MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 27, 2013
Prof. Alan Guth

PROBLEM SET 6

DUE DATE: Monday, November 4, 2013

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes*, Chapter 7 (*The First One-Hundredth Second*), and also Barbara Ryden, *Introduction to Cosmology*, Chapter 8 (*Dark Matter*).

UPCOMING QUIZ: Thursday, November 7, 2013.

PROBLEM 1: GEODESICS IN A FLAT UNIVERSE *(25 points)*

According to general relativity, in the absence of any non-gravitational forces a particle will travel along a spacetime geodesic. In this sense, gravity is reduced to a distortion in spacetime.

Consider the case of a flat (*i.e.*, $k = 0$) Robertson–Walker metric, which has the simple form

$$ds^2 = -c^2 dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right] \quad .$$

Since the spatial metric is flat, we have the option of writing it in terms of Cartesian rather than polar coordinates. Now consider a particle which moves along the x-axis. (Note that the galaxies are on the average at rest in this system, but one can still discuss the trajectory of a particle which moves through the model universe.)

(a) *(8 points)* Use the geodesic equation to show that the coordinate velocity computed with respect to proper time (*i.e.*, $dx/d\tau$) falls off as $1/a^2(t)$.

(b) *(8 points)* Use the expression for the spacetime metric to relate dx/dt to $dx/d\tau$.

(c) *(9 points)* The physical velocity of the particle relative to the galaxies that it is passing is given by

$$v = a(t)\frac{dx}{dt} \quad .$$

Show that the momentum of the particle, defined relativistically by

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

falls off as $1/a(t)$. (This implies, by the way, that if the particle were described as a quantum mechanical wave with wavelength $\lambda = h/|\vec{p}|$, then its wavelength would stretch with the expansion of the universe, in the same way that the wavelength of light is redshifted.)

PROBLEM 2: METRIC OF A STATIC GRAVITATIONAL FIELD *(25 points)*

In this problem we will consider the metric

$$ds^2 = -\left[c^2 + 2\phi(\vec{x})\right] dt^2 + \sum_{i=1}^{3} \left(dx^i\right)^2 \quad ,$$

which describes a static gravitational field. Here i runs from 1 to 3, with the identifications $x^1 \equiv x$, $x^2 \equiv y$, and $x^3 \equiv z$. The function $\phi(\vec{x})$ depends only on the spatial variables $\vec{x} \equiv (x^1, x^2, x^3)$, and not on the time coordinate t.

(a) *(5 points)* Suppose that a radio transmitter, located at $\vec{x}_e$, emits a series of evenly spaced pulses. The pulses are separated by a proper time interval ΔT_e, as measured by a clock at the same location. What is the coordinate time interval Δt_e between the emission of the pulses? (I.e., Δt_e is the difference between the time coordinate t at the emission of one pulse and the time coordinate t at the emission of the next pulse.)

(b) *(5 points)* The pulses are received by an observer at $\vec{x}_r$, who measures the time of arrival of each pulse. What is the ***coordinate*** time interval Δt_r between the reception of successive pulses?

(c) *(5 points)* The observer uses his own clocks to measure the proper time interval ΔT_r between the reception of successive pulses. Find this time interval, and also the redshift z, defined by

$$1 + z = \frac{\Delta T_r}{\Delta T_e} \quad .$$

First compute an exact expression for z, and then expand the answer to lowest order in $\phi(\vec{x})$ to obtain a weak-field approximation. (This weak-field approximation is in fact highly accurate in all terrestrial and solar system applications.)

(d) *(5 points)* A freely falling particle travels on a spacetime geodesic $x^\mu(\tau)$, where τ is the proper time. (I.e., τ is the time that would be measured by a clock moving with the particle.) The trajectory is described by the geodesic equation

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau} \quad ,$$

where the Greek indices ($\mu, \nu, \lambda, \sigma$, etc.) run from 0 to 3, and are summed over when repeated. Calculate an explicit expression for

$$\frac{d^2x^i}{d\tau^2} \; ,$$

valid for $i = 1, 2$, or 3. (It is acceptable to leave quantities such as $dt/d\tau$ or $dx^i/d\tau$ in the answer.)

(e) *(5 points)* In the weak-field nonrelativistic-velocity approximation, the answer to the previous part reduces to

$$\frac{d^2x^i}{dt^2} = -\partial_i \phi \ ,$$

so $\phi(\vec{x})$ can be identified as the Newtonian gravitational potential. Use this fact to estimate the gravitational redshift z of a photon that rises from the floor of this room to the ceiling (say 4 meters). (One significant figure will be sufficient.)

PROBLEM 3: CIRCULAR ORBITS IN A SCHWARZSCHILD METRIC *(30 points)*

READ THIS: *This problem was Problem 16 of Review Problems for Quiz 2 of 2011, and the solution is posted as http://web.mit.edu/8.286/www/quiz11/ecqr2-1.pdf. Like Problem 4 of Problem Set 3, but unlike all other homework problems so far, in this case you are encouraged to look at the solutions and benefit from them. When you write your solution, you can even copy it verbatim from these solutions if you wish, although obviously you will learn more if you think about the solution and write your own version.*

The Schwarzschild metric, which describes the external gravitational field of any spherically symmetric distribution of mass, is given by

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \ ,$$

where M is the total mass of the object, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, and $\phi = 2\pi$ is identified with $\phi = 0$. We will be concerned only with motion outside the Schwarzschild horizon $R_S = 2GM/c^2$, so we can take $r > R_S$. (This restriction allows us to avoid the complications of understanding the effects of the singularity at $r = R_S$.) In this problem we will use the geodesic equation to calculate the behavior of circular orbits in this metric. We will assume a perfectly circular orbit in the x-y plane: the radial coordinate r is fixed, $\theta = 90°$, and $\phi = \omega t$, for some angular velocity ω.

(a) *(7 points)* Use the metric to find the proper time interval $d\tau$ for a segment of the path corresponding to a coordinate time interval dt. Note that $d\tau$ represents the time that would actually be measured by a clock moving with the orbiting body. Your result should show that

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2GM}{rc^2} - \frac{r^2\omega^2}{c^2}} \ .$$

Note that for $M = 0$ this reduces to the special relativistic relation $d\tau/dt = \sqrt{1 - v^2/c^2}$, but the extra term proportional to M describes an effect that is new with general relativity— the gravitational field causes clocks to slow down, just as motion does.

(b) *(7 points)* Show that the geodesic equation of motion (Eq. (5.65)) for one of the coordinates takes the form

$$0 = \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial r}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{1}{2}\frac{\partial g_{tt}}{\partial r}\left(\frac{dt}{d\tau}\right)^2 \ .$$

(c) *(8 points)* Show that the above equation implies

$$r\left(\frac{d\phi}{d\tau}\right)^2 = \frac{GM}{r^2}\left(\frac{dt}{d\tau}\right)^2 \ ,$$

which in turn implies that

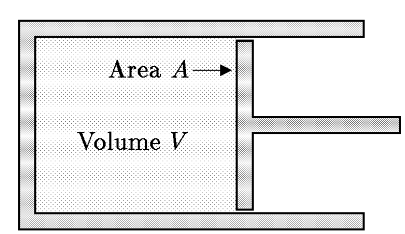
$$r\omega^2 = \frac{GM}{r^2} \ .$$

Thus, the relation between r and ω is exactly the same as in Newtonian mechanics. *[Note, however, that this does not really mean that general relativity has no effect. First, ω has been defined by $d\phi/dt$, where t is a time coordinate which is not the same as the proper time τ that would be measured by a clock on the orbiting body. Second, r does not really have the same meaning as in the Newtonian calculation, since it is not the measured distance from the center of motion. Measured distances, you will recall, are calculated by integrating the metric, as for example in Problem 1 of Problem Set 5, A Circle in a Non-Euclidean Geometry. Since the angular ($d\theta^2$ and $d\phi^2$) terms in the Schwarzschild metric are unaffected by the mass, however, it can be seen that the circumference of the circle is equal to $2\pi r$, as in the Newtonian calculation.]*

(d) *(8 points)* Show that circular orbits around a black hole have a minimum value of the radial coordinate r, which is larger than R_S. What is it?

PROBLEM 4: GAS PRESSURE AND ENERGY CONSERVATION *(25 points)*

In this problem we will pursue the implications of the conservation of energy. Consider first a gas contained in a chamber with a movable piston, as shown below:



Let U denote the total energy of the gas, and let p denote the pressure. Suppose that the piston is moved a distance dx to the right. (We suppose that the motion is slow, so that the gas particles have time to respond and to maintain a uniform pressure throughout the volume.) The gas exerts a force pA on the piston, so the gas does work $dW = pAdx$ as the piston is moved. Note that the volume increases by an amount $dV = Adx$, so $dW = pdV$. The energy of the gas decreases by this amount, so

$$dU = -pdV \; . \tag{P4.1}$$

It turns out that this relation is valid whenever the volume of a gas is changed, regardless of the shape of the volume.

Now consider a homogeneous, isotropic, expanding universe, described by a scale factor $a(t)$. Let u denote the energy density of the gas that fills it. (Remember that $u = \rho c^2$, where ρ is the mass density of the gas.) We will consider a fixed coordinate volume V_{coord}, so the physical volume will vary as

$$V_{\rm phys}(t) = a^3(t) V_{\rm coord} \; . \tag{P4.2}$$

The energy of the gas in this region is then given by

$$U = V_{\rm phys} u \; . \tag{P4.3}$$

(a) *(9 points)* Using these relations, show that

$$\frac{d}{dt}\left(a^3 \rho c^2\right) = -p \frac{d}{dt}(a^3) \; , \tag{P4.4}$$

and then that

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) \; , \tag{P4.5}$$

where the dot denotes differentiation with respect to t.

(b) *(8 points)* The scale factor evolves according to the relation

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \; . \tag{P4.6}$$

Using Eqs. (P4.5) and (P4.6), show that

$$\ddot a = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) a \; . \tag{P4.7}$$

This equation describes directly the deceleration of the cosmic expansion. Note that there are contributions from the mass density ρ, but also from the pressure p.

(c) *(8 points)* So far our equations have been valid for any sort of a gas, but let us now specialize to the case of black-body radiation. For this case we know that $\rho = bT^4$, where b is a constant and T is the temperature. We also know that as the universe expands, aT remains constant. Using these facts and Eq. (P4.5), find an expression for p in terms of ρ.

PROBLEM 5: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION *(25 points)*

A radiation-dominated universe behaves differently from a matter-dominated universe because the pressure of the radiation is significant. In this problem we explore the role of pressure for several fictitious forms of matter.

(a) *(8 points)* For the first fictitious form of matter, the mass density ρ decreases as the scale factor $a(t)$ grows, with the relation

$$\rho(t) \propto \frac{1}{a^6(t)} \; .$$

What is the pressure of this form of matter? *[Hint: the answer is proportional to the mass density.]*

(b) *(9 points)* Find the behavior of the scale factor $a(t)$ for a flat universe dominated by the form of matter described in part (a). You should be able to determine the function $a(t)$ up to a constant factor.

(c) *(8 points)* Now consider a universe dominated by a different form of fictitious matter, with a pressure given by

$$p = \frac{1}{2}\rho c^2 \; .$$

As the universe expands, the mass density of this form of matter behaves as

$$\rho(t) \propto \frac{1}{a^n(t)} \; .$$

Find the power n.

PROBLEM 6: TIME EVOLUTION OF A UNIVERSE WITH MYSTERIOUS STUFF *(15 points)*

Suppose that a model universe is filled with a peculiar form of matter for which

$$\rho \propto \frac{1}{a^5(t)} \; .$$

Assuming that the model universe is flat, calculate

(a) *(4 points)* The behavior of the scale factor, $a(t)$. You should be able to find $a(t)$ up to an arbitrary constant of proportionality.

(b) *(3 points)* The value of the Hubble parameter $H(t)$, as a function of t.

(c) *(4 points)* The physical horizon distance, $\ell_{p,\rm horizon}(t)$.

(d) *(4 points)* The mass density $\rho(t)$.

PROBLEM 7: EFFECT OF AN EXTRA NEUTRINO SPECIES *(15 points)*

According to the standard assumptions (which were used in the lecture notes), there are three species of effectively massless neutrinos. In the temperature range of 1 MeV $< kT <$ 100 MeV, the mass density of the universe is believed to have been dominated by the black-body radiation of photons, electron-positron pairs, and these neutrinos, all of which were in thermal equilibrium.

(a) *(5 points)* Under these assumptions, how long did it take (starting from the instant of the big bang) for the temperature to fall to the value such that $kT = 1$ MeV?

(b) *(5 points)* How much time would it have taken if there were one other species of massless neutrino, in addition to the three which we are currently assuming?

(c) *(5 points)* What would be the mass density of the universe when $kT = 1$ MeV under the standard assumptions, and what would it be if there were one other species of massless neutrino?

Total points for Problem Set 6: 160.