MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 6, 2018
Prof. Alan Guth

PROBLEM SET 4
Revised Version*

DUE DATE: Friday, October 12, 2018, at 5:00 pm.

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes*, Chapter 4; Barbara Ryden, *Introduction to Cosmology*, Chapters 4 and 5 and Sec. 6.1. In Weinberg's Chapter 4 (and, later, Chapter 5) there are a lot of numbers mentioned. You certainly do not need to learn all these numbers, but you should be familiar with the orders of magnitude. In Ryden's Chapters 4 and 5 (and, later, Chapter 6), the material parallels what we either have done or will be doing in lecture. For these chapters you should consider Ryden's book as an aid to understanding the lecture material, and not as a source of new material. On the upcoming quizzes, there will be no questions based specifically on the material in these chapters.

SHORT-TERM CALENDAR:

OCTOBER/NOVEMBER				
MON	TUES	WED	THURS	FRI
October 8 Columbus Day	9	10 Lecture 9	11	12 **PS 4 due**
October 15 Lecture 10	16	17 Lecture 11	18	19 **PS 5 due**
October 22 Lecture 12	23	24 Lecture 13	25	26
October 29 Lecture 14	30 **PS 6 due**	31 Lecture 15	November 1	2
November 5 Quiz 2 — in class	6	7	8	9

* Revised October 11, 2018: In Problem 1(h), $d\ell_{p,\gamma B}(t)/dt$ had been mistakenly mistyped (twice) as $d\ell_{p,\gamma B}(t)$.

QUIZ DATES FOR THE TERM:

Quiz 1: Wednesday, October 3, 2018
Quiz 2: Monday, November 5, 2018
Quiz 3: Wednesday, December 5, 2018

PROBLEM 1: PHOTON TRAJECTORIES AND HORIZONS IN A FLAT UNIVERSE WITH $a(t) = bt^{1/2}$ *(20 points)*

The following questions all pertain to a flat universe, with a scale factor given by

$$a(t) = bt^{1/2} ,$$

where b is a constant and t is the time. We will learn later that this is the behavior of a radiation-dominated flat universe.

(a) *(2 points)* If physical lengths are measured in meters, and coordinate lengths are measured in notches, what are the units of $a(t)$ and the constant b?

(b) *(2 points)* Find the Hubble expansion rate $H(t)$.

(c) *(2 points)* Find the physical horizon distance $\ell_{p,\text{hor}}(t)$. Your answer should give the horizon distance in physical units (e.g., meters) and not coordinate units (e.g., notches).

Consider two pieces of matter, A and B, at a coordinate distance ℓ_c from each other. We will consider a photon that is emitted by A at some early time t_A, traveling toward B. The physical distance between A and B at the time of emission is of course $\ell_{p,AB}(t_A) = bt_A^{1/2}\ell_c$, which approaches zero as $t_A \to 0$.

(d) *(2 points)* What is the rate of change of the physical distance between A and B, $d\ell_{p,AB}(t)/dt$, at $t = t_A$? Is the physical distance increasing or decreasing? Does the rate of change approach zero, infinity, negative infinity, or a nonzero finite number as $t_A \to 0$?

(e) *(3 points)* At what time t_B is the photon received by B? As $t_A \to 0$, does t_B approach zero, infinity, or a nonzero finite number?

(f) *(3 points)* Calculate $\ell_{p,\gamma B}(t)$, the physical distance between the photon and B at time t, for $t_A \leq t \leq t_B$.

(g) *(3 points)* What is the rate of change of the physical distance between the photon and B, $d\ell_{p,\gamma B}(t)/dt$, at the instant t_A when the photon is emitted?

(h) *(3 points)* At what value of t_A is this rate of change $d\ell_{p,\gamma B}(t)/dt$ equal to zero? For earlier values of t_A, is the physical distance between the photon and B increasing or decreasing at the time of emission? As $t_A \to 0$, does $d\ell_{p,\gamma B}(t)/dt$ at the time of emission approach zero, infinity, minus infinity, or a nonzero finite number?

PROBLEM 2: EVOLUTION OF AN OPEN, MATTER-DOMINATED UNIVERSE *(35 points)*

The following problem originated on Quiz 2 of 1992 (ancient history!), where it counted 30 points.

The equations describing the evolution of an open, matter-dominated universe were given in Lecture Notes 4 as

$$ct = \alpha\left(\sinh\theta - \theta\right)$$

and

$$\frac{a}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right) \ ,$$

where α is a constant with units of length. The following mathematical identities, which you should know, may also prove useful on parts (e) and (f):

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2} \quad , \quad \cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$$

$$e^\theta = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \dots \ .$$

a) *(5 points)* Find the Hubble expansion rate H as a function of α and θ.

b) *(5 points)* Find the mass density ρ as a function of α and θ.

c) *(5 points)* Find the mass density parameter Ω as a function of α and θ. As with part (c) of the previous problem, the answer to this part appears in Lecture Notes 4. However, you should show that you get the same answer by combining your answers to parts (a) and (b) of this question.

d) *(6 points)* Find the physical value of the horizon distance, $\ell_{p,\text{horizon}}$, as a function of α and θ.

e) *(7 points)* For very small values of t, it is possible to use the first nonzero term of a power-series expansion to express θ as a function of t, and then a as a function of t. Give the expression for $a(t)$ in this approximation. The approximation will be valid for $t \ll t^*$. Estimate the value of t^*.

f) *(7 points)* Even though these equations describe an open universe, one still finds that Ω approaches one for very early times. For $t \ll t^*$ (where t^* is defined in part (e)), the quantity $1-\Omega$ behaves as a power of t. Find the expression for $1-\Omega$ in this approximation.

PROBLEM 3: THE CRUNCH OF A CLOSED, MATTER-DOMINATED UNIVERSE *(25 points)*

This is Problem 6.5 from Barbara Ryden's ***Introduction to Cosmology****, with some paraphrasing to make it consistent with the language used in lecture.*

Consider a closed universe containing only nonrelativistic matter. This is the closed universe discussed in Lecture Notes 4, and it is also the "Big Crunch" model discussed in Ryden's section 6.1. At some time during the contracting phase (i.e., when $\theta > \pi$), an astronomer named Elbbuh Niwde discovers that nearby galaxies have blueshifts ($-1 \leq z < 0$) proportional to their distance. He then measures the present values of the Hubble expansion rate, H_0, and the mass density parameter, Ω_0. He finds, of course, that $H_0 < 0$ (because he is in the contracting phase) and $\Omega_0 > 1$ (because the universe is closed). In terms of H_0 and Ω_0, how long a time will elapse between Dr. Niwde's observation at $t = t_0$ and the final Big Crunch at $t = t_{\text{Crunch}} = 2\pi\alpha/c$? Assuming that Dr. Niwde is able to observe all objects within his horizon, what is the most blueshifted (i.e., most negative) value of z that Dr. Niwde is able to see? What is the lookback time to an object with this blueshift? (By lookback time, one means the difference between the time of observation t_0 and the time at which the light was emitted.)

PROBLEM 4: THE AGE OF A MATTER-DOMINATED UNIVERSE AS $\Omega \to 1$ *(15 points)*

The age t of a matter-dominated universe, for any value of Ω, was given in Lecture Notes 4 as

$$|H|\,t = \begin{cases} \dfrac{\Omega}{2(1-\Omega)^{3/2}}\left[\dfrac{2\sqrt{1-\Omega}}{\Omega} - \operatorname{arcsinh}\left(\dfrac{2\sqrt{1-\Omega}}{\Omega}\right)\right] & \text{if } \Omega < 1 \\ 2/3 & \text{if } \Omega = 1 \\ \dfrac{\Omega}{2(\Omega-1)^{3/2}}\left[\arcsin\left(\pm\dfrac{2\sqrt{\Omega-1}}{\Omega}\right) \mp \dfrac{2\sqrt{\Omega-1}}{\Omega}\right] & \text{if } \Omega > 1 \end{cases} \tag{4.47}$$

It was claimed that this formula is continuous at $\Omega = 1$. In this problem you are asked to show half of this statement. Specifically, you should show that as Ω approaches 1 from below, the expression for $|H|t$ approaches $2/3$. In doing this, you may find it useful to use the Taylor expansion for $\operatorname{arcsinh}(x)$ about $x = 0$:

$$\operatorname{arcsinh}(x) = x - \frac{(1)^2}{3!}x^3 + \frac{(3\cdot 1)^2}{5!}x^5 - \frac{(5\cdot 3\cdot 1)^2}{7!}x^7 + \dots \ .$$

The proof of continuity as $\Omega \to 0$ from above is of course very similar, and you are not asked to show it.

PROBLEM 5: ISOTROPY ABOUT TWO POINTS IN EUCLIDEAN SPACES

(This problem is not required, but can be done for 15 points extra credit. I'd like to give you two weeks to think about it, so you should turn it in with Problem Set 5 on October 19.)

In Steven Weinberg's *The First Three Minutes,* in Chapter 2 on page 24, he gives an argument to show that if a space is isotropic about two distinct points, then it is necessarily homogeneous. He is assuming Euclidean geometry, although he is not explicit about this point. (The statement is simply not true if one allows non-Euclidean spaces — we'll discuss this.) Furthermore, the argument is given in the context of a universe with only two space dimensions, but it could easily be generalized to three, and we will not concern ourselves with remedying this simplification. The statement is true for two-dimensional Euclidean spaces, but Weinberg's argument is not complete. To show that isotropy about two galaxies, 1 and 2, implies that the conditions at any two points A and B must be identical, he constructs two circles. One circle is centered on Galaxy 1 and goes through A, and the other is centered on Galaxy 2 and goes through B. He then argues that the conditions at A and B must both be identical to the conditions at the point C, where the circles intersect. The problem, however, is that the two circles need not intersect. One circle can be completely inside the other, or the two circles can be separated and disjoint. Thus Weinberg's proof is valid for some pairs of points A and B, but cannot be applied to all cases. For 15 points of extra credit, devise a proof that holds in all cases. We have not established axioms for Euclidean geometry, but you may use in your proof any well-known fact about Euclidean geometry.

Total points for Problem Set 4: 95, plus 15 points of extra credit.