PROBLEM 1: GEODESICS IN A FLAT UNIVERSE

A freely falling particle travels on a spacetime geodesic. A particle will move along a geodesic according to the geodesic equation.

\[ \frac{d^2 \mathbf{x}}{dt^2} = 0 \]

in a flat universe. The geodesic equation can be written in component form as follows:

\[ \frac{d^2 x_i}{dt^2} + \Gamma^i_{jk} \frac{dx_j}{dt} \frac{dx_k}{dt} = 0 \]

where \( \Gamma^i_{jk} \) are the Christoffel symbols of the second kind, which are defined as:

\[ \Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right) \]

Consider the case of a flat (\( \mathbb{R}^4 \)) Robertson-Walker metric, which has the simple form:

\[ ds^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right) \]

where \( a(t) \) is the scale factor of the universe.

The geodesic equation in this case is:

\[ \frac{d^2 x^i}{dt^2} = 0 \]

For a static gravitational field, the geodesic equation can be simplified to:

\[ \frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \]

where the Christoffel symbols vanish in a static field.

PROBLEM 2: METRIC OF A STATIC GRAVITATIONAL FIELD

\[ ds^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right) \]

where \( a(t) \) is the scale factor of the universe. The geodesic equation in this case is:

\[ \frac{d^2 x^i}{dt^2} = 0 \]

The geodesic equation can be simplified to:

\[ \frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \]

where the Christoffel symbols vanish in a static field.

Reading Assignment: Section 7.4 of Weinberg, Chapters 8, 9, and 10.

Due Date: Monday, November 4, 2013
Consider first a geodesic in a chamber with a movable piston, as shown below. In this problem we will pursue the implications of the conservation of energy.

**Problem 4: Gas Pressure and Energy Conservation**

(a) Show that the circumference of the circle is equal to $2\pi r$, where $r$ is the radius of the circle. For some $r > R$, where $R$ is the Schwarzschild radius, the gravitational potential is zero. Use this to calculate the gravitational potential at $r = R$. (7 points)

(b) Show that the equation of motion of a photon that rises from the boundary of the Schwarzschild horizon is a time coordinate. What is the Schwarzschild horizon? (6 points)

(c) Show that the final rest position of the photon is $r = \frac{2G\mu}{c^2}$. (4 points)

(d) Show that the proper time interval for a segment of the orbiting body is the time that would actually be measured by a clock moving with the orbiting body. Your result should show that $d\tau$ for a segment of the orbiting body is given by $d\tau = \frac{c^2}{GM^2} \omega^2$. Note that this equation is the same as in Newtonian mechanics. (5 points)

(e) Show that the proper time interval for a segment of the orbiting body is given by $d\tau = \frac{c^2}{GM^2} \omega^2$. Note that this equation is the same as in Newtonian mechanics. (5 points)

**Problem 3: Circular Orbits in a Schwarzschild Metric**

(a) Show that the equation of motion of a photon that rises from the boundary of the Schwarzschild horizon is a time coordinate. What is the Schwarzschild horizon? (6 points)

(b) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)

(c) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)

(d) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)

(e) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)

(f) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)

(g) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)

(h) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)

(i) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)

(j) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)

(k) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)

(l) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)

(m) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)

(n) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)

(o) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)

(p) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)

(q) Show that the Schwarzschild horizon can be identified as the Newtonian gravitational potential. Use this to calculate the Schwarzschild horizon. (6 points)
PROBLEM 6: TIME EVOLUTION OF A UNIVERSE WITH MYSTERIOUS MATTER.

Find the power \( n \) in

\[
\frac{(1)^n d \rho}{1} \propto (i) \rho
\]

As the universe expands, the mass density of the form of matter decreases as

\[
\frac{d}{dc} \rho = \frac{d}{dc}
\]

and then becomes proportional to

\[
\frac{(1)^n d \rho}{1} \propto (i) \rho
\]

What is the pressure of this form of matter?

\[
\frac{(1)^n d \rho}{1} \propto (i) \rho
\]

The scale factor evolves according to the relation

\[
\frac{(1)^n d \rho}{1} \propto (i) \rho
\]

The energy of the gas in this region is given by

\[
\frac{(1)^n d \rho}{1} \propto (i) \rho
\]

The energy of the gas is in the region is given by

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PROBLEM 7: EFFECT OF AN EXTRA NEUTRINO SPECIES

(15 points)
According to the standard assumptions (which were used in the lecture notes), there are three species of effectively massless neutrinos. In the temperature range of \(1 \text{ MeV} < kT < 100 \text{ MeV}\), the mass density of the universe is believed to have been dominated by the black-body radiation of photons, electron-positron pairs, and these neutrinos, all of which were in thermal equilibrium.

(a) (5 points) Under these assumptions, how long did it take (starting from the instant of the big bang) for the temperature to fall to the value such that \(kT = 1 \text{ MeV}\)?

(b) (5 points) How much time would it have taken if there were one other species of neutrino in addition to the three that we are currently assuming?

(c) (5 points) What would be the mass density of the universe when \(kT = 1 \text{ MeV}\) under these assumptions, and what would it be if there were one other species?