PROBLEM 1: ENTROPY AND THE BACKGROUND NEUTRINO TEMPERATURE

(15 points)

The formula for the entropy density of black-body radiation is given in Lecture Notes 6. The derivation of this formula has been left to the statistical mechanics course that you either have taken or are about to take. Since total entropy is conserved, the entropy density falls as $1/a^3(t)$.

When the electron-positron pairs disappear from the thermal equilibrium mixture as $kT$ falls below $m_e^c=0.511$ MeV, the weak interactions have such low cross sections that the neutrinos have essentially decoupled. To a good approximation, almost all of the energy and entropy released by the electron-positron pair annihilation takes place before the disappearance of the electron-positron pairs, and the neutrinos are unaffected. Use these facts to show that as electron-positron pair annihilation takes place,

$$aT_\gamma \text{ increases by a factor of }\left(\frac{11}{4}\right)^{\frac{1}{3}},$$

while $aT_\nu$ remains constant. It follows that after the disappearance of the electron-positron pairs,

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{\frac{1}{3}}.$$

As far as we know, nothing happens that significantly affects this ratio right up to the present day. So we expect today a background of thermal neutrinos which are slightly colder than the 2.7$^{\circ}$K background of photons.

PROBLEM 2: FREEZE-OUT OF MUONS

(25 points)

A particle called the muon seems to be essentially identical to the electron, except that it is heavier—the mass/energy of a muon is 106 MeV, compared to 0.511 MeV for the electron. The muon ($\mu^-$) has the same charge as an electron, denoted by $e^-$. There is also an antimuon ($\mu^+$), analogous to the positron, with charge $+e$. The muon and antimuon have the same spin as the electron. There is no known particle with a mass between that of an electron and that of a muon.

(a) The formula for the energy density of black-body radiation, as given by Eq. (6.48) of the lecture notes,

$$u = \frac{g\pi^2}{30} \left(\frac{kT}{\hbar c}\right)^4,$$

is written in terms of a normalization constant $g$. What is the value of $g$ for the muons, taking $\mu^-$ and $\mu^+$ together?

(b) When $kT$ is just above 106 MeV as the universe cools, what particles besides the muons are contained in the thermal radiation that fills the universe? What is the contribution to $g$ from each of these particles?

(c) As $kT$ falls below 106 MeV, the muons disappear from the thermal equilibrium radiation. At these temperatures all of the other particles in the black-body radiation are interacting fast enough to maintain equilibrium, so the heat given off from the muons is shared among all the other particles. Letting $a$ denote the Robertson-Walker scale factor, by what factor does $aT$ increase when the muons disappear?
where $h = 2\pi\bar{h}$ is Planck's original constant. 

$\rho(\nu, T) d\nu$ is the energy per unit volume carried by photons whose frequency is in the interval $[\nu, \nu + d\nu]$. In this problem we will assume that this formula holds at some initial time $t_1$, when the temperature had some value $T_1$. We will let $\tilde{\rho}(\nu, t)$ denote the spectral distribution for photons in the universe, which is a function of frequency $\nu$ and time $t$. Thus, our assumption about the initial condition can be expressed as

$$\tilde{\rho}(\nu, t_1) = \rho(\nu, T_1).$$

(P3.3)

The photons redshift as the universe expands, and to a good approximation the redshift and the dilution of photons due to the expansion are the only physical effects that cause the distribution of photons to evolve. Thus, using our knowledge of the redshift, we can calculate the spectral distribution $\tilde{\rho}(\nu, t_2)$ at some later time $t_2 > t_1$. It is not obvious that $\tilde{\rho}(\nu, t_2)$ will be a thermal distribution, but in fact we will be able to show that

$$\tilde{\rho}(\nu, t_2) = \rho(\nu, T(t_2)).$$

(P3.4)

To follow the evolution of the photons from time $t_1$ to time $t_2$, we can imagine selecting a region of comoving coordinates with coordinate volume $V_c$. Within this comoving volume, we can imagine tagging all the photons in a specified infinitesimal range of frequencies, those between $\nu_1$ and $\nu_1 + d\nu_1$. Recalling that the energy of each such photon is $h\nu$, then number $dN_1$ of tagged photons is then

$$dN_1 = \tilde{\rho}(\nu_1, t_1) a^3(t_1) V_c d\nu_1 h\nu_1.$$

(P3.5)

(a) We now wish to follow the photons in this frequency range from time $t_1$ to time $t_2$, during which time each photon redshifts. At the latter time we can denote the range of frequencies by $\nu_2$ to $\nu_2 + d\nu_2$. Express $\nu_2$ and $d\nu_2$ in terms of $\nu_1$ and $d\nu_1$, assuming that the scale factor $a(t)$ is given.

(b) At time $t_2$ we can imagine tagging all the photons in the frequency range $\nu_2$ to $\nu_2 + d\nu_2$ that are found in the original comoving region with coordinate volume $V_c$. Explain why the number $dN_2$ of such photons, on average, will equal $dN_1$ as calculated in Eq. (P3.5).

(c) Since $\tilde{\rho}(\nu, t_2)$ denotes the spectral energy density at time $t_2$, we can write

$$dN_2 = \tilde{\rho}(\nu_2, t_2) a^3(t_2) V_c d\nu_2 h\nu_2,$$

(P3.6)

using the same logic as in Eq. (P3.5). Use $dN_2 = dN_1$ to show that

$$\tilde{\rho}(\nu_2, t_2) = a^3(t_1/t_2)\tilde{\rho}(\nu_1, t_1).$$

(P3.7)

Use the above equation to show that Eq. (P3.3) is satisfied, for $L$ given by Eq. (4.3). Use $L = \tilde{\rho} \nu L^2 = e^{\Delta\nu} \int \tilde{\rho} \nu L^2 = \tilde{\rho} \nu L^2 e^{\Delta\nu}$ to show that

$$\frac{\tilde{\rho}(\nu_2, t_2) \nu_2 L^2}{\tilde{\rho}(\nu_1, t_1) \nu_1 L^2} = \frac{e^{\Delta\nu}}{e^{\Delta\nu}} = e^{\Delta\nu}.$$