds \quad .$$

Insert these expressions into the distance function, expand everything to second order in the infinitesimal quantities, and show that

$$ds^2 = a^2 \left\{ \frac{du^2}{(1 - u^2)^2} + \frac{u^2 d\theta^2}{1 - u^2} \right\} \quad .$$

(This part is rather messy, but you should be able to do it.)

(c) *(5 points)* Now find the relationship between r and u and show that the two metric functions are identical. Hint: The coefficients of $d\theta^2$ must be the same in the two cases. Can you now see why Klein had to impose the condition $x^2 + y^2 < 1$?

Total points for Problem Set 5: 85, plus up to 15 points extra credit.