MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 24, 2013
Prof. Alan Guth

PROBLEM SET 9

DUE DATE: Monday, December 2, 2013, at 5:00 pm. This is the last problem set before Quiz 3. There will also be a Problem Set 10, to be due Tuesday, December 10, 2013.

READING ASSIGNMENT: Barbara Ryden, *Introduction to Cosmology,* Chapter 11 *(Inflation and the Very Early Universe.)* Also read *Inflation and the New Era of High-Precision Cosmology,* by Alan Guth, written for the MIT Physics Department annual newsletter, 2002. It is available at

http://web.mit.edu/physics/news/physicsatmit/physicsatmit_02_cosmology.pdf

The data quoted in the article about the nonuniformities of the cosmic microwave background radiation has since been superceded by much better data, but the conclusions have not changed. They have only gotten stronger.

UPCOMING QUIZ: Thursday, December 5, 2013.

PROBLEMS 6 AND 7: These extra credit problems can be handed in anytime before Friday, December 9, and so the solutions will not be posted until that time.

PROBLEM 1: BIG BANG NUCLEOSYNTHESIS *(20 points)*

The calculations of big bang nucleosynthesis depend on a large number of measured parameters. Below you are asked to qualitatively describe the effects of changing some of these parameters. Include a sentence or two to explain each of your answers. (These topics have not been discussed in class, but you are expected to be able to answer the questions on the basis of your readings in Weinberg's and Ryden's books.)

(a) *(5 points)* Suppose an extra neutrino species is added to the calculation. Would the predicted helium abundance go up or down?

(b) *(5 points)* Suppose the weak interactions were stronger than they actually are, so that the thermal equilibrium distribution between neutrons and protons were maintained until $kT \approx 0.25$ MeV. Would the predicted helium abundance be larger or smaller than in the standard model?

(c) *(5 points)* Suppose the proton-neutron mass difference were larger than the actual value of $1.29\ \mathrm{MeV/c^2}$. Would the predicted helium abundance be larger or smaller than in the standard calculation?

(d) *(5 points)* The standard theory of big bang nucleosynthesis assumes that the matter in the universe was distributed homogeneously during the era of nucleosynthesis, but the alternative possibility of inhomogeneous big-bang nucleosynthesis has been discussed since the 1980s. Inhomogeneous nucleosynthesis hinges on the hypothesis that baryons became clumped during a phase transition at $t \approx 10^{-6}$ second, when the hot quark soup converted to a gas of mainly protons, neutrons, and in the early stages, pions. The baryons would then be concentrated in small nuggets, with a comparatively low density outside of these nuggets. After the phase transition but before nucleosynthesis, the neutrons would have the opportunity to diffuse away from these nuggets, becoming more or less uniformly distributed in space. The protons, however, since they are charged, interact electromagnetically with the plasma that fills the universe, and therefore have a much shorter mean free path than the neutrons. Most of the protons, therefore, remain concentrated in the nuggets. Does this scenario result in an increase or a decrease in the expected helium abundance?

PROBLEM 2: MASS DENSITY OF VACUUM FLUCTUATIONS *(25 points)*

The energy density of vacuum fluctuations has been discussed qualitatively in lecture. In this problem we will calculate in detail the energy density associated with quantum fluctuations of the electromagnetic field. To keep the problem finite, we will not consider all of space at once, but instead we will consider the electromagnetic field inside a cube of side L, defined by coordinates

$$0 \le x \le L \,, \qquad 0 \le y \le L \,, \qquad 0 \le z \le L \,.$$

Our goal, however, will be to compute the energy density in the limit as the size of the box is taken to infinity.

(a) *(10 points)* The electromagnetic waves inside the box can be decomposed into a Fourier sum of sinusoidal normal modes. Suppose we consider only modes that extend up to a maximum wave number $k_{\rm max}$, or equivalently modes that extend down to a minimum wavelength $\lambda_{\rm min}$, where

$$k_{\rm max} = \frac{2\pi}{\lambda_{\rm min}} \,.$$

How many such modes are there? I do not expect an exact answer, but your approximations should become arbitrarily accurate when $\lambda_{\rm min} \ll L$. (These mode counting techniques are probably familiar to many of you, but in case they are not I have attached an extended hint after part (c).)

(b) *(10 points)* When the electromagnetic field is described quantum mechanically, each normal mode behaves exactly as a harmonic oscillator: if the angular frequency of the mode is ω, then the quantized energy levels have energies given by

$$E_n = \left(n + \tfrac{1}{2}\right) \hbar\omega \ ,$$

where $\hbar$ is Planck's original constant divided by 2π, and n is an integer. The integer n is called the "occupation number," and is interpreted as the number of photons in the specified mode. The minimum energy is not zero, but instead is $\frac{1}{2}\hbar\omega$, which is the energy of the quantum fluctuations of the electromagnetic field. Assuming that the mode sum is cut off at $\lambda_{\rm min}$ equal to the Planck length (as defined in the Lecture Notes), what is the total mass density of these quantum fluctuations?

(c) *(5 points)* How does the mass density of the quantum fluctuations of the electromagnetic field compare with the critical density of our universe?

Extended Hint:

The electromagnetic fields inside a closed box can be expanded as the sum of modes, each of which has a sinusoidally varying time dependence, but the precise form of these modes depends on the nature of the boundary conditions on the walls of the box. Physically reasonable boundary conditions, such as total reflection, are in fact difficult to use in calculations. However, it is known that in the limit of an infinite-sized box, the nature of the boundary conditions will not make any difference. We are therefore free to choose the simplest boundary conditions that we can imagine, and for this purpose we will choose periodic boundary conditions. That is, we will assume that the fields and their normal derivatives on each wall are fixed to precisely match the fields and their normal derivatives on the opposite wall.

To begin, we consider a wave in one dimension, moving at the speed of light. Such waves are most easily described in terms of complex exponentials. If $A(x,t)$ represents the amplitude of the wave, then a sinusoidal wave moving in the positive x-direction can be written as

$$A(x,t) = \text{Re}\left[Be^{ik(x-ct)}\right] \ ,$$

where B is a complex constant and k is a real constant. Defining $\omega = c|k|$, waves in either direction can be written as

$$A(x,t) = \text{Re}\left[Be^{i(kx-\omega t)}\right] \ ,$$

where the sign of k determines the direction. To be periodic with period L, the parameter k must satisfy

$$kL = 2\pi n \ ,$$

where n is an integer. So the spacing between modes is $\Delta k = 2\pi/L$. The density of modes dN/dk (i.e., the number of modes per interval of k) is then one divided by the spacing, or $1/\Delta k$, so

$$\frac{dN}{dk} = \frac{L}{2\pi} \quad \text{(one dimension)} \ .$$

In three dimensions, a sinusoidal wave can be written as

$$A(\vec{x},t) = \text{Re}\left[Be^{i(\vec{k}\cdot\vec{x}-\omega t)}\right] \ ,$$

where $\omega = c|\vec{k}|$, and

$$k_x L = 2\pi n_x \ , \quad k_y L = 2\pi n_y \ , \quad k_z L = 2\pi n_z \ ,$$

where n_x, n_y, and n_z are integers. Thus, in three-dimensional $\vec{k}$-space the allowed values of $\vec{k}$ lie on a cubical lattice, with spacing $2\pi/L$. In counting the modes, one should also remember that for photons there is an extra factor of 2 associated with the fact that electromagnetic waves have two possible polarizations for each allowed value of $\vec{k}$.

PROBLEM 3: THE HORIZON PROBLEM *(20 points)*

The success of the big bang predictions for the abundances of the light elements suggests that the universe was already in thermal equilibrium at one second after the big bang. At this time, the region which later evolves to become the observed universe was many horizon distances across. Try to estimate how many. You may assume that the universe is flat.

PROBLEM 4: THE FLATNESS PROBLEM *(20 points)*

Although we now know that $\Omega_0 = 1$ to an accuracy of a few percent, let us pretend that the value of Ω today is 0.1. It nonetheless follows that at 10^{-37} second after the big bang (about the time of the grand unified theory phase transition), Ω must have been extraordinarily close to one. Writing $\Omega = 1 - \delta$, estimate the value of δ at $t = 10^{-37}$ sec (using the standard cosmological model).

PROBLEM 5: THE MAGNETIC MONOPOLE PROBLEM *(20 points)*

In Lecture Notes 9, we learned that Grand Unified Theories (GUTs) imply the existence of magnetic monopoles, which form as "topological defects" (topologically stable knots) in the configuration of the Higgs fields that are responsible for breaking the grand unified symmetry to the SU(3)×SU(2)×U(1) symmetry of the standard model of particle physics. At very high temperatures the Higgs fields oscillate wildly, so the fields average to zero. As the temperature T falls, however, the system undergoes a phase transition. The phase transition occurs at a temperature T_c, called the critical temperature, where $kT_c \approx 10^{16}$ GeV. At this phase transition the Higgs fields acquire nonzero expectation values, and the grand unified symmetry is thereby spontaneously broken. The monopoles are twists in the Higgs field expectation values, so the monopoles form at the phase transition. Each monopole is expected to have a mass $M_M c^2 \approx 10^{18}$ GeV, where the subscript "M" stands for "monopole." According to an estimate first proposed by T.W.B. Kibble, the number density n_M of monopoles formed at the phase transition is of order

$$n_M \sim 1/\xi^3 \; , \tag{P5.1}$$

where ξ is the correlation length of the field, defined roughly as the maximum distance over which the field at one point in space is correlated with the field at another point in space. The correlation length is certainly no larger than the physical horizon distance at the time of the phase transition, and it is believed to typically be comparable to this upper limit. Note that an upper limit on ξ is a lower limit on n_M — there must be at least of order one monopole per horizon-sized volume.

Assume that the particles of the grand unified theory form a thermal gas of blackbody radiation, as described by Eq. (6.48) of Lecture Notes 6,

$$u = g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} \; ,$$

with $g_{\rm GUT} \sim 200$. Further assume that the universe is flat and radiation-dominated from its beginning to the time of the GUT phase transition, $t_{\rm GUT}$.

For each of the following questions, first write the answer in terms of physical constants and the parameters T_c, M_M, and $g_{\rm GUT}$, and then evaluate the answers numerically.

(a) *(5 points)* Under the assumptions described above, at what time $t_{\rm GUT}$ does the phase transition occur? Express your answer first in terms of symbols, and then evaluate it in seconds.

(b) *(5 points)* Using Eq. (P5.1) and setting ξ equal to the horizon distance, estimate the number density n_M of magnetic monopoles just after the phase transition.

(c) *(5 points)* Calculate the ratio n_M/n_γ of the number of monopoles to the number of photons immediately after the phase Refer to Lecture Notes 6 to remind yourself about the number density of photons.

(d) *(5 points)* For topological reasons monopoles cannot disappear, but they form with an equal number of monopoles and antimonopoles, where the antimonopoles correspond to twists in the Higgs field in the opposite sense. Monopoles and antimonopoles can annihilate each other, but estimates of the rate for this process show that it is negligible. Thus, in the context of the conventional (non-inflationary) hot big bang model, the ratio of monopoles to photons would be about the same today as it was just after the phase transition. Use this assumption to estimate the contribution that these monopoles would make to the value of Ω today.

PROBLEM 6: A ZERO MASS DENSITY UNIVERSE— GENERAL RELATIVITY DESCRIPTION

(This problem is not required, but can be done for 20 points extra credit.)

In this problem and the next we will explore the connections between special relativity and the standard cosmological model which we have been discussing. Although we have not studied general relativity in detail, the description of the cosmological model that we have been using is precisely that of general relativity. In the limit of zero mass density the effects of gravity will become negligible, and the formulas must then be compatible with the special relativity which we discussed at the beginning of the course. The goal of these two problems is to see exactly how this happens.

These two problems will emphasize the notion that a coordinate system is nothing more than an arbitrary system of designating points in spacetime. A physical object might therefore look very different in two different coordinate systems, but the answer to any well-defined physical question must turn out the same regardless of which coordinate system is used in the calculation.

From the general relativity point of view, the model universe is described by the Robertson-Walker spacetime metric:

$$ds^2_{\rm ST} = -c^2 dt^2 + a^2(t) \left\{ \frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right\} \; .$$

I have included the subscript "ST" to remind us that this formula gives the full spacetime metric, as opposed to the purely spatial metric which we discussed earlier. This formula describes the analogue of the "invariant interval" of special relativity, measured between the spacetime points (t, r, θ, ϕ) and $(t+dt, r+dr, \theta+d\theta, \phi+d\phi)$.

The evolution of the model universe is governed by the general relation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} ,$$

except in this case the mass density term is to be set equal to zero.

(a) *(5 points)* Since the mass density is zero, it is certainly less than the critical mass density, so the universe is open. We can then choose $k = -1$. Derive an explicit expression for the scale factor $a(t)$.

(b) *(5 points)* Suppose that a light pulse is emitted by a comoving source at time t_e, and is received by a comoving observer at time t_o. Find the Doppler shift ratio z.

(c) *(5 points)* Consider a light pulse that leaves the origin at time t_e. In an infinitesimal time interval dt the pulse will travel a physical distance $ds = cdt$. Since the pulse is traveling in the radial direction (i.e., with $d\theta = d\phi = 0$), one has

$$cdt = a(t)\frac{dr}{\sqrt{1-kr^2}} .$$

Note that this is a slight generalization of Eq. (3.8), which applies for the case of a Euclidean geometry ($k = 0$). Derive a formula for the trajectory $r(t)$ of the light pulse. You may find the following integral useful:

$$\int \frac{dr}{\sqrt{1+r^2}} = \sinh^{-1} r .$$

(d) *(5 points)* Use these results to express the redshift z in terms of the coordinate r of the observer. If you have done it right, your answer will be independent of t_e. (In the special relativity description that will follow, it will be obvious why the redshift must be independent of t_e. Can you see the reason now?)

PROBLEM 7: A ZERO MASS DENSITY UNIVERSE— SPECIAL RELATIVITY DESCRIPTION

(This problem is also not required, but can be done for 20 points extra credit.)

In this problem we will describe the same model universe as in the previous problem, but we will use the standard formulation of special relativity. We will therefore use an inertial coordinate system, rather than the comoving system of the previous problem. Please note, however, that in the usual case in which gravity is significant, there is no inertial coordinate system. Only when gravity is absent does such a coordinate system exist.

To distinguish the two systems, we will use primes to denote the inertial coordinates: (t', x', y', z'). Since the problem is spherically symmetric, we will also introduce "polar inertial coordinates" (r', θ', ϕ') which are related to the Cartesian inertial coordinates by the usual relations:

$$\begin{aligned} x' &= r' \sin\theta' \cos\phi' \\ y' &= r' \sin\theta' \sin\phi' \\ z' &= r' \cos\theta' . \end{aligned}$$

In terms of these polar inertial coordinates, the invariant spacetime interval of special relativity can be written as

$$ds^2_{\rm ST} = -c^2 dt'^2 + dr'^2 + r'^2 \left(d\theta'^2 + \sin^2\theta' d\phi'^2\right) .$$

For purposes of discussion we will introduce a set of comoving observers which travel along with the matter in the universe, following the Hubble expansion pattern. (Although the matter has a negligible mass density, I will assume that enough of it exists to define a velocity at any point in space.) These trajectories must all meet at some spacetime point corresponding to the instant of the big bang, and we will take that spacetime point to be the origin of the coordinate system. Since there are no forces acting in this model universe, the comoving observers travel on lines of constant velocity (all emanating from the origin). The model universe is then confined to the future light-cone of the origin.

(a) *(5 points)* The cosmic time variable t used in the previous problem can be defined as the time measured on the clocks of the comoving observers, starting at the instant of the big bang. Using this definition and your knowledge of special relativity, find the value of the cosmic time t for given values of the inertial coordinates— i.e., find $t(t', r')$. [Hint: first find the velocity of a comoving observer who starts at the origin and reaches the spacetime point (t', r', θ', ϕ'). Note that the rotational symmetry makes θ' and ϕ' irrelevant, so one can examine motion along a single axis.]

(b) *(5 points)* Let us assume that angular coordinates have the same meaning in the two coordinate systems, so that $\theta = \theta'$ and $\phi = \phi'$. We will verify in part (d) below that this assumption is correct. Using this assumption, find the value of the comoving radial coordinate r in terms of the inertial coordinates— i.e., find $r(t', r')$. [Hint: consider an infinitesimal line segment which extends in the θ-direction, with constant values of t, r, and ϕ. Use the fact that this line segment must have the same physical length, regardless of which coordinate system is used to describe it.] Draw a graph of the t'-r' plane, and sketch in lines of constant t and lines of constant r.

(c) *(5 points)* Show that the radial coordinate r of the comoving system is related to the magnitude of the velocity in the inertial system by

$$r = \frac{v/c}{\sqrt{1 - v^2/c^2}} .$$

Suppose that a light pulse is emitted at the spatial origin ($r' = 0$, $t' =$ *anything*) and is received by another comoving observer who is traveling at speed v. With what redshift z is the pulse received? Express z as a function of r, and compare your answer to part (d) of the previous problem.

(d) *(5 points)* In this part we will show that the metric of the comoving coordinate system can be derived from the metric of special relativity, a fact which completely establishes the consistency of the two descriptions. To do this, first write out the equations of transformation in the form:

$$\begin{aligned} t' &= ? \\ r' &= ? \\ \theta' &= ? \\ \phi' &= ? \ , \end{aligned}$$

where the question marks denote expressions in t, r, θ, and ϕ. Now consider an infinitesimal spacetime line segment described in the comoving system by its two endpoints: (t, r, θ, ϕ) and $(t + dt, r + dr, \theta + d\theta, \phi + d\phi)$. Calculating to first order in the infinitesimal quantities, find the separation between the coordinates of the two endpoints in the inertial coordinate system— i.e., find dt', dr', $d\theta'$, and $d\phi'$. Now insert these expressions into the special relativity expression for the invariant interval ds^2_{ST} , and if you have made no mistakes you will recover the Robertson-Walker metric used in the previous problem.

DISCUSSION OF THE ZERO MASS DENSITY UNIVERSE:

The two problems above demonstrate how the general relativistic description of cosmology can reduce to special relativity when gravity is unimportant, but it provides a misleading picture of the big-bang singularity which I would like to clear up.

First, let me point out that the mass density of the universe increases as one looks backward in time. If the mass density parameter $\Omega \equiv \rho/\rho_c$ for our universe has a value of 0.2, at the low end of the empirically allowed range, then the universe today can be approximately modeled by the zero mass density universe. However, provided that Ω is greater than zero today, the zero mass density model cannot be taken as a valid model for the early history of the universe.

In the zero mass density model, the big-bang "singularity" is a single spacetime point which is in fact not singular at all. In the comoving description the scale factor $a(t)$ equals zero at this time, but in the inertial system one sees that the spacetime metric is really just the usual smooth metric of special relativity, expressed in a peculiar set of coordinates. In this model it is unnatural to think of $t = 0$ as really defining the beginning of anything, since the the future light-cone of the origin connects smoothly to the rest of the spacetime.

In the standard model of the universe with a nonzero mass density, the behavior of the singularity is very different. First of all, it really is singular— one can mathematically prove that there is no coordinate system in which the singularity disappears. Thus, the spacetime cannot be joined smoothly onto anything that may have happened earlier.

The differences between the singularities in the two models can also be seen by looking at the horizon distance. We learned in Lecture Notes 4 that light can travel only a finite distance from the time of the big bang to some arbitrary time t, and that this "horizon distance" is given by

$$\ell_p(t) = a(t) \int_0^t \frac{c}{a(t')} dt' .$$

For the scale factor of the zero mass density universe as found in the problem, one can see that this distance is infinite for any t— for the zero mass density model there is ***no*** horizon. For a radiation-dominated model, however, there is a finite horizon distance given by $2ct$.

Finally, in the zero mass density model the big bang occurs at a single point in spacetime, but for a nonzero mass density model it seems better to think of the big bang as occurring everywhere at once. In terms of the Robertson-Walker coordinates, the singularity occurs at $t = 0$, for all values of r, θ, and ϕ. There is a subtle issue, however, because with $a(t = 0) = 0$, all of these points have zero distance from each other. Mathematically the locus $t = 0$ in a nonzero mass density model is too singular to even be considered part of the space, which consists of all values of $t > 0$. Thus, the question of whether the singularity is a single point is not well defined. For any $t > 0$ the issue is of course clear— the space is homogeneous and infinite (for the case of the open universe). If one wishes to ignore the mathematical subtleties and call the singularity at $t = 0$ a single point, then one certainly must remember that the singularity makes it a very unusual point. Objects emanating from this "point" can achieve an infinite separation in an arbitrarily short length of time.

Total points for Problem Set 9: 105, plus an optional 40 points of extra credit.