MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe November 8, 2018
Prof. Alan Guth

PROBLEM SET 7

DUE DATE: Friday, November 16, 2018, at 4:00 pm.

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes*, Chapter 8 (*Epilogue: The Prospect Ahead*), and *Afterword: Cosmology Since 1977.* Barbara Ryden, *Introduction to Cosmology,* Chapter 9 (*The Cosmic Microwave Background*).

CALENDAR THROUGH THE END OF THE TERM:

NOVEMBER/DECEMBER				
MON	TUES	WED	THURS	FRI
November 5 **Quiz 2** — in class	6	7 Lecture 16	8	9
November 12 Veteran's Day	13	14 Lecture 17	15	16 **PS 7 due**
November 19 Lecture 18	20	21 Lecture 19	22 Thanksgiving	23 Thanksgiving
November 26 Lecture 20	27	28 Lecture 21	29	30 **PS 8 due**
December 3 Lecture 22	4	5 **Quiz 3** — in class	6	7
December 10 Lecture 23	11	12 Last Class **PS 9 due**	13	14

PROBLEM 1: EFFECT OF AN EXTRA NEUTRINO SPECIES *(15 points)*

According to the standard assumptions (which were used in the lecture notes), there are three species of effectively massless neutrinos. In the temperature range of 1 MeV $< kT <$ 100 MeV, the mass density of the universe is believed to have been dominated by the black-body radiation of photons, electron-positron pairs, and these neutrinos, all of which were in thermal equilibrium.

(a) *(5 points)* Under these assumptions, how long did it take (starting from the instant of the big bang) for the temperature to fall to the value such that $kT = 1$ MeV? (In this part and the next, you may assume that the period when $kT > 100$ MeV was so short that one can calculate as if the value of g that you find for 1 MeV $< kT <$ 100 MeV can be used for earlier times as well.)

(b) *(5 points)* How much time would it have taken if there were one other species of massless neutrino, in addition to the three which we are currently assuming?

(c) *(5 points)* What would be the mass density of the universe when $kT = 1$ MeV under the standard assumptions, and what would it be if there were one other species of massless neutrino?

PROBLEM 2: ENTROPY AND THE BACKGROUND NEUTRINO TEMPERATURE *(15 points)*

The formula for the entropy density of black-body radiation is given in Lecture Notes 6. The derivation of this formula has been left to the statistical mechanics course that you either have taken or hopefully will take. For our purposes, the important point is that the early universe remains very close to thermal equilibrium, and therefore entropy is conserved. The conservation of entropy applies even during periods when particles, such as electron-positron pairs, are "freezing out" of the thermal equilibrium mix. Since total entropy is conserved, the entropy density falls off as $1/a^3(t)$.

When the electron-positron pairs disappear from the thermal equilibrium mixture as kT falls below $m_e c^2 = 0.511$ MeV, the weak interactions have such low cross sections that the neutrinos have essentially decoupled. To a good approximation, all of the energy and entropy released by the annihilation of electrons and positrons is added to the photon gas, and the neutrinos are unaffected. Use the conservation of entropy to show that as electron-positron pair annihilation takes place, aT_γ increases by a factor of $(11/4)^{1/3}$, while aT_ν remains constant. It follows that after the disappearance of the electron-positron pairs, $T_\nu/T_\gamma = (4/11)^{1/3}$. As far as we know, nothing happens that significantly affects this ratio right up to the present day. So we expect today a background of thermal neutrinos which are slightly colder than the 2.7°K background of photons.

Added note: In principle the heating of the photon gas due to electron-positron annihilation can also be calculated by using energy conservation, but it is much more difficult. Since

$$\dot{\rho} = -3H\left(\rho + \frac{p}{c^2}\right)$$

(this was Eq. (6.36) of Lecture Notes 6), one needs to know $p(t)$ to understand the changes in energy density. But as the electron-positron pairs are disappearing, kT is comparable to the electon rest mass $m_e c^2$, and the formula for the thermal equilibrium pressure under these circumstances is complicated.

PROBLEM 3: FREEZE-OUT OF MUONS *(25 points)*

A particle called the muon seems to be essentially identical to the electron, except that it is heavier— the mass/energy of a muon is 106 MeV, compared to 0.511 MeV for the electron. The muon (μ^-) has the same charge as an electron, denoted by $-e$. There is also an antimuon (μ^+), analogous to the positron, with charge $+e$. The muon and antimuon have the same spin as the electron. There is no known particle with a mass between that of an electron and that of a muon.

(a) The formula for the energy density of black-body radiation, as given by Eq. (6.48) of the lecture notes,

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3}\ ,$$

is written in terms of a normalization constant g. What is the value of g for the muons, taking μ^+ and μ^- together?

(b) When kT is just above 106 MeV as the universe cools, what particles besides the muons are contained in the thermal radiation that fills the universe? What is the contribution to g from each of these particles?

(c) As kT falls below 106 MeV, the muons disappear from the thermal equilibrium radiation. At these temperatures all of the other particles in the black-body radiation are interacting fast enough to maintain equilibrium, so the heat given off from the muons is shared among all the other particles. Letting a denote the Robertson-Walker scale factor, by what factor does the quantity aT increase when the muons disappear?

PROBLEM 4: THE REDSHIFT OF THE COSMIC MICROWAVE BACKGROUND *(25 points)*

It was mentioned in Lecture Notes 6 that the black-body spectrum has the peculiar feature that it maintains its form under uniform redshift. That is, as the universe expands, even if the photons do not interact with anything, they will continue to be described by a black-body spectrum, although at a temperature that decreases as the universe expands. Thus, even though the cosmic microwave background (CMB) has not been interacting significantly with matter since 350,000 years after the big bang, the radiation today still has a black-body spectrum. In this problem we will demonstrate this important property of the black-body spectrum.

The spectral energy density $\rho_\nu(\nu,T)$ for the thermal (black-body) radiation of photons at temperature T was stated in Lecture Notes 6 as Eq. (6.71), which we can rewrite as

$$\rho_\nu(\nu,T) = \frac{16\pi^2\hbar\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}\ , \tag{4.1}$$

where $h = 2\pi\hbar$ is Planck's original constant. $\rho_\nu(\nu,T)\,d\nu$ is the energy per unit volume carried by photons whose frequency is in the interval $[\nu,\nu+d\nu]$. In this problem we will assume that this formula holds at some initial time t_1, when the temperature had some value T_1. We will let $\tilde\rho(\nu,t)$ denote the spectral distribution for photons in the universe, which is a function of frequency ν and time t. Thus, our assumption about the initial condition can be expressed as

$$\tilde\rho(\nu,t_1) = \rho_\nu(\nu,T_1)\ . \tag{4.2}$$

The photons redshift as the universe expands, and to a good approximation the redshift and the dilution of photons due to the expansion are the only physical effects that cause the distribution of photons to evolve. Thus, using our knowledge of the redshift, we can calculate the spectral distribution $\tilde\rho(\nu,t_2)$ at some later time $t_2 > t_1$. It is not obvious that $\tilde\rho(\nu,t_2)$ will be a thermal distribution, but in fact we will be able to show that

$$\tilde\rho(\nu,t_2) = \rho\big(\nu,T(t_2)\big)\ , \tag{4.3}$$

where in fact $T(t_2)$ will agree with what we already know about the evolution of T in a radiation-dominated universe:

$$T(t_2) = \frac{a(t_1)}{a(t_2)}\,T_1\ . \tag{4.4}$$

To follow the evolution of the photons from time t_1 to time t_2, we can imagine selecting a region of comoving coordinates with coordinate volume V_c. Within this comoving volume, we can imagine tagging all the photons in a specified infinitesimal range of frequencies, those between ν_1 and $\nu_1 + d\nu_1$. Recalling that the energy of each such photon is $h\nu$, the number dN_1 of tagged photons is then

$$dN_1 = \frac{\tilde\rho(\nu_1,t_1)\,a^3(t_1)\,V_c\,d\nu_1}{h\nu_1}\ . \tag{4.5}$$

(a) We now wish to follow the photons in this frequency range from time t_1 to time t_2, during which time each photon redshifts. At the latter time we can denote the range of frequencies by ν_2 to $\nu_2 + d\nu_2$. Express ν_2 and $d\nu_2$ in terms of ν_1 and $d\nu_1$, assuming that the scale factor $a(t)$ is given.

(b) At time t_2 we can imagine tagging all the photons in the frequency range ν_2 to $\nu_2 + d\nu_2$ that are found in the original comoving region with coordinate volume

V_c. Explain why the number dN_2 of such photons, on average, will equal dN_1 as calculated in Eq. (4.5).

(c) Since $\tilde{\rho}(\nu, t_2)$ denotes the spectral energy density at time t_2, we can write

$$dN_2 = \frac{\tilde{\rho}(\nu_2, t_2)\, a^3(t_2)\, V_c\, d\nu_2}{h\nu_2} \,, \tag{4.6}$$

using the same logic as in Eq. (4.5). Use $dN_2 = dN_1$ to show that

$$\tilde{\rho}(\nu_2, t_2) = \frac{a^3(t_1)}{a^3(t_2)}\, \tilde{\rho}(\nu_1, t_1) \ . \tag{4.7}$$

Use the above equation to show that Eq. (4.3) is satisfied, for $T(t)$ given by Eq. (4.4).

Total points for Problem Set 7: 80.