MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 29, 2016
Prof. Alan Guth

PROBLEM SET 7

DUE DATE: Friday, November 4, 2016, at 4:00 pm.

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes*, Chapter 8 (*Epilogue: The Prospect Ahead*), and *Afterword: Cosmology Since 1977*. There is no reading assignment from Ryden this week, but if you would like to read ahead, you will be asked to read Chapter 9 (*The Cosmic Microwave Background*) and Chapter 11 (*Inflation and the Very Early Universe*) before the end of the semester.

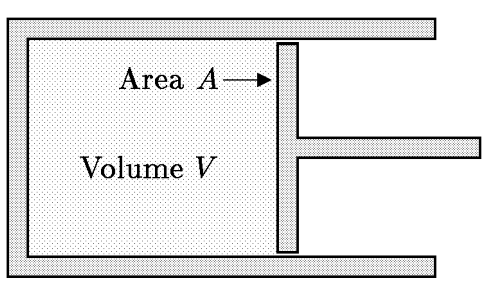
UPCOMING QUIZ: Quiz 2, Wednesday, November 9, 2016.

SHORT-TERM CALENDAR:

OCTOBER/NOVEMBER				
MON	TUES	WED	THURS	FRI
October 31 Lecture 14	November 1	2 Lecture 15	3	4 **PS 7 due**
November 7 Lecture 16	8	9 **Quiz 2** — in class	10	11

PROBLEM 1: GAS PRESSURE AND ENERGY CONSERVATION *(25 points)*

In this problem we will pursue the implications of the conservation of energy. Consider first a gas contained in a chamber with a movable piston, as shown below:



Let U denote the total energy of the gas, and let p denote the pressure. Suppose that the piston is moved a distance dx to the right. (We suppose that the motion is slow, so that the gas particles have time to respond and to maintain a uniform pressure throughout the volume.) The gas exerts a force pA on the piston, so the gas does work $dW = pAdx$ as the piston is moved. Note that the volume increases by an amount $dV = Adx$, so $dW = pdV$. The energy of the gas decreases by this amount, so

$$dU = -pdV \; . \tag{P1.1}$$

It turns out that this relation is valid whenever the volume of a gas is changed, regardless of the shape of the volume.

Now consider a homogeneous, isotropic, expanding universe, described by a scale factor $a(t)$. Let u denote the energy density of the gas that fills it. (Remember that $u = \rho c^2$, where ρ is the mass density of the gas.) We will consider a fixed coordinate volume V_{coord}, so the physical volume will vary as

$$V_{\rm phys}(t) = a^3(t) V_{\rm coord} \; . \tag{P1.2}$$

The energy of the gas in this region is then given by

$$U = V_{\rm phys} u \; . \tag{P1.3}$$

(a) *(9 points)* Using these relations, show that

$$\frac{d}{dt}\left(a^3 \rho c^2\right) = -p\frac{d}{dt}(a^3) \; , \tag{P1.4}$$

and then that

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) \; , \tag{P1.5}$$

where the dot denotes differentiation with respect to t.

(b) *(8 points)* The scale factor evolves according to the relation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \; . \tag{P1.6}$$

Using Eqs. (P1.5) and (P1.6), show that

$$\ddot{a} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) a \; . \tag{P1.7}$$

This equation describes directly the deceleration of the cosmic expansion. Note that there are contributions from the mass density ρ, but also from the pressure p.

(c) *(8 points)* So far our equations have been valid for any sort of a gas, but let us now specialize to the case of black-body radiation. For this case we know that $\rho = bT^4$, where b is a constant and T is the temperature. We also know that as the universe expands, aT remains constant. Using these facts and Eq. (P1.5), find an expression for p in terms of ρ.

PROBLEM 2: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION *(25 points)*

A radiation-dominated universe behaves differently from a matter-dominated universe because the pressure of the radiation is significant. In this problem we explore the role of pressure for several fictitious forms of matter.

(a) *(8 points)* For the first fictitious form of matter, the mass density ρ decreases as the scale factor $a(t)$ grows, with the relation

$$\rho(t) \propto \frac{1}{a^6(t)} \ .$$

What is the pressure of this form of matter? *[Hint: the answer is proportional to the mass density.]*

(b) *(9 points)* Find the behavior of the scale factor $a(t)$ for a flat universe dominated by the form of matter described in part (a). You should be able to determine the function $a(t)$ up to a constant factor.

(c) *(8 points)* Now consider a universe dominated by a different form of fictitious matter, with a pressure given by

$$p = \frac{1}{2}\rho c^2 \ .$$

As the universe expands, the mass density of this form of matter behaves as

$$\rho(t) \propto \frac{1}{a^n(t)} \ .$$

Find the power n.

PROBLEM 3: TIME EVOLUTION OF A UNIVERSE WITH MYSTERIOUS STUFF *(15 points)*

Suppose that a model universe is filled with a peculiar form of matter for which

$$\rho \propto \frac{1}{a^5(t)} \ .$$

Assuming that the model universe is flat, calculate

(a) *(4 points)* The behavior of the scale factor, $a(t)$. You should be able to find $a(t)$ up to an arbitrary constant of proportionality.

(b) *(3 points)* The value of the Hubble parameter $H(t)$, as a function of t.

(c) *(4 points)* The physical horizon distance, $\ell_{p,\text{horizon}}(t)$.

(d) *(4 points)* The mass density $\rho(t)$.

PROBLEM 4: EFFECT OF AN EXTRA NEUTRINO SPECIES *(15 points)*

According to the standard assumptions (which were used in the lecture notes), there are three species of effectively massless neutrinos. In the temperature range of 1 MeV $< kT <$ 100 MeV, the mass density of the universe is believed to have been dominated by the black-body radiation of photons, electron-positron pairs, and these neutrinos, all of which were in thermal equilibrium.

(a) *(5 points)* Under these assumptions, how long did it take (starting from the instant of the big bang) for the temperature to fall to the value such that $kT = 1$ MeV?

(b) *(5 points)* How much time would it have taken if there were one other species of massless neutrino, in addition to the three which we are currently assuming?

(c) *(5 points)* What would be the mass density of the universe when $kT = 1$ MeV under the standard assumptions, and what would it be if there were one other species of massless neutrino?

Total points for Problem Set 7: 80.