MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — August 30, 2020

Prof. Alan Guth

PROBLEM SET 1

DUE DATE: Friday, September 11, 2020, 5:00 pm.

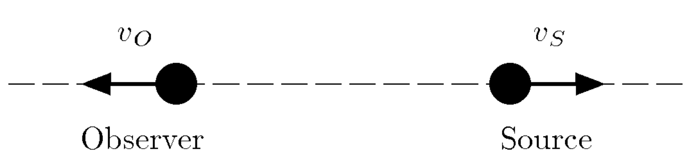
READING ASSIGNMENT: *The First Three Minutes*, Chapters 1 and 2.

NOTE ABOUT EXTRA CREDIT: This problem set contains 40 points of regular problems and 15 points extra credit, so it is probably worthwhile for me to clarify the operational definition of "extra credit". We will keep track of the extra credit grades separately, and at the end of the course I will first assign provisional grades based solely on the regular coursework. I will consult with our teaching assistant, Bruno Scheihing, and we will try to make sure that these grades are reasonable. Then I will add in the extra credit, allowing the grades to change upwards accordingly. Finally, Bruno and I will look at each student's grades individually, and we might decide to give a higher grade to some students who are slightly below a borderline. Students whose grades have improved significantly during the term, students whose average has been pushed down by single low grade, and students who have been affected by adverse personal or medical problems will be the ones most likely to be boosted.

The bottom line is that the extra credit problems are OPTIONAL. You should feel free to skip them, and you will still get an excellent grade in the course if you do well on the regular problems. However, if you have some time and enjoy an extra challenge, then I hope that you will find the extra credit problems interesting and worthwhile.

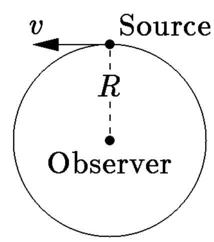
PROBLEM 1: NONRELATIVISTIC DOPPLER SHIFT, SOURCE AND OBSERVER IN MOTION *(15 points)*

Consider the Doppler shift of sound waves, for a case in which both the source and the observer are moving. Suppose the source is moving with a speed v_s relative to the air, while the observer is receding from the source, moving in the opposite direction with speed v_o relative to the air. Calculate the Doppler shift z. (Recall that z is defined by $1 + z \equiv \lambda_o/\lambda_s$, where λ_o and λ_s are the wavelengths as measured by the observer and by the source, respectively.) *Hint:* while this problem can be solved directly, you can save time by finding a way to determine the answer by using the cases that are already calculated in Lecture Notes 1.



PROBLEM 2: THE TRANSVERSE DOPPLER SHIFT *(25 points)*

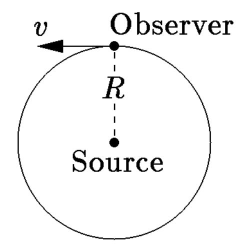
Consider the Doppler shift observed by a stationary observer, from a source that travels in a circular orbit of radius R about the observer. Let the speed of the source be v.



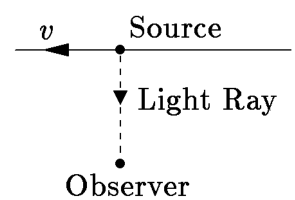
(a) *(5 points)* If the wave in question is sound, and both the source speed v and the wave speed u are very small compared to the speed of light c, what is the Doppler shift z? Assume that the observer is at rest relative to the air.

(b) *(5 points)* If the wave is light, traveling with speed c, and v is not small compared to c, what is the Doppler shift z? This is called the *transverse Doppler shift*, since the velocity of the light ray is perpendicular to the velocity of the source at the time of emission, as seen in the reference frame of the observer.

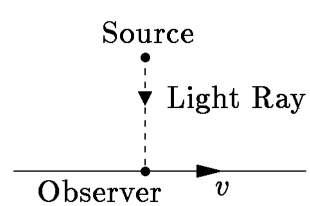
(c) *(5 points)* Still considering light waves and the same pattern of motion as shown in the figure, suppose that the source and the observer were reversed. That is, suppose a light ray is sent from the person at the center of the circle to the person traveling around the circle at speed v. In this case, what would be the Doppler shift z?



(d) *(5 points)* Now suppose that the motion is linear instead of circular. Again we consider light rays, and as in part (b) we assume that the source is moving with a speed v that is not small compared to c. If the light ray is emitted by the source at the moment of its closest approach to the observer, as shown in the diagram, what is the Doppler shift z?



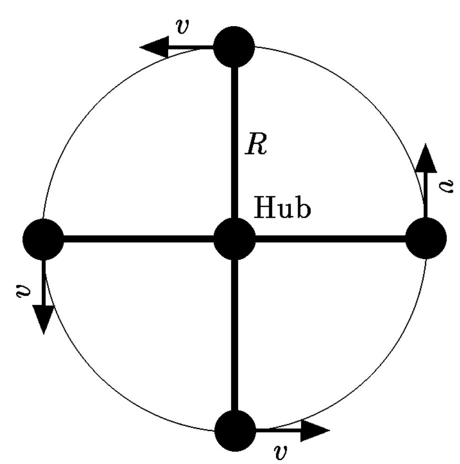
(e) *(5 points)* Again consider linear motion, with light rays. As in part (c), assume that the observer is moving with a speed v that is not small compared to c. If the light ray is received by the observer at the moment of its closest approach to the source, as shown in the diagram, what is the Doppler shift z?



PROBLEM 3: A HIGH-SPEED MERRY-GO-ROUND

(This problem is not required, but can be done for 15 points extra credit.)

Now consider the Doppler shift as it would be observed in a high-speed "merry-go-round." Four evenly-spaced cars travel around a central hub at speed v, all at a distance R from a central hub. Each car is sending waves to all three of the other cars.



(a) If the wave in question is sound, and both the source speed v and the wave speed u are very small compared to the speed of light c, with what Doppler shift z does a given car receive the sound from (i) the car in front of it; (ii) the car behind it; and (iii) the car opposite it?

(b) In the relativistic situation, where the wave is light and the speed v may be comparable to c, what is the answer to the same three parts (i)-(iii) above?

Total points for Problem Set 1: 40, plus 15 points of extra credit.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — September 12, 2020
Prof. Alan Guth

PROBLEM SET 2

DUE DATE: Friday, September 18, 2020, 5:00 pm.

SEPTEMBER/OCTOBER				
MON	TUES	WED	THURS	FRI
September 14 Class 3	15	16 Class 4	17	18
21 Class 5 PS 2 due	22	23 Class 6	24	25 PS 3 due
28 Class 7	29	30 Quiz 1 — "in class"	October 1	2

READING ASSIGNMENT: Barbara Ryden, *Introduction to Cosmology*, Chapters 1-3.

PLANNING AHEAD: If you want to read ahead, the reading assignment with Problem Set 3 will be Weinberg, *The First Three Minutes,* Chapter 3. Problem Sets 1 through 3, including the reading assignments, will be included in the material covered on Quiz 1, on Wednesday, September 30.

INTRODUCTION TO THE PROBLEM SET

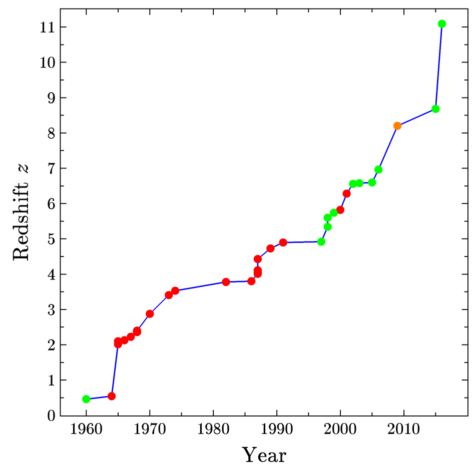
In this problem set we will consider a universe in which the scale factor is given by

$$a(t) = bt^{2/3} \ ,$$

where b is an arbitrary constant of proportionality which should not appear in the answers to any of the questions below. (We will see in Lecture Notes 3 that this is the behavior of a flat universe with a mass density that is dominated by nonrelativistic matter.) We will suppose that a distant galaxy is observed with a redshift z. As a concrete example we will consider the most distant known object with a well-determined redshift, the galaxy GN-z11, which has a redshift $z = 11.1$. The discovery of this galaxy was announced in March 2016 by an international group of astronomers, using the Hubble Space Telescope*

* P. A. Oesh et al., "A Remarkably Luminous Galaxy at $z = 11.1$ Measured with Hubble Space Telescope Grism Spectroscopy," The Astrophysical Journal **819**, 129 (2016), https://arxiv.org/abs/1603.00461.

The rate at which the highest measured redshift has been growing has been dramatic. In 1960 the highest measured redshift was only z=0.461. The diagram at the right shows a graph of the highest confirmed redshift by year of discovery, using the listing in the Wikipedia‡ The red circles represent quasars, the green circles represent galaxies, and the one orange circle at 2009 is a gamma ray burst. The search for high redshift objects continues to be an exciting area of research, as astronomers try to sort out the conditions in the universe when the first galaxies began to form.



PROBLEM 1: DISTANCE TO THE GALAXY *(10 points)*

Let t_0 denote the present time, and let t_e denote the time at which the light that we are currently receiving was emitted by the galaxy. In terms of these quantities, find the present value of the physical distance ℓ_p between this distant galaxy and us.

PROBLEM 2: TIME OF EMISSION *(10 points)*

Express the redshift z in terms of t_0 and t_e. Find the ratio t_e/t_0 for the $z = 11.1$ galaxy.

PROBLEM 3: DISTANCE IN TERMS OF REDSHIFT z *(10 points)*

Express the present value of the physical distance in terms of the present value of the Hubble expansion rate H_0 and the redshift z. Taking $H_0 \approx 67$ km-sec^{-1}-Mpc^{-1}, how far away is the galaxy? Express your answer both in light-years and in Mpc.

PROBLEM 4: SPEED OF RECESSION *(10 points)*

Find the present rate at which the physical distance ℓ_p between the distant galaxy and us is changing. Express your answer in terms of the redshift z and the speed of light c, and evaluate it numerically for the case $z = 11.1$. Express your answer as a fraction of the speed of light. [If you get it right, this "fraction" is greater than one! Our expanding universe violates special relativity, but is consistent with general relativity.]

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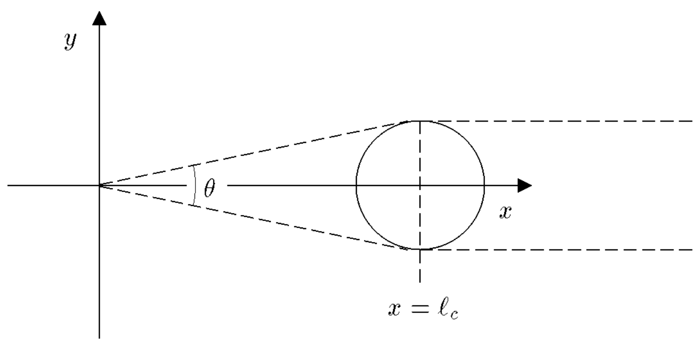
‡ "List of the most distant astronomical objects." In *Wikipedia, The Free Encyclopedia.* Retrieved 16:35, September 12, 2020, from https://en.wikipedia.org/wiki/List_of_the_most_distant_astronomical_objects.

PROBLEM 5: APPARENT ANGULAR SIZES *(20 points)*

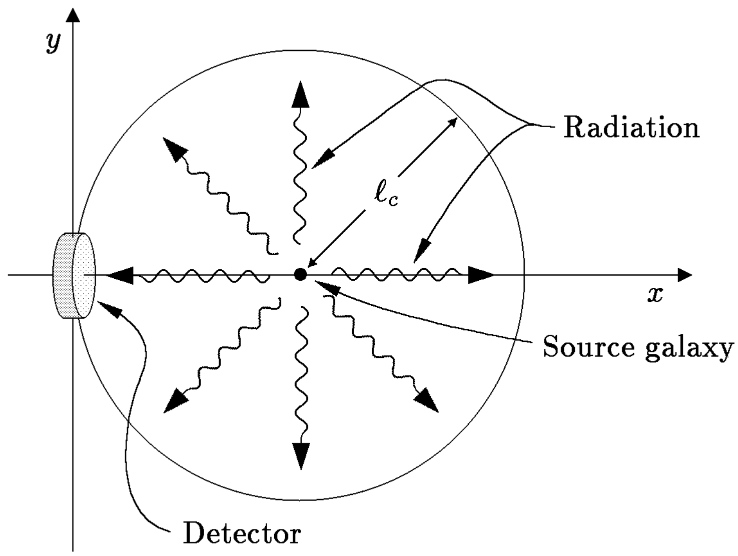


Now suppose for simplicity that the galaxy is spherical, and that its physical diameter was w at the time it emitted the light. (The actual galaxy is seen as an unresolved point source, so we don't know it's actual size and shape.) Find the apparent angular size θ (measured from one edge to the other) of the galaxy as it would be observed from Earth today. Express your answer in terms of w, z, H_0, and c. You may assume that $\theta \ll 1$. Compare your answer to the apparent angular size of a circle of diameter w in a static Euclidean space, at a distance equal to the present value of the physical distance to the galaxy, as found in Problem 1. [Hint: draw diagrams which trace the light rays in the ***comoving*** coordinate system. If you have it right, you will find that θ has a minimum value for $z = 1.25$, and that θ increases for larger z. This phenomenon makes sense if you think about the distance to the galaxy at the time of emission. If the galaxy is ***very*** far away today, then the light that we now see must have left the object very early, when it was rather close to us!]

PROBLEM 6: RECEIVED RADIATION FLUX

(This problem is not required, but can be done for 15 points extra credit.)

At the time of emission, the galaxy had a power output P (measured, say, in ergs/sec) which was radiated uniformly in all directions. This power was emitted in the form of photons. What is the radiation energy flux J from this galaxy at the earth today? Energy flux (which might be measured in ergs-cm^{-2}-sec^{-1}) is defined as the energy per unit area per unit time striking a surface that is orthogonal to the direction of energy flow. The easiest way to solve this problem is to consider the trajectories of the photons, as viewed in comoving coordinates. You must calculate the rate at which photons arrive at the detector, and you must also use the fact that the energy of each photon is proportional to its frequency, and is therefore decreased by the redshift. You may find it useful to think of the detector as a small part of a sphere that is centered on the source, as shown in the following diagram:



Total points for Problem Set 2: 60, plus 15 points of extra credit.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — September 19, 2020
Prof. Alan Guth

PROBLEM SET 3

DUE DATE: Friday, September 25, 2020, 5:00 pm.

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes*, Chapter 3.

SHORT-TERM CALENDAR:

SEPTEMBER/OCTOBER				
MON	TUES	WED	THURS	FRI
September 14 Class 3	15	16 Class 4	17	18
21 Class 5 PS 2 due	22	23 Class 6	24	25 PS 3 due
28 Class 7	29	30 Quiz 1 — "in class"	October 1	2

QUIZ DATES FOR THE TERM:

Quiz 1: Wednesday, September 30, 2020
Quiz 2: Wednesday, October 28, 2020
Quiz 3: Wednesday, December 2, 2020

FIRST QUIZ: The first of three quizzes for the term will be given on Wednesday, September 30, 2020.

Coverage: Lecture Notes 1, 2, and 3; Problem Sets 1, 2, and 3; Weinberg, Chapters 1, 2, and 3; Ryden, Chapters 1, 2, and 3. While all of Ryden's Chapter 3 has been assigned, questions on the quiz will be limited to Sections 3.1 (*The Way of Newton*) and 3.3 (*The General Way of Einstein*). Section 3.2 (*The Special Way of Einstein*) describes special relativity. Ryden's approach is somewhat different from our Lecture Notes 1 — for the quiz, you will be responsible only for the issues discussed in Lecture Notes 1. The material in Sections 3.4–3.6 will be discussed in lecture later in the course, and you will not be responsible for it until then.

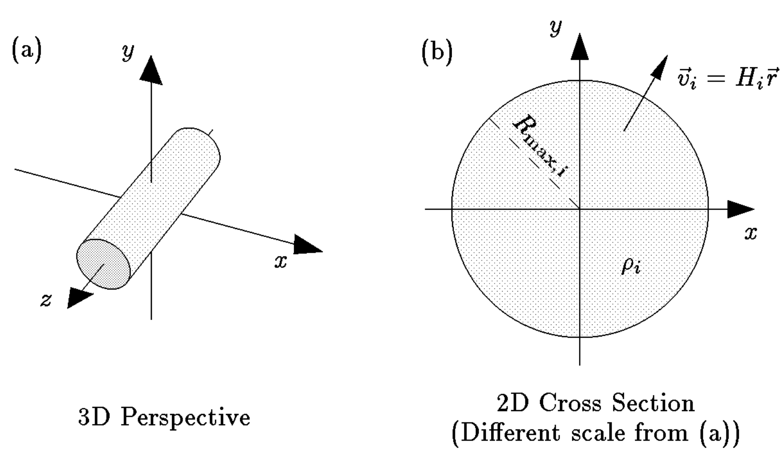
Quiz Logistics: The quiz will be closed book, no calculators, no internet, and 85 minutes long. I assume that most of you will take it during our regular class time on September 30, but you will have the option of starting it any time during a 24-hour window from

11:05 am EDT on September 30 to 11:05 am EDT on Thursday, October 1. If you want to start later than 11:05 am 9/30/2020, you should email me your choice of starting time by 11:59 pm on 9/29/2020. The quiz will be contained in a PDF file, which I am planning to distribute by email. You will each be expected to spend up to 85 minutes working on it, and then you will upload your answers to Canvas as a PDF file. I won't place any precise time limit on scanning or photographing and uploading, because the time needed for that can vary. If you have questions about the meaning of the questions, I will be available on Zoom during the September 30 class time, and we will arrange for either Bruno or me to be available by email as much as possible during the other quiz times. If you have any special circumstances that might make this procedure difficult, or if you need a postponement beyond the 24-hour window, please let me (guth@ctp.mit.edu) know.

PROBLEM 1: A CYLINDRICAL UNIVERSE *(25 points)*

The following problem originated on Quiz 2 of 1994, where it counted 30 points.

The lecture notes showed a construction of a Newtonian model of the universe that was based on a uniform, expanding, sphere of matter. In this problem we will construct a model of a cylindrical universe, one which is expanding in the x and y directions but which has no motion in the z direction. Instead of a sphere, we will describe an infinitely long cylinder of radius $R_{\rm max,i}$, with an axis coinciding with the z-axis of the coordinate system:



We will use cylindrical coordinates, so

$$r = \sqrt{x^2 + y^2}$$

and

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} \; ; \qquad \hat{r} = \frac{\vec{r}}{r} \; ,$$

where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the usual unit vectors along the x, y, and z axes. We will assume that at the initial time t_i, the initial density of the cylinder is ρ_i, and the initial velocity of a particle at position $\vec{r}$ is given by the Hubble relation

$$\vec{v}_i = H_i \vec{r} \ .$$

(a) *(5 points)* By using Gauss' law of gravity, it is possible to show that the gravitational acceleration at any point is given by

$$\vec{g} = -\frac{A\mu}{r}\hat{r} \ ,$$

where A is a constant and μ is the total mass per length contained within the radius r. Evaluate the constant A.

(b) *(5 points)* As in the lecture notes, we let $r(r_i, t)$ denote the trajectory of a particle that starts at radius r_i at the initial time t_i. Find an expression for $\ddot{r}(r_i, t)$, expressing the result in terms of r, r_i, ρ_i, and any relevant constants. (Here an overdot denotes a time derivative.)

(c) *(5 points)* Defining

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} \ ,$$

show that $u(r_i, t)$ is in fact independent of r_i. This implies that the cylinder will undergo uniform expansion, just as the sphere did in the case discussed in the lecture notes. As before, we define the scale factor $a(t) \equiv u(r_i, t)$.

(d) *(5 points)* Express the mass density $\rho(t)$ in terms of the initial mass density ρ_i and the scale factor $a(t)$. Use this expression to obtain an expression for $\ddot{a}$ in terms of a, ρ, and any relevant constants.

(e) *(5 points)* Find an expression for a conserved quantity of the form

$$E = \frac{1}{2}\dot{a}^2 + V(a) \ .$$

What is $V(a)$? Will this universe expand forever, or will it collapse?

PROBLEM 2: A FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION *(10 points)*

Consider a ***flat*** universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$a(t) = bt^{3/4} ,$$

where b is a constant.

(a) *(5 points)* For this universe, find the value of the Hubble expansion rate $H(t)$.

(b) *(5 points)* What is the mass density of the universe, $\rho(t)$? (In answering this question, you will need to know that the equation for $\dot{a}/a$ in Lecture Notes 3,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} ,$$

holds for all forms of matter, while the equation for $\ddot{a}$,

$$\ddot{a} = -\frac{4\pi}{3}G\rho(t)a ,$$

requires modification if the matter has a significant pressure. The $\ddot{a}$ equation is therefore not applicable to this problem.)

PROBLEM 3: ENERGY AND THE FRIEDMANN EQUATION *(30 points)*

The Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} , \tag{1}$$

was derived in Lecture Notes 3 as a first integral of the equations of motion. The equation was first derived in a different form,

$$E = \frac{1}{2}\dot{a}^2 - \frac{4\pi}{3}\frac{G\rho_i}{a} = \text{constant}, \tag{2}$$

where $k = -2E/c^2$. In this form the equation looks more like a conservation of energy relation, although the constant E does not have the dimensions of energy. There are two ways, however, in which the quantity E can be connected to the conservation of energy. It is related the energy of a test particle that moves with the Hubble expansion, and it is also related to the total energy of the entire expanding sphere of radius R_{max}, which was discussed in Lecture Notes 3 as a method of deriving the Friedmann equations. In this problem you will derive these relations.

First, to see the relation with the energy of a test particle moving with the Hubble expansion, define a physical energy $E_{\rm phys}$ by

$$E_{\rm phys} \equiv m r_i^2 E \ , \tag{3}$$

where m is the mass of the test particle and r_i is its initial radius. Note that the gravitational force on this particle is given by

$$\vec{F} = -\frac{GmM(r_i)}{r^2}\hat{r} = -\vec{\nabla} V_{\rm eff}(r) \ , \tag{4}$$

where $M(r_i)$ is the total mass initially contained within a radius r_i of the origin, r is the present distance of the test particle from the origin, and the "effective" potential energy $V_{\rm eff}(r)$ is given by

$$V_{\rm eff}(r) = -\frac{GmM(r_i)}{r} \ . \tag{5}$$

The motivation for calling this quantity the "effective" potential energy will be explained below.

(a) *(10 points)* Show that $E_{\rm phys}$ is equal to the "effective" energy of the test particle, defined by

$$E_{\rm eff} = \frac{1}{2} m v^2 + V_{\rm eff}(r) \ . \tag{6}$$

We understand that $E_{\rm eff}$ is conserved because it is the energy in an analogue problem in which the test particle moves in the gravitational field of a point particle of mass $M(r_i)$, located at the origin, with potential energy function $V_{\rm eff}(r)$. In this analogue problem the force on the test particle is exactly the same as in the real problem, but in the analogue problem the energy of the test particle is conserved.

We call (6) the "effective" energy because it is really the energy of the analogue problem, and not the real problem. The true potential energy $V(r,t)$ of the test particle is defined to be the amount of work we must supply to move the particle to its present location from some fixed reference point, which we might take to be $r = \infty$. We will not bother to write $V(r,t)$ explicitly, since we will not need it, but we point out that it depends on the time t and on $R_{\rm max}$, and when differentiated gives the correct gravitational force at any radius. By contrast, $V_{\rm eff}(r)$ gives the correct force only at the radius of the test particle, $r = a(t) r_i$. The true potential energy function $V(r,t)$ gives no conservation law, since it is explicitly time-dependent, which is why the quantity $V_{\rm eff}(r)$ is useful.

To relate E to the total energy of the expanding sphere, we need to integrate over the sphere to determine its total energy. These integrals are most easily carried out by dividing the sphere into shells of radius r, and thickness dr, so that each shell has a volume

$$dV = 4\pi r^2 \, dr \ . \tag{7}$$

(b) *(10 points)* Show that the total kinetic energy K of the sphere is given by

$$K = c_K M R_{\max,i}^2 \left\{ \frac{1}{2} \dot{a}^2(t) \right\} , \tag{8}$$

where c_K is a numerical constant, M is the total mass of the sphere, and $R_{\max,i}$ is the initial radius of the sphere. Evaluate the numerical constant c_K.

(c) *(10 points)* Show that the total potential energy of the sphere can similarly be written as

$$U = c_U M R_{\max,i}^2 \left\{ -\frac{4\pi}{3} G \frac{\rho_i}{a} \right\} . \tag{9}$$

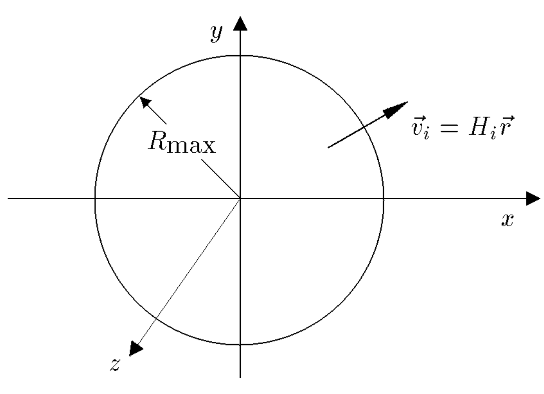
(*Suggestion:* calculate the total energy needed to assemble the sphere by bringing in one shell of mass at a time from infinity.) Show that $c_U = c_K$, so that the total energy of the sphere is given by

$$E_{\text{total}} = c_K M R_{\max,i}^2 E . \tag{10}$$

PROBLEM 4: A POSSIBLE MODIFICATION OF NEWTON'S LAW OF GRAVITY *(20 points)*

READ THIS: *This problem was Problem 2 of Quiz 1 of 2011, and the solution is posted as http://web.mit.edu/8.286/www/quiz11/ecqs1-1.pdf. Unlike the situation with other problems, in this case you are encouraged to look at these solutions and benefit from them. When you write your solution, you can even copy it verbatim from these solutions if you wish, although obviously you will learn more if you think about the solution and write your own version.*

In Lecture Notes 3 we developed a Newtonian model of cosmology, by considering a uniform sphere of mass, centered at the origin, with initial mass density ρ_i and an initial pattern of velocities corresponding to Hubble expansion: $\vec{v}_i = H_i \vec{r}$:



We denoted the radius at time t of a particle which started at radius r_i by the function $r(r_i,t)$. Assuming Newton's law of gravity, we concluded that each particle would experience an acceleration given by

$$\vec{g} = -\frac{GM(r_i)}{r^2(r_i,t)}\,\hat{r}\ ,$$

where $M(r_i)$ denotes the total mass contained initially in the region $r < r_i$, given by

$$M(r_i) = \frac{4\pi}{3} r_i^3 \rho_i\ .$$

Suppose that the law of gravity is modified to contain a new, repulsive term, producing an acceleration which grows as the nth power of the distance, with a strength that is independent of the mass. That is, suppose $\vec{g}$ is given by

$$\vec{g} = -\frac{GM(r_i)}{r^2(r_i,t)}\,\hat{r} + \gamma r^n(r_i,t)\,\hat{r}\ ,$$

where γ is a constant. The function $r(r_i,t)$ then obeys the differential equation

$$\ddot{r} = -\frac{GM(r_i)}{r^2(r_i,t)} + \gamma r^n(r_i,t)\ .$$

(a) *(4 points)* As done in the lecture notes, we define

$$u(r_i,t) \equiv r(r_i,t)/r_i\ .$$

Write the differential equation obeyed by u. *(Hint: be sure that u is the only time-dependent quantity in your equation; r, ρ, etc. must be rewritten in terms of u, ρ_i, etc.)*

(b) *(4 points)* For what value of the power n is the differential equation found in part (a) independent of r_i?

(c) *(4 points)* Write the initial conditions for u which, when combined with the differential equation found in (a), uniquely determine the function u.

(d) *(8 points)* If all is going well, then you have learned that for a certain value of n, the function $u(r_i,t)$ will in fact not depend on r_i, so we can define

$$a(t) \equiv u(r_i,t)\ .$$

Show, for this value of n, that the differential equation for a can be integrated once to obtain an equation related to the conservation of energy. The desired equation should include terms depending on a and $\dot{a}$, but not $\ddot{a}$ or any higher derivatives.

Total points for Problem Set 3: 85.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 4, 2020
Prof. Alan Guth

PROBLEM SET 4

DUE DATE: Friday, October 9, 2020, at 5:00 pm.

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes*, Chapter 4; Barbara Ryden, *Introduction to Cosmology*, Chapters 4 and Chapter 5 through Section 5.4.1 (*Matter + Curvature*). (In the first edition, Chapters 4, 5, and Section 6.1.) In Weinberg's Chapter 4 (and, later, Chapter 5) there are a lot of numbers mentioned. You certainly do not need to learn all these numbers, but you should be familiar with the orders of magnitude. In Ryden's Chapters 4 and 5, the material parallels what we either have done or will be doing in lecture. For these chapters you should consider Ryden's book as an aid to understanding the lecture material, and not as a source of new material. On the upcoming quizzes, there will be no questions based specifically on the material in these chapters.

SHORT-TERM CALENDAR:

OCTOBER/NOVEMBER				
MON	TUES	WED	THURS	FRI
October 5 Class 9	6	7 Class 10	8	9 **PS 4 due**
October 12 Columbus Day	13 Class 11	14 Class 12	15	16 **PS 5 due**
October 19 Class 13	20	21 Class 14	22	23 **PS 6 due**
October 26 Class 15	27	28 Class 16 Quiz 2	29	30

QUIZ DATES FOR THE TERM:

Quiz 1: Wednesday, September 30, 2020
Quiz 2: Wednesday, October 28, 2020
Quiz 3: Wednesday, December 2, 2020

PROBLEM 1: PHOTON TRAJECTORIES AND HORIZONS IN A FLAT UNIVERSE WITH $a(t) = bt^{1/2}$ *(20 points)*

The following questions all pertain to a flat universe, with a scale factor given by

$$a(t) = bt^{1/2} \ ,$$

where b is a constant and t is the time. We will learn later that this is the behavior of a radiation-dominated flat universe.

(a) *(2 points)* If physical lengths are measured in meters, and coordinate lengths are measured in notches, what are the units of $a(t)$ and the constant b?

(b) *(2 points)* Find the Hubble expansion rate $H(t)$.

(c) *(2 points)* Find the physical horizon distance $\ell_{p,\mathrm{hor}}(t)$. Your answer should give the horizon distance in physical units (e.g., meters) and not coordinate units (e.g., notches).

Consider two pieces of comoving matter, A and B, at a coordinate distance ℓ_c from each other. We will consider a photon that is emitted by A at some early time t_A, traveling toward B. The physical distance between A and B at the time of emission is of course $\ell_{p,AB}(t_A) = bt_A^{1/2}\ell_c$, which approaches zero as $t_A \to 0$.

(d) *(2 points)* What is the rate of change of the physical distance between A and B, $\mathrm{d}\ell_{p,AB}(t)/\mathrm{d}t$, at $t = t_A$? Is the physical distance increasing or decreasing? Does the rate of change approach zero, infinity, negative infinity, or a nonzero finite number as $t_A \to 0$?

(e) *(3 points)* At what time t_B is the photon received by B? As $t_A \to 0$, does t_B approach zero, infinity, or a nonzero finite number?

(f) *(3 points)* Calculate $\ell_{p,\gamma B}(t)$, the physical distance between the photon and B at time t, for $t_A \le t \le t_B$.

(g) *(3 points)* What is the rate of change of the physical distance between the photon and B, $\mathrm{d}\ell_{p,\gamma B}(t)/\mathrm{d}t$, at the instant t_A when the photon is emitted?

(h) *(3 points)* At what value of t_A is this rate of change $\mathrm{d}\ell_{p,\gamma B}(t)/\mathrm{d}t$ equal to zero? For earlier values of t_A, is the physical distance between the photon and B increasing or decreasing at the time of emission? As $t_A \to 0$, does $\mathrm{d}\ell_{p,\gamma B}(t)/\mathrm{d}t$ at the time of emission approach zero, infinity, minus infinity, or a nonzero finite number?

PROBLEM 2: EVOLUTION OF AN OPEN, MATTER-DOMINATED UNIVERSE *(35 points)*

The following problem originated on Quiz 2 of 1992 (ancient history!), where it counted 30 points.

The equations describing the evolution of an open, matter-dominated universe were given in Lecture Notes 4 as

$$ct = \alpha\,(\sinh\theta - \theta)$$

and

$$\frac{a}{\sqrt{\kappa}} = \alpha\,(\cosh\theta - 1)\ ,$$

where α is a constant with units of length. The following mathematical identities, which you should know, may also prove useful on parts (e) and (f):

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} \quad , \quad \cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

$$e^{\theta} = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots\ .$$

a) *(5 points)* Find the Hubble expansion rate H as a function of α and θ.

b) *(5 points)* Find the mass density ρ as a function of α and θ.

c) *(5 points)* Find the mass density parameter Ω as a function of α and θ. As with part (c) of the previous problem, the answer to this part appears in Lecture Notes 4. However, you should show that you get the same answer by combining your answers to parts (a) and (b) of this question.

d) *(6 points)* Find the physical value of the horizon distance, $\ell_{p,\text{horizon}}$, as a function of α and θ.

e) *(7 points)* For very small values of t, it is possible to use the first nonzero term of a power-series expansion to express θ as a function of t, and then a as a function of t. Give the expression for $a(t)$ in this approximation. The approximation will be valid for $t \ll t^*$. Estimate the value of t^*.

f) *(7 points)* Even though these equations describe an open universe, one still finds that Ω approaches one for very early times. For $t \ll t^*$ (where t^* is defined in part (e)), the quantity $1 - \Omega$ behaves as a power of t. Find the expression for $1 - \Omega$ in this approximation.

PROBLEM 3: THE CRUNCH OF A CLOSED, MATTER-DOMINATED UNIVERSE *(25 points)*

This is Problem 5.7 (Problem 6.5 in the first edition) from Barbara Ryden's ***Introduction to Cosmology****, with some paraphrasing to make it consistent with the language used in lecture.*

Consider a closed universe containing only nonrelativistic matter. This is the closed universe discussed in Lecture Notes 4, and it is also the "Big Crunch" model discussed in Ryden's section Section 5.4.1 (Section 6.1 in the first edition). At some time during the contracting phase (i.e., when $\theta > \pi$), an astronomer named Elbbuh Niwde discovers that nearby galaxies have blueshifts ($-1 \leq z < 0$) proportional to their distance. He then measures the present values of the Hubble expansion rate, H_0, and the mass density parameter, Ω_0. He finds, of course, that $H_0 < 0$ (because he is in the contracting phase) and $\Omega_0 > 1$ (because the universe is closed). In terms of H_0 and Ω_0, how long a time will elapse between Dr. Niwde's observation at $t = t_0$ and the final Big Crunch at $t = t_{\text{Crunch}} = 2\pi\alpha/c$? Assuming that Dr. Niwde is able to observe all objects within his horizon, what is the most blueshifted (i.e., most negative) value of z that Dr. Niwde is able to see? What is the lookback time to an object with this blueshift? (By lookback time, one means the difference between the time of observation t_0 and the time at which the light was emitted.)

PROBLEM 4: THE AGE OF A MATTER-DOMINATED UNIVERSE AS $\boldsymbol{\Omega \to 1}$ *(15 points)*

The age t of a matter-dominated universe, for any value of Ω, was given in Lecture Notes 4 as

$$|H|\,t = \begin{cases} \dfrac{\Omega}{2(1-\Omega)^{3/2}}\left[\dfrac{2\sqrt{1-\Omega}}{\Omega} - \operatorname{arcsinh}\left(\dfrac{2\sqrt{1-\Omega}}{\Omega}\right)\right] & \text{if } \Omega < 1 \\ 2/3 & \text{if } \Omega = 1 \\ \dfrac{\Omega}{2(\Omega-1)^{3/2}}\left[\arcsin\left(\pm\dfrac{2\sqrt{\Omega-1}}{\Omega}\right) \mp \dfrac{2\sqrt{\Omega-1}}{\Omega}\right] & \text{if } \Omega > 1 \end{cases} \tag{4.47}$$

It was claimed that this formula is continuous at $\Omega = 1$. In this problem you are asked to show half of this statement. Specifically, you should show that as Ω approaches 1 from below, the expression for $|H|t$ approaches $2/3$. In doing this, you may find it useful to use the Taylor expansion for $\operatorname{arcsinh}(x)$ about $x = 0$:

$$\operatorname{arcsinh}(x) = x - \frac{(1)^2}{3!}x^3 + \frac{(3\cdot 1)^2}{5!}x^5 - \frac{(5\cdot 3\cdot 1)^2}{7!}x^7 + \ldots\ .$$

The proof of continuity as $\Omega \to 0$ from above is of course very similar, and you are not asked to show it.

PROBLEM 5: ISOTROPY ABOUT TWO POINTS IN EUCLIDEAN SPACES

(This problem is not required, but can be done for 15 points extra credit. I'd like to give you two weeks to think about it, so you should turn it in with Problem Set 5 on October 16.)

In Steven Weinberg's *The First Three Minutes,* in Chapter 2 on page 24, he gives an argument to show that if a space is isotropic about two distinct points, then it is necessarily homogeneous. He is assuming Euclidean geometry, although he is not explicit about this point. (The statement is simply not true if one allows non-Euclidean spaces — we'll discuss this.) Furthermore, the argument is given in the context of a universe with only two space dimensions, but it could easily be generalized to three, and we will not concern ourselves with remedying this simplification. The statement is true for two-dimensional Euclidean spaces, but Weinberg's argument is not complete. To show that isotropy about two galaxies, 1 and 2, implies that the conditions at any two points A and B must be identical, he constructs two circles. One circle is centered on Galaxy 1 and goes through A, and the other is centered on Galaxy 2 and goes through B. He then argues that the conditions at A and B must both be identical to the conditions at the point C, where the circles intersect. The problem, however, is that the two circles need not intersect. One circle can be completely inside the other, or the two circles can be separated and disjoint. Thus Weinberg's proof is valid for some pairs of points A and B, but cannot be applied to all cases. For 15 points of extra credit, devise a proof that holds in all cases. We have not established axioms for Euclidean geometry, but you may use in your proof any well-known fact about Euclidean geometry.

Total points for Problem Set 4: 95, plus 15 points of extra credit.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 10, 2020
Prof. Alan Guth

PROBLEM SET 5

DUE DATE: Friday, October 16, 2020, at 5:00 pm.

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes,* Chapters 5 and 6, and also Barbara Ryden, *Introduction to Cosmology,* Chapter 7 (*Dark Matter*) (First edition: Chapter 8).

SHORT-TERM CALENDAR:

OCTOBER				
MON	TUES	WED	THURS	FRI
October 12 Indigenous Peoples Day	13 Class 11	14 Class 12	15	16 **PS 5 due**
19 Class 13	20	21 Class 14	22	23 **PS 6 due**
26 Class 15	27	28 Class 16 **Quiz 2**	29	30

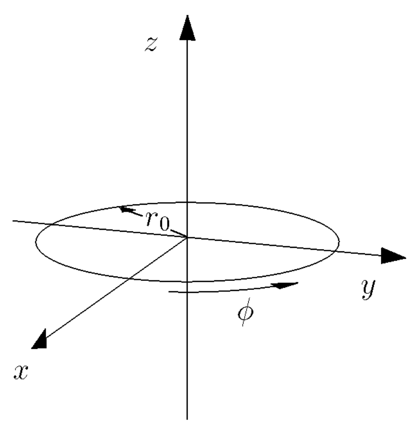
PROBLEM 1: A CIRCLE IN A NON-EUCLIDEAN GEOMETRY
(15 points)

Consider a three-dimensional space described by the following metric:

$$ds^2 = R^2 \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right\} .$$

Here R and k are constants, where k will always have one of the values 1, -1, or 0. θ and ϕ are angular coordinates with the usual properties: $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, where $\phi = 2\pi$ and $\phi = 0$ are identified. r is a radial coordinate, which runs from 0 to 1 if $k = 1$, and otherwise from 0 to ∞. (This is the Robertson-Walker metric of Eq. (5.27) of Lecture Notes 5, evaluated at some particular time t, with $R \equiv a(t)$. You should be able to work this problem, however, whether or not you have gotten that far. The problem requires only that you understand what a metric means.) Consider a circle described by the equations

$$z = 0$$
$$x^2 + y^2 = r_0^2 \quad ,$$



or equivalently by the angular coordinates

$$r = r_0$$
$$\theta = \pi/2 \quad .$$

(a) *(5 points)* Find the circumference S of this circle. Hint: break the circle into infinitesimal segments of angular size $d\phi$, calculate the arc length of such a segment, and integrate.

(b) *(5 points)* Find the radius ρ of this circle. Note that ρ is the length of a line which runs from the origin to the circle ($r = r_0$), along a trajectory of $\theta = \pi/2$ and $\phi =$ constant. Hint: Break the line into infinitesimal segments of coordinate length dr, calculate the length of such a segment, and integrate. Consider the case of open and closed universes separately, and take $k = \pm 1$. (If you don't remember why we can take $k = \pm 1$, see the section called "Units" in Lecture Notes 3,). You will want the following integrals:

$$\int \frac{dr}{\sqrt{1-r^2}} = \sin^{-1} r$$

and

$$\int \frac{dr}{\sqrt{1+r^2}} = \sinh^{-1} r \quad .$$

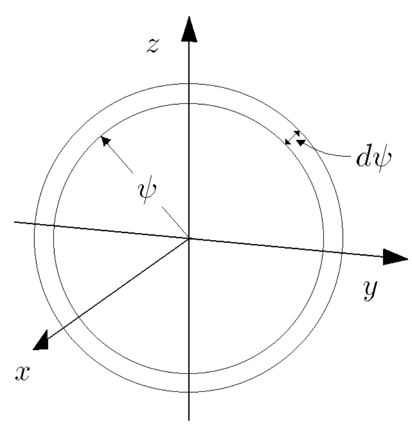
(c) *(5 points)* Express the circumference S in terms of the radius ρ. This result is independent of the coordinate system which was used for the calculation, since S and ρ are both measurable quantities. Since the space described by this metric is homogeneous and isotropic, the answer does not depend on where the circle is located or on how it is oriented. For the two cases of open and closed universes, state whether S is larger or smaller than the value it would have for a Euclidean circle of radius ρ.

PROBLEM 2: VOLUME OF A CLOSED UNIVERSE *(15 points)*

Calculate the total volume of a closed universe, as described by the metric of Eq. (5.14) of Lecture Notes 5:

$$ds^2 = R^2 \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] \quad .$$

Break the volume up into spherical shells of infinitesimal thickness, extending from ψ to $\psi + d\psi$:



By comparing Eq. (5.14) with Eq. (5.8),

$$\mathrm{d}s^2 = R^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \ ,$$

the metric for the surface of a sphere, one can see that as long as ψ is held fixed, the metric for varying θ and ϕ is the same as that for a spherical surface of radius $R\sin\psi$. Thus the area of the spherical surface is $4\pi R^2\sin^2\psi$. To find the volume, multiply this area by the thickness of the shell (which you can read off from the metric), and then integrate over the full range of ψ, from 0 to π.

PROBLEM 3: SURFACE BRIGHTNESS IN A CLOSED UNIVERSE *(25 points)*

The spacetime metric for a homogeneous, isotropic, closed universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ ,$$

where I have taken $k = 1$. To discuss motion in the radial direction, it is more convenient to work with an alternative radial coordinate ψ, related to r by

$$r = \sin\psi \ .$$

Then

$$\frac{dr}{\sqrt{1-r^2}} = d\psi \ ,$$

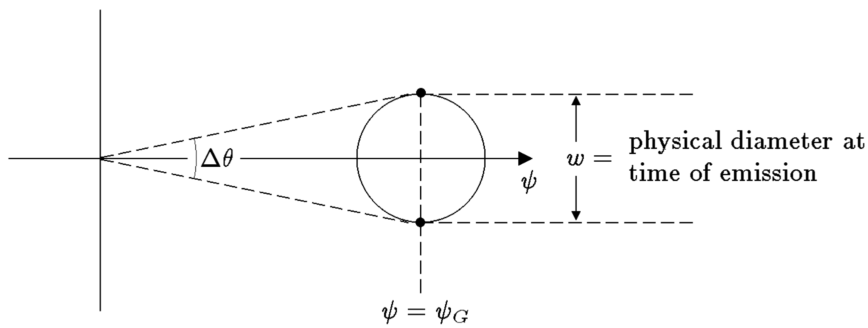
so the metric simplifies to

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + a^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ .$$

The form of $a(t)$ depends on the nature of the matter in the universe, but for this problem you should consider $a(t)$ to be an arbitrary function. You should simplify your answers as far as it is possible without knowing the function $a(t)$.

(a) *(10 points)* Suppose that the Earth is at the center of these coordinates, and that we observe a spherical galaxy that is located at $\psi = \psi_G$. The light that we see was emitted from the galaxy at time t_G, and is being received today, at a time that we call t_0. At the time of emission, the galaxy had a power output P (which could be measured, for example, in watts, where 1 watt = 1 joule/sec). The power was radiated uniformly in all directions, in the form of photons. What is the radiation energy flux J from this galaxy at the Earth today? Energy flux (which might be measured in joule-m^{-2}-sec^{-1}) is defined as the energy per unit area per unit time striking a surface that is orthogonal to the direction of the energy flow. [Hint: it is easiest to use a comoving coordinate system with the radiating galaxy at the origin.]

(b) *(10 points)* Suppose that the physical diameter of the galaxy at time t_G was w. Find the apparent angular size $\Delta\theta$ (measured from one edge to the other) of the galaxy as it would be observed from Earth today.



(c) *(5 points)* The *surface brightness* σ of the distant galaxy is defined to be the energy flux J per solid angle subtended by the galaxy.* Calculate the surface brightness σ of the galaxy described in parts (a) and (b). [Hint: if you have the right answer, it can be written in terms of P, w, and the redshift z, without any reference to ψ_G. The rapid decrease in σ with z means that high-z galaxies are difficult to distinguish from the night sky.]

PROBLEM 4: TRAJECTORIES AND DISTANCES IN AN OPEN UNIVERSE *(30 points)*

The spacetime metric for a homogeneous, isotropic, open universe is given by the Robertson-Walker formula:

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + a^2(t) \left\{ \frac{dr^2}{1+r^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\} ,$$

where I have taken $k = -1$. As in Problem 3, for the discussion of radial motion it is convenient to introduce an alternative radial coordinate ψ, which in this case is related to r by

$$r = \sinh\psi \ .$$

* *Definition of solid angle:* To define the solid angle subtended by the galaxy, imagine surrounding the observer by a sphere of arbitrary radius r. The sphere should be small compared to cosmological distances, so that Euclidean geometry is valid within the sphere. If a picture of the galaxy is painted on the surface of the sphere so that it just covers the real image, then the solid angle, in steradians, is the area of the picture on the sphere, divided by r^2.

Then

$$\frac{dr}{\sqrt{1+r^2}} = d\psi \ ,$$

so the metric simplifies to

$$ds^2 = -c^2\, d\tau^2 = -c^2\, dt^2 + a^2(t)\left\{d\psi^2 + \sinh^2\psi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\} \ .$$

You should treat the function $a(t)$ as a given function. You should simplify your answers as far as it is possible without knowing explicitly the function $a(t)$.

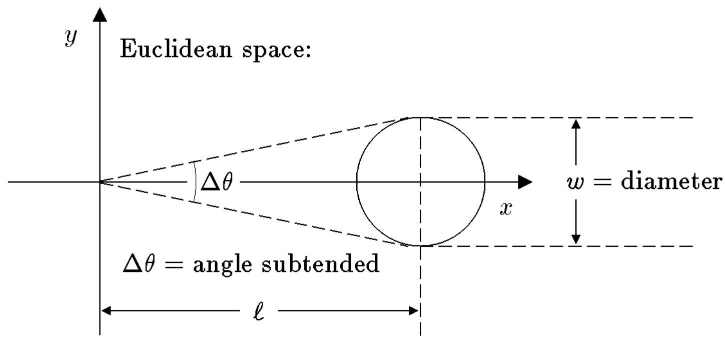
(a) *(5 points)* Suppose that the Earth is at the origin of the coordinate system ($\psi = 0$), and that at the present time, t_0, we receive a light pulse from a distant galaxy G, located at $\psi = \psi_G$. Write down an equation which determines the time t_G at which the light pulse left the galaxy. (You may assume that the light pulse travels on a "null" trajectory, which means that $d\tau = 0$ for any segment of it. Since you don't know $a(t)$ you cannot solve this equation, so please do not try.)

(b) *(5 points)* What is the redshift z_G of the light from galaxy G? (Your answer may depend on t_G, as well as ψ_G, t_0, or any property of the function $a(t)$.)

(c) *(5 points)* To estimate the number of galaxies that one expects to see in a given range of redshifts, it is necessary to know the volume of the region of space that corresponds to this range. Write an expression for the present value of the volume that corresponds to redshifts smaller than that of galaxy G. (You may leave your answer in the form of a definite integral, which may be expressed in terms of ψ_G, t_G, t_0, z_G, or the function $a(t)$.)

(d) *(5 points)* There are a number of different ways of defining distances in cosmology, and generally they are not equal to each other. One choice is called ***proper distance***, which corresponds to the distance that one could in principle measure with rulers. The proper distance is defined as the total length of a network of rulers that are laid end to end from here to the distant galaxy. The rulers have different velocities, because each is at rest with respect to the matter in its own vicinity. They are arranged so that, at the present instant of time, each ruler just touches its neighbors on either side. Write down an expression for the proper distance ℓ_{prop} of galaxy G.

(e) *(5 points)* Another common definition of distance is ***angular size distance,*** determined by measuring the apparent size of an object of known physical size. In a static, Euclidean space, a small sphere of diameter w at a distance ℓ will subtend an angle $\Delta\theta = w/\ell$:

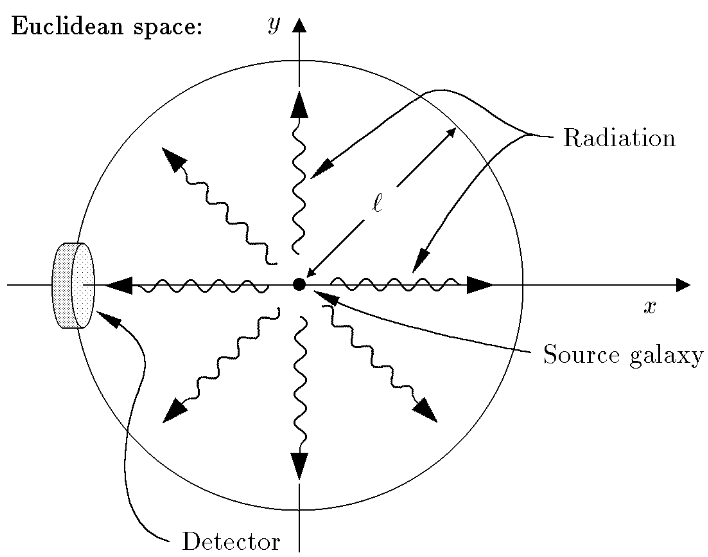


Motivated by this relation, cosmologists define the angular size distance $\ell_{\rm ang}$ of an object by

$$\ell_{\rm ang} \equiv \frac{w}{\Delta\theta} \ .$$

What is the angular size distance $\ell_{\rm ang}$ of galaxy G?

(f) *(5 points)* A third common definition of distance is called ***luminosity distance***, which is determined by measuring the apparent brightness of an object for which the actual total power output is known. In a static, Euclidean space, the energy flux J received from a source of power P at a distance ℓ is given by $J = P/(4\pi\ell^2)$:



Cosmologists therefore define the luminosity distance by

$$\ell_{\rm lum} \equiv \sqrt{\frac{P}{4\pi J}} \ .$$

Find the luminosity distance $\ell_{\rm lum}$ of galaxy G. (Hint: the Robertson-Walker coordinates can be shifted so that the galaxy G is at the origin.)

PROBLEM 5: THE KLEIN DESCRIPTION OF THE G-B-L GEOMETRY

(This problem is not required, but can be done for 15 points extra credit.)

I stated in Lecture Notes 5 that the space invented by Klein, described by the distance relation

$$\cosh\left[\frac{d(1,2)}{a}\right] = \frac{1 - x_1 x_2 - y_1 y_2}{\sqrt{1 - x_1^2 - y_1^2}\,\sqrt{1 - x_2^2 - y_2^2}} \quad ,$$

where

$$x^2 + y^2 < 1 \quad ,$$

is a two-dimensional space of constant negative curvature. In other words, this is just a two-dimensional Robertson–Walker metric, as would be described by a two-dimensional version of Eq. (5.27), with $k = -1$:

$$ds^2 = a^2 \left\{ \frac{dr^2}{1 + r^2} + r^2 d\theta^2 \right\} \quad .$$

The problem is to prove the equivalence.

(a) *(5 points)* As a first step, show that if x and y are replaced by the polar coordinates defined by

$$x = u \cos\theta$$

$$y = u \sin\theta \quad ,$$

then the distance equation can be rewritten as

$$\cosh\left[\frac{d(1,2)}{a}\right] = \frac{1 - u_1 u_2 \cos(\theta_1 - \theta_2)}{\sqrt{1 - u_1^2}\,\sqrt{1 - u_2^2}} \quad .$$

(b) *(5 points)* The next step is to derive the metric from the distance function above. Let

$$u_1 = u \qquad\qquad \theta_1 = \theta \quad ,$$

$$u_2 = u + du \qquad\qquad \theta_2 = \theta + d\theta \quad ,$$

and

$$d(1,2) = ds \quad .$$

Insert these expressions into the distance function, expand everything to second order in the infinitesimal quantities, and show that

$$ds^2 = a^2 \left\{ \frac{du^2}{(1 - u^2)^2} + \frac{u^2 d\theta^2}{1 - u^2} \right\} \quad .$$

(This part is rather messy, but you should be able to do it.)

(c) *(5 points)* Now find the relationship between r and u and show that the two metric functions are identical. Hint: The coefficients of $d\theta^2$ must be the same in the two cases. Can you now see why Klein had to impose the condition $x^2 + y^2 < 1$?

REMINDER: The following extra credit problem from Problem Set 4 is to be turned in with this problem set, if you choose to do it:

PROBLEM 5 (PROBLEM SET 4): ISOTROPY ABOUT TWO POINTS IN EUCLIDEAN SPACES

(This problem is not required, but can be done for 15 points extra credit. It was first posted with Problem Set 4, but is to be turned in with Problem Set 5.)

In Steven Weinberg's *The First Three Minutes,* in Chapter 2 on page 24, he gives an argument to show that if a space is isotropic about two distinct points, then it is necessarily homogeneous. He is assuming Euclidean geometry, although he is not explicit about this point. (The statement is simply not true if one allows non-Euclidean spaces — we'll discuss this.) Furthermore, the argument is given in the context of a universe with only two space dimensions, but it could easily be generalized to three, and we will not concern ourselves with remedying this simplification. The statement is true for two-dimensional Euclidean spaces, but Weinberg's argument is not complete. To show that isotropy about two galaxies, 1 and 2, implies that the conditions at any two points A and B must be identical, he constructs two circles. One circle is centered on Galaxy 1 and goes through A, and the other is centered on Galaxy 2 and goes through B. He then argues that the conditions at A and B must both be identical to the conditions at the point C, where the circles intersect. The problem, however, is that the two circles need not intersect. One circle can be completely inside the other, or the two circles can be separated and disjoint. Thus Weinberg's proof is valid for some pairs of points A and B, but cannot be applied to all cases. For 15 points of extra credit, devise a proof that holds in all cases. We have not established axioms for Euclidean geometry, but you may use in your proof any well-known fact about Euclidean geometry.

Total points for Problem Set 5: 85, plus up to 15 points extra credit.
Also up to 15 points extra credit for Problem Set 4.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 17, 2020
Prof. Alan Guth

PROBLEM SET 6

DUE DATE: Friday, October 23, 2020, at 5:00 pm.

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes,* Chapter 7 (*The First One-Hundredth Second*), and also Barbara Ryden, *Introduction to Cosmology,* Chapter 9 (*Nucleosynthesis and the Early Universe*) [First edition: Chapter 10]. For now we are skipping Chapter 8 (*The Cosmic Microwave Background*), but we will come back to it. Chapter 9 contains some references back to Eqs. (8.26) and (8.28) of Section 8.3 (*The Physics of Recombination*), but I will post some notes that will give you the necessary information. Ryden's treatment in Sections 9.2 and 9.3 of particle number densities in thermal equilibrium is not quite correct, as she leaves out the role of chemical potentials. Her statement in footnote 3 on p. 170 that "chemical potentials are small enough to be safely neglected" is in fact wildly incorrect. Equations for number densities such as Eqs. (9.11) and (9.12) [(10.11) and (10.12) in the first edition] are incorrect, but all of the equations for ratios of number densities (such as (9.13), (9.25), or (9.26) [(10.13), (10.25), or (10.26) in the first edition])) are correct, with the chemical potential factors canceling. The notes that will be posted will correct these errors, and will hopefully give a clear explanation of how chemical potentials are treated.

CALENDAR FOR THE REST OF THE TERM:

OCTOBER/NOVEMBER				
MON	TUES	WED	THURS	FRI
19 Class 13	20	21 Class 14	22	23 **PS 6 due**
26 Class 15	27	28 Class 16 **Quiz 2**	29	30
November 2 Class 17	3	4 Class 18	5	6 **PS 7 due**

NOVEMBER/DECEMBER				
MON	TUES	WED	THURS	FRI
9 Class 19	10	11 **Veteran's Day**	12	13
16 Class 20	17	18 Class 21	19	20 **PS 8 due**
23 **Thanksgiving Week**	24 —	25 —	26 —	27 —
30 Class 22	December 1	2 Class 23 **Quiz 3**	3	4
7 Class 24	8	9 Class 25 **PS 9 due** Last Class	10	11

PROBLEM 1: GEODESICS IN A FLAT UNIVERSE *(25 points)*

According to general relativity, in the absence of any non-gravitational forces a particle will travel along a spacetime geodesic. In this sense, gravity is reduced to a distortion in spacetime.

Consider the case of a flat (*i.e.*, $k = 0$) Robertson–Walker metric, which has the simple form

$$ds^2 = -c^2 dt^2 + a^2(t) \left[dx^2 + dy^2 + dz^2 \right] \quad .$$

Since the spatial metric is flat, we have the option of writing it in terms of Cartesian rather than polar coordinates. Now consider a particle which moves along the x-axis. (Note that the galaxies are on the average at rest in this system, but one can still discuss the trajectory of a particle which moves through the model universe.)

(a) *(8 points)* Use the geodesic equation to show that the coordinate velocity computed with respect to proper time (*i.e.*, $dx/d\tau$) falls off as $1/a^2(t)$.

(b) *(8 points)* Use the expression for the spacetime metric to relate dx/dt to $dx/d\tau$.

(c) *(9 points)* The physical velocity of the particle relative to the galaxies that it is passing is given by

$$v = a(t)\frac{dx}{dt} \quad .$$

(Note that this is just a generalization of what we have previously said for photons, $dx/dt = c/a(t)$.) Show that the momentum of the particle, defined relativistically by

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

falls off as $1/a(t)$. (This implies, by the way, that if the particle were described as a quantum mechanical wave with wavelength $\lambda = h/|\vec{p}|$, then its wavelength would stretch with the expansion of the universe, in the same way that the wavelength of light is redshifted.)

PROBLEM 2: METRIC OF A STATIC GRAVITATIONAL FIELD *(25 points)*

In this problem we will consider the metric

$$ds^2 = -\left[c^2 + 2\phi(\vec{x})\right]\, dt^2 + \sum_{i=1}^{3}\left(dx^i\right)^2 \quad ,$$

which describes a static gravitational field. Here i runs from 1 to 3, with the identifications $x^1 \equiv x$, $x^2 \equiv y$, and $x^3 \equiv z$. The function $\phi(\vec{x})$ depends only on the spatial variables $\vec{x} \equiv (x^1, x^2, x^3)$, and not on the time coordinate t.

(a) *(5 points)* Suppose that a radio transmitter, located at $\vec{x}_e$, emits a series of evenly spaced pulses. The pulses are separated by a proper time interval ΔT_e, as measured by a clock at the same location. What is the coordinate time interval Δt_e between the emission of the pulses? (I.e., Δt_e is the difference between the time coordinate t at the emission of one pulse and the time coordinate t at the emission of the next pulse.)

(b) *(5 points)* The pulses are received by an observer at $\vec{x}_r$, who measures the time of arrival of each pulse. What is the ***coordinate*** time interval Δt_r between the reception of successive pulses?

(c) *(5 points)* The observer uses his own clocks to measure the proper time interval ΔT_r between the reception of successive pulses. Find this time interval, and also the redshift z, defined by

$$1 + z = \frac{\Delta T_r}{\Delta T_e} \quad .$$

First compute an exact expression for z, and then expand the answer to lowest order in $\phi(\vec{x})$ to obtain a weak-field approximation. (This weak-field approximation is in fact highly accurate in all terrestrial and solar system applications.)

(d) *(5 points)* A freely falling particle travels on a spacetime geodesic $x^\mu(\tau)$, where τ is the proper time. (I.e., τ is the time that would be measured by a clock moving with the particle.) The trajectory is described by the geodesic equation

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau} \quad ,$$

where the Greek indices ($\mu, \nu, \lambda, \sigma$, etc.) run from 0 to 3, and are summed over when repeated. Calculate an explicit expression for

$$\frac{d^2x^i}{d\tau^2} \; ,$$

valid for $i = 1, 2$, or 3. (It is acceptable to leave quantities such as $dt/d\tau$ or $dx^i/d\tau$ in the answer.)

(e) *(5 points)* In the weak-field nonrelativistic-velocity approximation, the answer to the previous part reduces to

$$\frac{d^2x^i}{dt^2} = -\partial_i\phi \; ,$$

so $\phi(\vec{x})$ can be identified as the Newtonian gravitational potential. Use this fact to estimate the gravitational redshift z of a photon that rises from the floor of this room to the ceiling (say 4 meters). (One significant figure will be sufficient.)

PROBLEM 3: CIRCULAR ORBITS IN A SCHWARZSCHILD METRIC *(30 points)*

READ THIS: *This problem was Problem 16 of Review Problems for Quiz 2 of 2011, and the solution is posted as http://web.mit.edu/8.286/www/quiz11/ecqr2-1.pdf. Like Problem 4 of Problem Set 3, but unlike all other homework problems so far, in this case you are encouraged to look at the solutions and benefit from them. When you write your solution, you can even copy it verbatim from these solutions if you wish, although obviously you will learn more if you think about the solution and write your own version.*

The Schwarzschild metric, which describes the external gravitational field of any spherically symmetric distribution of mass (including black holes), is given by

$$ds^2 = -c^2d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2 \; ,$$

where M is the total mass of the object, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, and $\phi = 2\pi$ is identified with $\phi = 0$. We will be concerned only with motion outside the Schwarzschild horizon $R_S = 2GM/c^2$, so we can take $r > R_S$. (This restriction allows us to avoid the complications of understanding the effects of the singularity at $r = R_S$.) In this problem we will use the geodesic equation to calculate the behavior of circular orbits in this metric. We will assume a perfectly circular orbit in the x-y plane: the radial coordinate r is fixed, $\theta = 90°$, and $\phi = \omega t$, for some angular velocity ω.

(a) *(7 points)* Use the metric to find the proper time interval $d\tau$ for a segment of the path corresponding to a coordinate time interval dt. Note that $d\tau$ represents the time that would actually be measured by a clock moving with the orbiting body. Your result should show that

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2GM}{rc^2} - \frac{r^2\omega^2}{c^2}} \; .$$

Note that for $M = 0$ this reduces to the special relativistic relation $d\tau/dt = \sqrt{1 - v^2/c^2}$, but the extra term proportional to M describes an effect that is new with general relativity— the gravitational field causes clocks to slow down, just as motion does.

(b) *(7 points)* Show that the geodesic equation of motion (Eq. (5.65)) for one of the coordinates takes the form

$$0 = \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial r}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{1}{2}\frac{\partial g_{tt}}{\partial r}\left(\frac{dt}{d\tau}\right)^2 \; .$$

(c) *(8 points)* Show that the above equation implies

$$r\left(\frac{d\phi}{d\tau}\right)^2 = \frac{GM}{r^2}\left(\frac{dt}{d\tau}\right)^2 \; ,$$

which in turn implies that

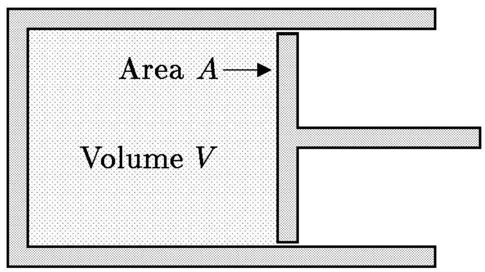
$$r\omega^2 = \frac{GM}{r^2} \; .$$

Thus, the relation between r and ω is exactly the same as in Newtonian mechanics. *[Note, however, that this does not really mean that general relativity has no effect. First, ω has been defined by $d\phi/dt$, where t is a time coordinate which is not the same as the proper time τ that would be measured by a clock on the orbiting body. Second, r does not really have the same meaning as in the Newtonian calculation, since it is not the measured distance from the center of motion. Measured distances, you will recall, are calculated by integrating the metric, as for example in Problem 1 of Problem Set 5, A Circle in a Non-Euclidean Geometry. Since the angular ($d\theta^2$ and $d\phi^2$) terms in the Schwarzschild metric are unaffected by the mass, however, it can be seen that the circumference of the circle is equal to $2\pi r$, as in the Newtonian calculation.]*

(d) *(8 points)* Show that circular orbits around a black hole have a minimum value of the radial coordinate r, which is larger than R_S. What is it?

PROBLEM 4: GAS PRESSURE AND ENERGY CONSERVATION *(25 points)*

In this problem we will pursue the implications of the conservation of energy. Consider first a gas contained in a chamber with a movable piston, as shown below:



Let U denote the total energy of the gas, and let p denote the pressure. Suppose that the piston is moved a distance dx to the right. (We suppose that the motion is slow, so that the gas particles have time to respond and to maintain a uniform pressure throughout the volume.) The gas exerts a force pA on the piston, so the gas does work $dW = pAdx$ as the piston is moved. Note that the volume increases by an amount $dV = Adx$, so $dW = pdV$. The energy of the gas decreases by this amount, so

$$dU = -pdV \ . \tag{P4.1}$$

It turns out that this relation is valid whenever the volume of a gas is changed, regardless of the shape of the volume.

Now consider a homogeneous, isotropic, expanding universe, described by a scale factor $a(t)$. Let u denote the energy density of the gas that fills it. (Remember that $u = \rho c^2$, where ρ is the mass density of the gas.) We will consider a fixed coordinate volume V_{coord}, so the physical volume will vary as

$$V_{\text{phys}}(t) = a^3(t) V_{\text{coord}} \ . \tag{P4.2}$$

The energy of the gas in this region is then given by

$$U = V_{\text{phys}} u \ . \tag{P4.3}$$

(a) *(9 points)* Using these relations, show that

$$\frac{d}{dt}\left(a^3 \rho c^2\right) = -p \frac{d}{dt}(a^3) \ , \tag{P4.4}$$

and then that

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) , \tag{P4.5}$$

where the dot denotes differentiation with respect to t.

(b) *(8 points)* The scale factor evolves according to the relation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} . \tag{P4.6}$$

Using Eqs. (P4.5) and (P4.6), show that

$$\ddot{a} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)a . \tag{P4.7}$$

This equation describes directly the deceleration of the cosmic expansion. Note that there are contributions from the mass density ρ, but also from the pressure p.

(c) *(8 points)* So far our equations have been valid for any sort of a gas, but let us now specialize to the case of black-body radiation. For this case we know that $\rho = bT^4$, where b is a constant and T is the temperature. We also know that as the universe expands, aT remains constant. Using these facts and Eq. (P4.5), find an expression for p in terms of ρ.

PROBLEM 5: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION *(25 points)*

A radiation-dominated universe behaves differently from a matter-dominated universe because the pressure of the radiation is significant. In this problem we explore the role of pressure for several fictitious forms of matter.

(a) *(8 points)* For the first fictitious form of matter, the mass density ρ decreases as the scale factor $a(t)$ grows, with the relation

$$\rho(t) \propto \frac{1}{a^6(t)} .$$

What is the pressure of this form of matter? *[Hint: the answer is proportional to the mass density.]*

(b) *(9 points)* Find the behavior of the scale factor $a(t)$ for a flat universe dominated by the form of matter described in part (a). You should be able to determine the function $a(t)$ up to a constant factor.

(c) *(8 points)* Now consider a universe dominated by a different form of fictitious matter, with a pressure given by

$$p = \frac{1}{2}\rho c^2 .$$

As the universe expands, the mass density of this form of matter behaves as

$$\rho(t) \propto \frac{1}{a^n(t)} .$$

Find the power n.

PROBLEM 6: TIME EVOLUTION OF A UNIVERSE WITH MYSTERIOUS STUFF *(15 points)*

Suppose that a model universe is filled with a peculiar form of matter for which

$$\rho \propto \frac{1}{a^5(t)} \ .$$

Assuming that the model universe is flat, calculate

(a) *(4 points)* The behavior of the scale factor, $a(t)$. You should be able to find $a(t)$ up to an arbitrary constant of proportionality.

(b) *(3 points)* The value of the Hubble parameter $H(t)$, as a function of t.

(c) *(4 points)* The physical horizon distance, $\ell_{p,\mathrm{horizon}}(t)$.

(d) *(4 points)* The mass density $\rho(t)$.

Total points for Problem Set 6: 145.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — October 31, 2020
Prof. Alan Guth

PROBLEM SET 7

DUE DATE: Friday, November 6, 2020, at 5:00 pm.

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes,* Chapter 8 (*Epilogue: The Prospect Ahead*), and *Afterword: Cosmology Since 1977.* Barbara Ryden, *Introduction to Cosmology,* Chapter 8 [First edition: Chapter 7] (*The Cosmic Microwave Background*).

CALENDAR FOR THE REST OF THE TERM:

NOVEMBER				
MON	TUES	WED	THURS	FRI
November 2 Class 17	3	4 Class 18	5	6 **PS 7 due**
9 Class 19	10	11 **Veteran's Day**	12	13
16 Class 20	17	18 Class 21	19	20 **PS 8 due**
23 **Thanksgiving Week**	24 —	25 —	26 —	27 —
30 Class 22	December 1	2 Class 23 **Quiz 3**	3	4
7 Class 24	8	9 Class 25 **PS 9 due** Last Class	10	11

PROBLEM 1: EFFECT OF AN EXTRA NEUTRINO SPECIES *(15 points)*

According to the standard assumptions (which were used in the lecture notes), there are three species of effectively massless neutrinos. In the temperature range of 1 MeV $<$ $kT < 100$ MeV, the mass density of the universe is believed to have been dominated by the black-body radiation of photons, electron-positron pairs, and these neutrinos, all of which were in thermal equilibrium.

(a) *(5 points)* Under these assumptions, how long did it take (starting from the instant of the big bang) for the temperature to fall to the value such that $kT = 1$ MeV? (In this part and the next, you may assume that the period when $kT > 100$ MeV was so short that one can calculate as if the value of g that you find for 1 MeV $< kT < 100$ MeV can be used for earlier times as well.)

(b) *(5 points)* How much time would it have taken if there were one other species of massless neutrino, in addition to the three which we are currently assuming?

(c) *(5 points)* What would be the mass density of the universe when $kT = 1$ MeV under the standard assumptions, and what would it be if there were one other species of massless neutrino?

PROBLEM 2: ENTROPY AND THE BACKGROUND NEUTRINO TEMPERATURE *(15 points)*

The formula for the entropy density of black-body radiation is given in Lecture Notes 6. The derivation of this formula has been left to the statistical mechanics course that you either have taken or hopefully will take. For our purposes, the important point is that the early universe remains very close to thermal equilibrium, and therefore entropy is conserved. The conservation of entropy applies even during periods when particles, such as electron-positron pairs, are "freezing out" of the thermal equilibrium mix. Since total entropy is conserved, the entropy density falls off as $1/a^3(t)$.

When the electron-positron pairs disappear from the thermal equilibrium mixture as kT falls below $m_e c^2 = 0.511$ MeV, the weak interactions have such low cross sections that the neutrinos have essentially decoupled. To a good approximation, all of the energy and entropy released by the annihilation of electrons and positrons is added to the photon gas, and the neutrinos are unaffected. Use the conservation of entropy to show that as electron-positron pair annihilation takes place, aT_γ increases by a factor of $(11/4)^{1/3}$, while aT_ν remains constant. It follows that after the disappearance of the electron-positron pairs, $T_\nu/T_\gamma = (4/11)^{1/3}$. As far as we know, nothing happens that significantly affects this ratio right up to the present day. So we expect today a background of thermal neutrinos which are slightly colder than the 2.7°K background of photons.

Added note: In principle the heating of the photon gas due to electron-positron annihilation can also be calculated by using energy conservation, but it is much more difficult. Since

$$\dot{\rho} = -3H\left(\rho + \frac{p}{c^2}\right)$$

(this was Eq. (6.36) of Lecture Notes 6), one needs to know $p(t)$ to understand the changes in energy density. But as the electron-positron pairs are disappearing, kT is comparable to the electon rest mass $m_e c^2$, and the formula for the thermal equilibrium pressure under these circumstances is complicated.

PROBLEM 3: FREEZE-OUT OF MUONS *(25 points)*

A particle called the muon seems to be essentially identical to the electron, except that it is heavier— the mass/energy of a muon is 106 MeV, compared to 0.511 MeV for the electron. The muon (μ^-) has the same charge as an electron, denoted by $-e$. There is also an antimuon (μ^+), analogous to the positron, with charge $+e$. The muon and antimuon have the same spin as the electron. There is no known particle with a mass between that of an electron and that of a muon.

(a) The formula for the energy density of black-body radiation, as given by Eq. (6.48) of the lecture notes,

$$u = g\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3} \ ,$$

is written in terms of a normalization constant g. What is the value of g for the muons, taking μ^+ and μ^- together?

(b) When kT is just above 106 MeV as the universe cools, what particles besides the muons are contained in the thermal radiation that fills the universe? What is the contribution to g from each of these particles?

(c) As kT falls below 106 MeV, the muons disappear from the thermal equilibrium radiation. At these temperatures all of the other particles in the black-body radiation are interacting fast enough to maintain equilibrium, so the heat given off from the muons is shared among all the other particles. Letting a denote the Robertson-Walker scale factor, by what factor does the quantity aT increase when the muons disappear?

PROBLEM 4: THE REDSHIFT OF THE COSMIC MICROWAVE BACKGROUND *(25 points)*

It was mentioned in Lecture Notes 6 that the black-body spectrum has the peculiar feature that it maintains its form under uniform redshift. That is, as the universe expands, even if the photons do not interact with anything, they will continue to be described by a black-body spectrum, although at a temperature that decreases as the universe expands. Thus, even though the cosmic microwave background (CMB) has not been interacting significantly with matter since 350,000 years after the big bang, the radiation today still has a black-body spectrum. In this problem we will demonstrate this important property of the black-body spectrum.

The spectral energy density $\rho_\nu(\nu, T)$ for the thermal (black-body) radiation of photons at temperature T was stated in Lecture Notes 6 as Eq. (6.75), which we can rewrite as

$$\rho_\nu(\nu, T) = \frac{16\pi^2\hbar\nu^3}{c^3}\frac{1}{e^{h\nu/kT} - 1} , \tag{4.1}$$

where $h = 2\pi\hbar$ is Planck's original constant. $\rho_\nu(\nu, T)\, d\nu$ is the energy per unit volume carried by photons whose frequency is in the interval $[\nu, \nu + d\nu]$. In this problem we will assume that this formula holds at some initial time t_1, when the temperature had some value T_1. We will let $\tilde{\rho}(\nu, t)$ denote the spectral distribution for photons in the universe, which is a function of frequency ν and time t. Thus, our assumption about the initial condition can be expressed as

$$\tilde{\rho}(\nu, t_1) = \rho_\nu(\nu, T_1) . \tag{4.2}$$

The photons redshift as the universe expands, and to a good approximation the redshift and the dilution of photons due to the expansion are the only physical effects that cause the distribution of photons to evolve. Thus, using our knowledge of the redshift, we can calculate the spectral distribution $\tilde{\rho}(\nu, t_2)$ at some later time $t_2 > t_1$. It is not obvious that $\tilde{\rho}(\nu, t_2)$ will be a thermal distribution, but in fact we will be able to show that

$$\tilde{\rho}(\nu, t_2) = \rho\big(\nu, T(t_2)\big) , \tag{4.3}$$

where in fact $T(t_2)$ will agree with what we already know about the evolution of T in a radiation-dominated universe:

$$T(t_2) = \frac{a(t_1)}{a(t_2)}\, T_1 . \tag{4.4}$$

To follow the evolution of the photons from time t_1 to time t_2, we can imagine selecting a region of comoving coordinates with coordinate volume V_c. Within this comoving volume, we can imagine tagging all the photons in a specified infinitesimal range of frequencies, those between ν_1 and $\nu_1 + d\nu_1$. Recalling that the energy of each such photon is $h\nu$, the number dN_1 of tagged photons is then

$$dN_1 = \frac{\tilde{\rho}(\nu_1, t_1)\, a^3(t_1)\, V_c\, d\nu_1}{h\nu_1} . \tag{4.5}$$

(a) We now wish to follow the photons in this frequency range from time t_1 to time t_2, during which time each photon redshifts. At the latter time we can denote the range of frequencies by ν_2 to $\nu_2 + d\nu_2$. Express ν_2 and $d\nu_2$ in terms of ν_1 and $d\nu_1$, assuming that the scale factor $a(t)$ is given.

(b) At time t_2 we can imagine tagging all the photons in the frequency range ν_2 to $\nu_2 + d\nu_2$ that are found in the original comoving region with coordinate volume V_c. Explain why the number dN_2 of such photons, on average, will equal dN_1 as calculated in Eq. (4.5).

(c) Since $\tilde{\rho}(\nu, t_2)$ denotes the spectral energy density at time t_2, we can write

$$dN_2 = \frac{\tilde{\rho}(\nu_2, t_2)\, a^3(t_2)\, V_c\, d\nu_2}{h\nu_2} \ , \tag{4.6}$$

using the same logic as in Eq. (4.5). Use $dN_2 = dN_1$ to show that

$$\tilde{\rho}(\nu_2, t_2) = \frac{a^3(t_1)}{a^3(t_2)}\, \tilde{\rho}(\nu_1, t_1) \ . \tag{4.7}$$

Use the above equation to show that Eq. (4.3) is satisfied, for $T(t)$ given by Eq. (4.4).

Total points for Problem Set 7: 80.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe November 7, 2020
Prof. Alan Guth

PROBLEM SET 8

DUE DATE: Friday, November 20, 2020, at 5:00 pm. This is the last problem set before Quiz 3, which will be Wednesday, December 2. There will also be a Problem Set 9, to be due Wednesday, December 9, the last day of classes.

READING ASSIGNMENT: Barbara Ryden, *Introduction to Cosmology,* Chapter 10 *(Inflation and the Very Early Universe)* [First edition: Chapter 11]. Also read *Inflation and the New Era of High-Precision Cosmology,* by Alan Guth, written for the MIT Physics Department annual newsletter, 2002. It is available at

http://web.mit.edu/physics/news/physicsatmit/physicsatmit_02_cosmology.pdf

The data quoted in the article about the nonuniformities of the cosmic microwave background radiation has since been superceded by much better data, but the conclusions have not changed. They have only gotten stronger.

CALENDAR FOR THE REST OF THE TERM:

NOVEMBER/DECEMBER				
MON	TUES	WED	THURS	FRI
9 Class 19	10	11 **Veteran's Day**	12	13
16 Class 20	17	18 Class 21	19	20 **PS 8 due**
23 **Thanksgiving Week**	24 —	25 —	26 —	27 —
30 Class 22	December 1	2 Class 23 **Quiz 3**	3	4
7 Class 24	8	9 Class 25 **PS 9 due** Last Class	10	11

PROBLEM 1: BIG BANG NUCLEOSYNTHESIS *(20 points)*

The calculations of big bang nucleosynthesis depend on a large number of measured parameters. Below you are asked to qualitatively describe the effects of changing some of these parameters. Include a sentence or two to explain each of your answers. (These topics have not been discussed in class, but you are expected to be able to answer the questions on the basis of your readings in Weinberg's and Ryden's books.)

(a) *(5 points)* Suppose an extra neutrino species is added to the calculation. Would the predicted helium abundance go up or down?

(b) *(5 points)* Suppose the weak interactions were stronger than they actually are, so that the thermal equilibrium distribution between neutrons and protons were maintained until $kT \approx 0.25$ MeV. Would the predicted helium abundance be larger or smaller than in the standard model?

(c) *(5 points)* Suppose the proton-neutron mass difference were larger than the actual value of 1.29 MeV/c^2. Would the predicted helium abundance be larger or smaller than in the standard calculation?

(d) *(5 points)* The standard theory of big bang nucleosynthesis assumes that the matter in the universe was distributed homogeneously during the era of nucleosynthesis, but the alternative possibility of inhomogeneous big-bang nucleosynthesis has been discussed since the 1980s. Inhomogeneous nucleosynthesis hinges on the hypothesis that baryons became clumped during a phase transition at $t \approx 10^{-6}$ second, when the hot quark soup converted to a gas of mainly protons, neutrons, and in the early stages, pions. The baryons would then be concentrated in small nuggets, with a comparatively low density outside of these nuggets. After the phase transition but before nucleosynthesis, the neutrons would have the opportunity to diffuse away from these nuggets, becoming more or less uniformly distributed in space. The protons, however, since they are charged, interact electromagnetically with the plasma that fills the universe, and therefore have a much shorter mean free path than the neutrons. Most of the protons, therefore, remain concentrated in the nuggets. Does this scenario result in an increase or a decrease in the expected helium abundance?

PROBLEM 2: BRIGHTNESS VS. REDSHIFT WITH A POSSIBLE COSMOLOGICAL CONSTANT *(25 points)*

In Lecture Notes 7, we derived the relation between the power output P of a source and the energy flux J, for the case of a closed universe:

$$J = \frac{PH_0^2|\Omega_{k,0}|}{4\pi(1+z_S)^2c^2\sin^2\psi(z_S)} ,$$

where

$$\psi(z_S) = \sqrt{|\Omega_{k,0}|}\int_0^{z_S} \frac{dz}{\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\rm rad,0}(1+z)^4 + \Omega_{\rm vac,0} + \Omega_{k,0}(1+z)^2}} .$$

Here z_S denotes the observed redshift, H_0 denotes the present value of the Hubble expansion rate, $\Omega_{m,0}$, $\Omega_{\rm rad,0}$, and $\Omega_{\rm vac,0}$ denote the present contributions to Ω from nonrelativistic matter, radiation, and vacuum energy, respectively, and $\Omega_{k,0} \equiv 1 - \Omega_{m,0} - \Omega_{\rm rad,0} - \Omega_{\rm vac,0}$.

(a) Derive the corresponding formula for the case of an open universe. You can of course follow the same logic as the derivation in the lecture notes, but the solution you write should be complete and self-contained. (I.e., you should **NOT** say "the derivation is the same as the lecture notes except for . . . .")

(b) Derive the corresponding formula for the case of a flat universe. Here there is of course no need to repeat anything that you have already done in part (a). If you wish you can start with the answer for an open or closed universe, taking the limit as $k \to 0$. The limit is delicate, however, because both the numerator and denominator of the equation for J vanish as $\Omega_{k,0} \to 0$.

PROBLEM 3: AGE OF A UNIVERSE WITH MYSTERIOUS STUFF *(20 points)*

READ THIS: *This problem was Problem 8 of Review Problems for Quiz 3 of 2011, and the solution is posted as http://web.mit.edu/8.286/www/quiz11/ecqr3-1.pdf. Like Problem 4 of Problem Set 3 and Problem 3 of Problem Set 6, but unlike all other homework problems so far, in this case you are encouraged to look at the solutions and benefit from them. When you write your solution, you can even copy it verbatim from these solutions if you wish, although obviously you will learn more if you think about the solution and write your own version.*

Consider a universe that contains nonrelativistic matter, radiation, vacuum energy, and the same ***mysterious stuff*** that was introduced in Problem 7 of Review Problems for Quiz 3, from 2011. Since the mass density of mysterious stuff falls off as $1/\sqrt{V}$, where V is the volume, it follows that in an expanding universe the mass density of mysterious stuff falls off as $1/a^{3/2}(t)$.

Suppose that you are given the present value of the Hubble expansion rate H_0, and also the present values of the contributions to $\Omega \equiv \rho/\rho_c$ from each of the constituents: $\Omega_{m,0}$ (nonrelativistic matter), $\Omega_{r,0}$ (radiation), $\Omega_{v,0}$ (vacuum energy density), and $\Omega_{\mathrm{ms},0}$ (mysterious stuff). Our goal is to express the age of the universe t_0 in terms of these quantities.

(a) *(10 points)* Let $x(t)$ denote the ratio

$$x(t) \equiv \frac{a(t)}{a(t_0)}$$

for an arbitrary time t. Write an expression for the total mass density of the universe $\rho(t)$ in terms of $x(t)$ and the given quantities described above.

(b) *(10 points)* Write an integral expression for the age of the universe t_0. The expression should depend only on H_0 and the various contributions to Ω_0 listed above ($\Omega_{m,0}$, $\Omega_{r,0}$, etc.), but it might include x as a variable of integration.

PROBLEM 4: SHARED CAUSAL PAST *(20 points)*

Recently several of my colleagues published a paper (Andrew S. Friedman, David I. Kaiser, and Jason Gallicchio, "The Shared Causal Pasts and Futures of Cosmological Events," http://arxiv.org/abs/arXiv:1305.3943, *Physical Review D,* Vol. 88, article 044038 (2013)) in which they investigated the causal connections in the standard cosmological model. In particular, they calculated the present redshift z of a distant quasar which has the property that a light signal, if sent from our own location at the instant of the big bang, would have just enough time to reach the quasar and return to us, so that we could see the reflection of the signal at the present time. They found $z = 3.65$, using $\Omega_{\mathrm{matter},0} = 0.315$, $\Omega_{\mathrm{rad},0} = 9.29 \times 10^{-5}$, $\Omega_{\mathrm{vac},0} = 0.685 - \Omega_{\mathrm{rad},0}$, and $H_0 = 67.3$ km-s^{-1}-Mpc^{-1}. Feel free to read their paper if you like. Your job, however, is to carry out an independent calculation to find out if they got it right.

(a) *(15 points)* Write an equation that determines this redshift z. The equation may involve one or more integrals which are not evaluated, and the equation itself does not have to be solved.

(b) *(5 points)* The integrals that should appear in your answer to part (a) can be evaluated numerically, and the whole equation you found in part (a) can be solved numerically. Do this, and see how your z compares with 3.65.

PROBLEM 5: MASS DENSITY OF VACUUM FLUCTUATIONS *(25 points)*

The energy density of vacuum fluctuations has been discussed qualitatively in lecture. In this problem we will calculate in detail the energy density associated with quantum fluctuations of the electromagnetic field. To keep the problem finite, we will not consider all of space at once, but instead we will consider the electromagnetic field inside a cube of side L, defined by coordinates

$$0 \leq x \leq L \ ,$$

$$0 \leq y \leq L \ ,$$

$$0 \leq z \leq L \ .$$

Our goal, however, will be to compute the energy density in the limit as the size of the box is taken to infinity.

(a) *(10 points)* The electromagnetic waves inside the box can be decomposed into a Fourier sum of sinusoidal normal modes. Suppose we consider only modes that extend up to a maximum wave number $k_{\rm max}$, or equivalently modes that extend down to a minimum wavelength $\lambda_{\rm min}$, where

$$k_{\rm max} = \frac{2\pi}{\lambda_{\rm min}} \ .$$

How many such modes are there? I do not expect an exact answer, but your approximations should become arbitrarily accurate when $\lambda_{\rm min} \ll L$. (These mode counting techniques are probably familiar to many of you, but in case they are not I have attached an extended hint after part (c).)

(b) *(10 points)* When the electromagnetic field is described quantum mechanically, each normal mode behaves exactly as a harmonic oscillator: if the angular frequency of the mode is ω, then the quantized energy levels have energies given by

$$E_n = \left(n + \tfrac{1}{2}\right) \hbar\omega \ ,$$

where $\hbar$ is Planck's original constant divided by 2π, and n is an integer. The integer n is called the "occupation number," and is interpreted as the number of photons in the specified mode. The minimum energy is not zero, but instead is $\frac{1}{2}\hbar\omega$, which is the energy of the quantum fluctuations of the electromagnetic field. Assuming that the mode sum is cut off at $\lambda_{\rm min}$ equal to the Planck length (as defined in the Lecture Notes), what is the total mass density of these quantum fluctuations?

(c) *(5 points)* How does the mass density of the quantum fluctuations of the electromagnetic field compare with the critical density of our universe?

Extended Hint:

The electromagnetic fields inside a closed box can be expanded as the sum of modes, each of which has a sinusoidally varying time dependence, but the precise form of these modes depends on the nature of the boundary conditions on the walls of the box. Physically reasonable boundary conditions, such as total reflection, are in fact difficult to use in calculations. However, it is known that in the limit of an infinite-sized box, the nature of the boundary conditions will not make any difference. We are therefore free to choose the simplest boundary conditions that we can imagine, and for this purpose we will choose periodic boundary conditions. That is, we will assume that the fields and their normal derivatives on each wall are fixed to precisely match the fields and their normal derivatives on the opposite wall.

To begin, we consider a wave in one dimension, moving at the speed of light. Such waves are most easily described in terms of complex exponentials. If $A(x,t)$ represents the amplitude of the wave, then a sinusoidal wave moving in the positive x-direction can be written as

$$A(x,t) = \mathrm{Re}\left[Be^{ik(x-ct)}\right] \ ,$$

where B is a complex constant and k is a real constant. Defining $\omega = c|k|$, waves in either direction can be written as

$$A(x,t) = \mathrm{Re}\left[Be^{i(kx-\omega t)}\right] \ ,$$

where the sign of k determines the direction. To be periodic with period L, the parameter k must satisfy

$$kL = 2\pi n \ ,$$

where n is an integer. So the spacing between modes is $\Delta k = 2\pi/L$. The density of modes dN/dk (i.e., the number of modes per interval of k) is then one divided by the spacing, or $1/\Delta k$, so

$$\frac{dN}{dk} = \frac{L}{2\pi} \quad \text{(one dimension)} \ .$$

In three dimensions, a sinusoidal wave can be written as

$$A(\vec{x},t) = \mathrm{Re}\left[Be^{i(\vec{k}\cdot\vec{x}-\omega t)}\right] \ ,$$

where $\omega = c|\vec{k}|$, and

$$k_x L = 2\pi n_x \ , \quad k_y L = 2\pi n_y \ , \quad k_z L = 2\pi n_z \ ,$$

where n_x, n_y, and n_z are integers. Thus, in three-dimensional $\vec{k}$-space the allowed values of $\vec{k}$ lie on a cubical lattice, with spacing $2\pi/L$. In counting the modes, one should also remember that for photons there is an extra factor of 2 associated with the fact that electromagnetic waves have two possible polarizations for each allowed value of $\vec{k}$.

PROBLEM 6: PLOTTING THE SUPERNOVA DATA *(EXTRA CREDIT, 20 pts)*

The original data on the Hubble diagram based on Type Ia supernovae are found in two papers. One paper is authored by the High Z Supernova Search Team,* and the other is by the Supernova Cosmology Project.† More recent data from the High Z team, which includes many more data points, can be found in Riess *et al.*, http://arXiv.org/abs/astro-ph/0402512.¶ (By the way, the lead author Adam Riess was an MIT undergraduate physics major, graduating in 1992.)

You are asked to plot the data from either the 2nd or 3rd of these papers, and to include on the graph the theoretical predictions for several cosmological models.

The plot will be similar to the plots contained in these papers, and to the plot on p. 121 of Ryden's book, showing a graph of (corrected) magnitude m vs. redshift z. Your graph should include the error bars. If you plot the Perlmutter *et al.* data, you will be plotting "effective magnitude" m vs. redshift z. The magnitude is related to the flux J of the observed radiation by $m = -\frac{5}{2}\log_{10}(J) + \text{const}$. The value of the constant in this expression will not be needed. The word "corrected" refers both to corrections related to the spectral sensitivity of the detectors and to the brightness of the supernova explosions themselves. That is, the supernova at various distances are observed with different redshifts, and hence one must apply corrections if the detectors used to measure the radiation do not have the same sensitivity at all wavelengths. In addition, to improve the uniformity of the supernova as standard candles, the astronomers apply a correction based on the duration of the light output. Note that our ignorance of the absolute brightness of the supernova, of the precise value of the Hubble constant, and of the constant that appears in the definition of magnitude all combine to give an unknown but constant contribution to the predicted magnitudes. The consequence is that you will be able to move your predicted curves up or down (i.e., translate them by a fixed distance along the m axis). You should choose the vertical positioning of your curve to optimize your fit, either by eyeball or by some more systematic method.

If you choose to plot the data from the 3rd paper, Riess *et al.* 2004, then you should see the note at the end of this problem.

For your convenience, the magnitudes and redshifts for the Supernova Cosmology Project paper, from Tables 1 and 2, are summarized in a text file on the 8.286 web page. The data from Table 5 of the Riess *et al.* 2004 paper, mentioned above, is also posted on the 8.286 web page.

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* http://arXiv.org/abs/astro-ph/9805201, later published as Riess *et al.*, *Astronomical Journal* **116**, 1009 (1998).

† http://arXiv.org/abs/astro-ph/9812133, later published as Perlmutter *et al.*, *Astrophysical Journal* **517**:565–586 (1999).

¶ Published as *Astrophysical Journal* **607**:665-687 (2004).

For the cosmological models to plot, you should include:

(i) A matter-dominated universe with $\Omega_m = 1$.

(ii) An open universe, with $\Omega_{m,0} = 0.3$.

(iii) A universe with $\Omega_{m,0} = 0.3$ and a cosmological constant, with $\Omega_{\text{vac},0} = 0.7$. (If you prefer to avoid the flat case, you can use $\Omega_{\text{vac},0} = 0.6$. Or, if you want to compare directly with Figure 4 of the Riess *et al.* (2004) paper, you should use $\Omega_{m,0} = 0.29$, $\Omega_{\text{vac},0} = 0.71$.)

You may include any other models if they interest you. You can draw the plot with either a linear or a logarithmic scale in z. I would recommend extending your theoretical plot to $z = 3$, if you do it logarithmically, or $z = 2$ if you do it linearly, even though the data does not go out that far. That way you can see what possible knowledge can be gained by data at higher redshift.

NOTE FOR THOSE PLOTTING DATA FROM RIESS ET AL. 2004:

Unlike the Perlmutter *et al.* data, the Riess *et al.* data is expressed in terms of the distance modulus, which is a direct measure of the luminosity distance. The distance modulus is defined both in the Riess *et al.* paper and in Ryden's book (p. 118) [First edition: p. 120] as

$$\mu = 5 \log_{10} \left(\frac{d_L}{1 \text{ Mpc}} \right) + 25 ,$$

where Ryden uses the notation $m - M$ for the distance modulus, and d_L is the luminosity distance. The luminosity distance, in turn, is really a measure of the observed brightness of the object. It is defined as the distance that the object would have to be located to result in the observed brightness, if we were living in a static Euclidean universe. More explicitly, if we lived in a static Euclidean universe and an object radiated power P in a spherically symmetric pattern, then the energy flux J at a distance d would be

$$J = \frac{P}{4\pi d^2} .$$

That is, the power would be distributed uniformly over the surface of a sphere at radius d. The luminosity distance is therefore defined as

$$d_L = \sqrt{\frac{P}{4\pi J}} .$$

Thus, a specified value of the distance modulus μ implies a definite value of the ratio J/P.

In plotting a theoretical curve, you will need to choose a value for H_0. Riess *et al.* do not specify what value they used, but I found that their curve is most closely reproduced if I choose $H_0 = 66$ km-sec^{-1}-Mpc^{-1}. This seems a little on the low side, since the value is usually estimated as 70–72 km-sec^{-1}-Mpc^{-1}, but Riess *et al.* emphasize that they were not concerned with this value. They were concerned with the relative values of the distance moduli, and hence the shape of the graph of the distance modulus vs. z. In their own words, from Appendix A, "The zeropoint, distance scale, absolute magnitude of the fiducial SN Ia or Hubble constant derived from Table 5 are all closely related (or even equivalent) quantities which were arbitrarily set for the sample presented here. Their correct value is not relevant for the analyses presented which only make use of differences between SN Ia magnitudes. Thus the analysis are independent of the aforementioned normalization parameters."

Total points for Problem Set 8: 110, plus an optional 20 points of extra credit.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — November 28, 2020
Prof. Alan Guth

PROBLEM SET 9 (The Last!)

DUE DATE: Wednesday, December 9, 2020, at 12:30 pm.

READING ASSIGNMENT: None.

CALENDAR FOR THE REST OF THE TERM:

NOVEMBER/DECEMBER				
MON	TUES	WED	THURS	FRI
23 **Thanksgiving Week**	24 —	25 —	26 —	27 —
30 Class 22	December 1	2 Class 23 **Quiz 3**	3	4
7 Class 24	8	9 Class 25 **PS 9 due** Last Class	10	11

PROBLEM 1: THE HORIZON PROBLEM *(20 points)*

The success of the big bang predictions for the abundances of the light elements suggests that the universe was already in thermal equilibrium at one second after the big bang. At this time, the region that later evolves to become the observed universe was, in the context of the conventional (non-inflationary) cosmological model, many horizon distances across. Try to estimate how many. You may assume that the universe is flat, that it was radiation-dominated for $t \lesssim 50{,}000$ yr, and for this crude estimate you can also assume that it has been matter-dominated for all $t \gtrsim 50{,}000$ yr, and that $a(t)T(t)$ was about equal to its present value for the whole period from 1 second to the present.

PROBLEM 2: THE FLATNESS PROBLEM *(20 points)*

Although we now know that $\Omega_0 = 1$ to an accuracy of about half a percent, let us pretend that the value of Ω today is 0.1. It nonetheless follows that at 10^{-37} second after the big bang (about the time of the grand unified theory phase transition), Ω must have been extraordinarily close to one. Writing $\Omega = 1 - \delta$, estimate the value of δ at $t = 10^{-37}$ sec (using the standard cosmological model). You may again use any of the approximations mentioned in Problem 1.

PROBLEM 3: THE MAGNETIC MONOPOLE PROBLEM *(20 points)*

In Lecture Notes 9, we learned that Grand Unified Theories (GUTs) imply the existence of magnetic monopoles, which form as "topological defects" (topologically stable knots) in the configuration of the Higgs fields that are responsible for breaking the grand unified symmetry to the SU(3)×SU(2)×U(1) symmetry of the standard model of particle physics. It was stated that if grand unified theories and the conventional (non-inflationary) cosmological model were both correct, then far too many magnetic monopoles would have been produced in the big bang. In this problem we will fill in the mathematical steps of that argument.

At very high temperatures the Higgs fields oscillate wildly, so the fields average to zero. As the temperature T falls, however, the system undergoes a phase transition. The phase transition occurs at a temperature T_c, called the critical temperature, where $kT_c \approx 10^{16}$ GeV. At this phase transition the Higgs fields acquire nonzero expectation values, and the grand unified symmetry is thereby spontaneously broken. The monopoles are twists in the Higgs field expectation values, so the monopoles form at the phase transition. Each monopole is expected to have a mass $M_M c^2 \approx 10^{18}$ GeV, where the subscript "M" stands for "monopole." According to an estimate first proposed by T.W.B. Kibble, the number density n_M of monopoles formed at the phase transition is of order

$$n_M \sim 1/\xi^3 \ , \tag{3.1}$$

where ξ is the correlation length of the field, defined roughly as the maximum distance over which the field at one point in space is correlated with the field at another point in space. The correlation length is certainly no larger than the physical horizon distance at the time of the phase transition, and it is believed to typically be comparable to this upper limit. Note that an upper limit on ξ is a lower limit on n_M — there must be at least of order one monopole per horizon-sized volume.

Assume that the particles of the grand unified theory form a thermal gas of blackbody radiation, as described by Eq. (6.48) of Lecture Notes 6,

$$u = g\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3} \ , \tag{3.2}$$

with $g_{\rm GUT} \sim 200$. Further assume that the universe is flat and radiation-dominated from its beginning to the time of the GUT phase transition, $t_{\rm GUT}$.

For each of the following questions, first write the answer in terms of physical constants and the parameters T_c, M_M, and $g_{\rm GUT}$, and then evaluate the answers numerically.

(a) *(5 points)* Under the assumptions described above, at what time $t_{\rm GUT}$ does the phase transition occur? Express your answer first in terms of symbols, and then evaluate it in seconds.

(b) *(5 points)* Using Eq. (3.1) and setting ξ equal to the horizon distance, estimate the number density n_M of magnetic monopoles just after the phase transition.

(c) *(5 points)* Calculate the ratio n_M/n_γ of the number of monopoles to the number of photons immediately after the phase transition. Refer to Lecture Notes 6 to remind yourself about the number density of photons. You may assume that the temperature after the phase transition is still approximately T_c.

(d) *(5 points)* For topological reasons monopoles cannot disappear, but they form with an equal number of monopoles and antimonopoles, where the antimonopoles correspond to twists in the Higgs field in the opposite sense. Monopoles and antimonopoles can annihilate each other, but estimates of the rate for this process show that it is negligible. Thus, in the context of the conventional (non-inflationary) hot big bang model, the ratio of monopoles to photons would be about the same today as it was just after the phase transition. Use this assumption to estimate the contribution that these monopoles would make to the value of Ω today.

PROBLEM 4: EXPONENTIAL EXPANSION OF THE INFLATIONARY UNIVERSE *(15 points)*

Recall that the evolution of a Robertson-Walker universe is described by the equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}. \tag{4.1}$$

Suppose that the mass density ρ is given by the constant mass density ρ_f of the false vacuum. For the case $k = 0$, the growing solution is given simply by

$$a(t) = \text{const } e^{\chi t}, \tag{4.2}$$

where

$$\chi = \sqrt{\frac{8\pi}{3}G\rho_f} \tag{4.3}$$

and *const* is an arbitrary constant. Find the growing solution to this equation for an arbitrary value of k. Be sure to consider both possibilities for the sign of k. You may find the following integrals useful:

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x \ . \tag{4.4a}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \ . \tag{4.4b}$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x \ . \tag{4.4c}$$

Show that for large times one has

$$a(t) \propto e^{\chi t} \tag{4.5}$$

for all choices of k.

PROBLEM 5: THE HORIZON DISTANCE FOR THE PRESENT UNIVERSE *(25 points)*

We have not discussed horizon distances since the beginning of Lecture Notes 4, when we found that

$$\ell_{p,\text{horizon}}(t) = a(t) \int_0^t \frac{c}{a(t')} dt' \ . \tag{5.1}$$

This formula was derived before we discussed curved spacetimes, but the formula is valid for any Robertson-Walker universe, whether it is open, closed, or flat.

(a) *(5 points)* Show that the formula above is valid for closed universes. Hint: write the closed universe metric as it was written in Eq. (7.27):

$$ds^2 = -c^2\, dt^2 + \tilde{a}^2(t) \left\{ d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\} \ , \tag{5.2}$$

where

$$\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{k}} \tag{5.3}$$

and ψ is related to the usual Robertson-Walker coordinate r by

$$\sin\psi \equiv \sqrt{k}\, r \ . \tag{5.4}$$

Use the fact that the physical speed of light is c, or equivalently the fact that $ds^2 = 0$ for any segment of the light ray's trajectory.

(b) *(20 points)* The evaluation of the formula depends of course on the form of the function $a(t)$, which is governed by the Friedmann equations. For the Planck 2018 best fit to the parameters (see Table 7.1 of Lecture Notes 7, and Eq. (6.23) of Lecture Notes 6),

$$H_0 = 67.7 \text{ km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1} \tag{5.5a}$$

$$\Omega_{m,0} = 0.311 \tag{5.5b}$$

$$\begin{aligned}\Omega_{r,0} &= 4.15 \times 10^{-5} h_0^{-2} \quad (T_{\gamma,0} = 2.725\,\text{K}) \\ &= 9.05 \times 10^{-5}\end{aligned} \tag{5.5c}$$

$$\Omega_{\text{vac},0} = 1 - \Omega_{m,0} - \Omega_{r,0} \ , \tag{5.5d}$$

find the current horizon distance, expressed both in light-years and in Mpc. Hint: find an integral expression for the horizon distance, similar to Eq. (7.23a) for the age of the universe. Then do the integral numerically.

Note that the model for which you are calculating does not explicitly include inflation. If it did, the horizon distance would turn out to be vastly larger. By ignoring the inflationary era in calculating the integral of Eq. (5.1), we are finding an effective horizon distance, defined as the present distance of the most distant objects that we can in principle observe by using only photons that have left their sources after the end of inflation. Photons that left their sources earlier than the end of inflation have undergone incredibly large redshifts, so it is reasonable to consider them to be completely unobservable in practice.

PROBLEM 6: THE DEUTERIUM BOTTLENECK *(30 points)*

The "deuterium bottleneck" plays a major role in the description of big bang nucleosynthesis: all of the nuclear reactions involved in nucleosynthesis depend on deuterium forming at the start, but, due to its small binding energy, deuterium does not become stable until the temperature reaches a rather low value. In this problem we will explore the statistical mechanics of the deuterium bottleneck. Ryden discusses this in Section 9.3 [First Edition: Section 10.3], *Deuterium Synthesis*, but the calculation here will be a little more complete.

As discussed in *Notes on Thermal Equilibrium* (which I will refer to as NTE), a dilute ideal gas of classical nonrelativistic particles of type X, in thermal equilibrium, has a number density given by

$$n_X = \bar{g}_X \left(\frac{m_X kT}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{m_X c^2}{kT}\right) \exp\left(\frac{\mu_X}{kT}\right) , \qquad (6.1)$$

where $\hbar = h/2\pi$, c, and k have their usual meanings: Planck's constant, the speed of light, and the Boltzmann constant. Here $\bar{g}_X$ is the number of spin degrees of freedom associated with the particle, m_X is the mass of the particle, T is the temperature, and μ_X is the chemical potential of the particle. Because the gas is dilute there are no corrections associated with fermions and the Pauli exclusion principle, since it is highly unlikely in any case for two particles to occupy the same quantum state. The dilute gas approximation is valid provided that

$$n_X \ll \left(\frac{m_X kT}{2\pi\hbar^2}\right)^{3/2} . \qquad (6.2)$$

Eq. (6.1) is equivalent to Eq. (23) of NTE, or Eq. (8.26) of Ryden's 2nd edition.

(a) *(5 points)* I mentioned in NTE that our textbook sometimes writes Eq. (6.1) incorrectly, omitting the chemical potential factor. See for example Eqs. (9.11) and (9.12). The author has a footnote about this (p. 170), which concludes that "in most cosmological contexts, as it turns out, the chemical potential is small enough to be safely neglected." [First edition: Eqs. (10.11) and (10.12, footnote on p. 156.] We can check this statement by using the author's formula to calculate the proton density at 3 minutes into the big bang, at the time of Steven Weinberg's Fifth Frame, from chapter 5 of *The First Three Minutes.* At that time the temperature was $T = 10^9$ K. To find the right answer for comparison, we make use of the fact that the ratio of the number density n_b of baryons to the number density n_γ of photons is given by*

$$\eta \equiv \frac{n_b}{n_\gamma} = (6.15 \pm 0.35) \times 10^{-10} \quad \text{(95\% confidence)}. \qquad (6.3)$$

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* Particle Data Group, Section 2: *Astrophysical Constants and Parameters* (2020), https://pdg.lbl.gov/2020/reviews/rpp2020-rev-astrophysical-constants.pdf.

According to Weinberg, at that time 14% of the baryons were neutrons, with 86% protons. At the risk of appearing impertinent toward the author (but physicists are known for their impertinence), I will phrase the question this way: By how many kilo-orders of magnitude is the author's formula for n_p in error?† (Be prepared to have your calculators overflow — if they do, calculate the logarithm of the answer.)

(b) *(15 points)* For deuterium production, the relevant reaction is

$$n + p \longleftrightarrow D + \gamma \ . \tag{6.4}$$

Recall that chemical potentials are defined initially in terms of conserved quantities, so the chemical potentials on both sides of any allowed process much match. Since the photon carries no conserved quantities, its chemical potential must vanish. It follows that $\mu_n + \mu_p = \mu_D$. This equality implies that if we form the ratio

$$\frac{n_D}{n_p\, n_n} \ , \tag{6.5}$$

expressing each number density as in Eq. (6.1), then the chemical potential factors will cancel out. (This is how the formula is normally used, and this is how Ryden uses it on pp. 175–176 [First edition: pp. 180–181]. From here on Ryden's treatment is correct, but we will proceed with slightly more detail.) To describe the bookkeeping for the reaction of Eq. (6.4), we need to define our variables. I am using n_n, n_p, and n_D to mean the number densities of free neutrons, free protons, and deuterium nuclei. n_b denotes the total baryon number density, so

$$n_b = n_n + n_p + 2n_D \ . \tag{6.6}$$

Finally, I will use $n_n^{\rm TOT}$ and $n_p^{\rm TOT}$ to denote the total number densities of neutrons and protons respectively, whether free or bound inside deuterium. We assume that deuterium production happens fast enough so that there is no further change in the neutron-proton balance while deuterium is forming, so the ratio

$$f \equiv \frac{n_n^{\rm TOT}}{n_b} \tag{6.7}$$

is fixed. We will describe the extent to which the reaction has proceeded by specifying the fraction x of neutrons that remain free,

$$x \equiv \frac{n_n}{n_n^{\rm TOT}} \ . \tag{6.8}$$

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† I exchanged email with Barbara Ryden about this after the first edition came out, and she said she would fix it in the next edition. She corrected Section 8.3, *The Physics of Recombination*, but did not follow through consistently.

Using these definitions, write the equation that equates the ratio $n_D/(n_p\, n_n)$ to a function of temperature, using Eq. (6.1) for each of the number densities. The deuteron is spin-1, with $g = 3$, and the proton and neutron are each spin-$\frac{1}{2}$, with $g = 2$. Except in the exponential factor, you may approximate $m_n = m_p = m_D/2$. Manipulate this formula so that it has the form

$$F(\eta, f, x) = G(T) \ , \tag{6.9}$$

where F and G are functions that you must determine. You will need the binding energy of deuterium,

$$B = (m_p + m_n - m_D)c^2 \approx 2.22 \text{ MeV}. \tag{6.10}$$

Eq. (6.9) determines x as a function of T, or vice versa, but we will not try to write $x(T)$ or $T(x)$ explicitly.

(c) *(5 points)* Using your result in part (b), and taking $f = 0.14$ from Weinberg's book, find the value of x, the fraction of neutrons that have been bound in deuterium, at the time of the Fifth Frame, when $T = 10^9$ K. You will probably want to solve the equation numerically. Two significant figures will be sufficient.

(d) *(5 points)* Again using your result from part (b), and assuming that $f = 0.14$ is still accurate, find the temperature at which $x = \frac{1}{2}$, i.e., the temperature for which half of the neutrons have become combined into deuterium. Again you will presumably find the answer numerically, and 2 significant figures will be sufficient. What is the value of kT at this temperature? Qualitatively, what feature of the calculation causes this number to be small compared to B?

PROBLEM 7: A ZERO MASS DENSITY UNIVERSE— GENERAL RELATIVITY DESCRIPTION

(This problem is not required, but can be done for 20 points extra credit.)

In this problem and the next we will explore the connections between special relativity and the standard cosmological model which we have been discussing. Although we have not studied general relativity in detail, the description of the cosmological model that we have been using is precisely that of general relativity. In the limit of zero mass density the effects of gravity will become negligible, and the formulas must then be compatible with the special relativity which we discussed at the beginning of the course. The goal of these two problems is to see exactly how this happens.

These two problems will emphasize the notion that a coordinate system is nothing more than an arbitrary system of designating points in spacetime. A physical object might therefore look very different in two different coordinate systems, but the answer to any well-defined physical question must turn out the same regardless of which coordinate system is used in the calculation.

From the general relativity point of view, the model universe is described by the Robertson-Walker spacetime metric:

$$ds^2 = -c^2 dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right\} . \tag{7.1}$$

This formula describes the analogue of the "invariant interval" of special relativity, measured between the spacetime points (t, r, θ, ϕ) and $(t + dt, r + dr, \theta + d\theta, \phi + d\phi)$.

The evolution of the model universe is governed by the general relation

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} , \tag{7.2}$$

except in this case the mass density term is to be set equal to zero.

(a) *(5 points)* Since the mass density is zero, it is certainly less than the critical mass density, so the universe is open. We can then choose $k = -1$. Derive an explicit expression for the scale factor $a(t)$.

(b) *(5 points)* Suppose that a light pulse is emitted by a comoving source at time t_e, and is received by a comoving observer at time t_o. Find the Doppler shift ratio z.

(c) *(5 points)* Consider a light pulse that leaves the origin at time t_e. In an infinitesimal time interval dt the pulse will travel a physical distance $ds = cdt$. Since the pulse is traveling in the radial direction (i.e., with $d\theta = d\phi = 0$), one has

$$cdt = a(t) \frac{dr}{\sqrt{1 - kr^2}} . \tag{7.3}$$

Note that this is a slight generalization of Eq. (2.9), which applies for the case of a Euclidean geometry ($k = 0$). Derive a formula for the trajectory $r(t)$ of the light pulse. You may find the following integral useful:

$$\int \frac{dr}{\sqrt{1 + r^2}} = \sinh^{-1} r . \tag{7.4}$$

(d) *(5 points)* Use these results to express the redshift z in terms of the coordinate r of the observer. If you have done it right, your answer will be independent of t_e. (In the special relativity description that will follow, it will be obvious why the redshift must be independent of t_e. Can you see the reason now?)

PROBLEM 8: A ZERO MASS DENSITY UNIVERSE— SPECIAL RELATIVITY DESCRIPTION

(This problem is also not required, but can be done for 20 points extra credit.)

In this problem we will describe the same model universe as in the previous problem, but we will use the standard formulation of special relativity. We will therefore use an inertial coordinate system, rather than the comoving system of the previous problem. Please note, however, that in the usual case in which gravity is significant, there is no inertial coordinate system. Only when gravity is absent does such a coordinate system exist.

To distinguish the two systems, we will use primes to denote the inertial coordinates: (t', x', y', z'). Since the problem is spherically symmetric, we will also introduce "polar inertial coordinates" (r', θ', ϕ') which are related to the Cartesian inertial coordinates by the usual relations:

$$\begin{aligned} x' &= r' \sin\theta' \cos\phi' \\ y' &= r' \sin\theta' \sin\phi' \\ z' &= r' \cos\theta' \ . \end{aligned} \tag{8.1}$$

In terms of these polar inertial coordinates, the invariant spacetime interval of special relativity can be written as

$$ds^2 = -c^2 dt'^2 + dr'^2 + r'^2 \left(d\theta'^2 + \sin^2\theta' d\phi'^2 \right) \ . \tag{8.2}$$

For purposes of discussion we will introduce a set of comoving observers which travel along with the matter in the universe, following the Hubble expansion pattern. (Although the matter has a negligible mass density, I will assume that enough of it exists to define a velocity at any point in space.) These trajectories must all meet at some spacetime point corresponding to the instant of the big bang, and we will take that spacetime point to be the origin of the coordinate system. Since there are no forces acting in this model universe, the comoving observers travel on lines of constant velocity (all emanating from the origin). The model universe is then confined to the future light-cone of the origin.

(a) *(5 points)* The cosmic time variable t used in the previous problem can be defined as the time measured on the clocks of the comoving observers, starting at the instant of the big bang. Using this definition and your knowledge of special relativity, find the value of the cosmic time t for given values of the inertial coordinates— i.e., find $t(t', r')$. [Hint: first find the velocity of a comoving observer who starts at the origin and reaches the spacetime point (t', r', θ', ϕ'). Note that the rotational symmetry makes θ' and ϕ' irrelevant, so one can examine motion along a single axis.]

(b) *(5 points)* Let us assume that angular coordinates have the same meaning in the two coordinate systems, so that $\theta = \theta'$ and $\phi = \phi'$. We will verify in part (d) below that this assumption is correct. Using this assumption, find the value of the comoving radial coordinate r in terms of the inertial coordinates— i.e., find $r(t', r')$. [Hint: consider an infinitesimal line segment which extends in the θ-direction, with constant values of t, r, and ϕ. Use the fact that this line segment must have the same physical length, regardless of which coordinate system is used to describe it.] Draw a graph of the t'-r' plane, and sketch in lines of constant t and lines of constant r.

(c) *(5 points)* Show that the radial coordinate r of the comoving system is related to the magnitude of the velocity in the inertial system by

$$r = \frac{v/c}{\sqrt{1 - v^2/c^2}} \ . \tag{8.3}$$

Suppose that a light pulse is emitted at the spatial origin ($r' = 0$, $t' =$ *anything*) and is received by another comoving observer who is traveling at speed v. With what redshift z is the pulse received? Express z as a function of r, and compare your answer to part (d) of the previous problem.

(d) *(5 points)* In this part we will show that the metric of the comoving coordinate system can be derived from the metric of special relativity, a fact which completely establishes the consistency of the two descriptions. To do this, first write out the equations of transformation in the form:

$$\begin{aligned} t' &= ? \\ r' &= ? \\ \theta' &= ? \\ \phi' &= ? \ , \end{aligned} \tag{8.4}$$

where the question marks denote expressions in t, r, θ, and ϕ. Now consider an infinitesimal spacetime line segment described in the comoving system by its two endpoints: (t, r, θ, ϕ) and $(t + dt, r + dr, \theta + d\theta, \phi + d\phi)$. Calculating to first order in the infinitesimal quantities, find the separation between the coordinates of the two endpoints in the inertial coordinate system— i.e., find dt', dr', $d\theta'$, and $d\phi'$. Now insert these expressions into the special relativity expression for the invariant interval ds^2 , and if you have made no mistakes you will recover the Robertson-Walker metric used in the previous problem.

DISCUSSION OF THE ZERO MASS DENSITY UNIVERSE:

The two problems above demonstrate how the general relativistic description of cosmology can reduce to special relativity when gravity is unimportant, but it provides a misleading picture of the big-bang singularity which I would like to clear up.

First, let me point out that the mass density of the universe increases as one looks backward in time. So, if we imagine a model universe with $\Omega = 0.01$ at a given time, it could be well-approximated by the zero mass density universe at this time. However, no matter how small Ω is at a given time, the mass density will increase as one follows the model to earlier times, and the behavior of the model near $t = 0$ will be very different from the zero mass density model.

In the zero mass density model, the big-bang "singularity" is a single spacetime point which is in fact not singular at all. In the comoving description the scale factor $a(t)$ equals zero at this time, but in the inertial system one sees that the spacetime metric is really just the usual smooth metric of special relativity, expressed in a peculiar set of coordinates. In this model it is unnatural to think of $t = 0$ as really defining the beginning of anything, since the the future light-cone of the origin connects smoothly to the rest of the spacetime.

In the standard model of the universe with a nonzero mass density, the behavior of the singularity is very different. First of all, it really is singular— one can mathematically prove that there is no coordinate system in which the singularity disappears. Thus, the spacetime cannot be joined smoothly onto anything that may have happened earlier.

The differences between the singularities in the two models can also be seen by looking at the horizon distance. We learned in Lecture Notes 4 that light can travel only a finite distance from the time of the big bang to some arbitrary time t, and that this "horizon distance" is given by

$$\ell_p(t) = a(t) \int_0^t \frac{c}{a(t')} dt' \; . \tag{8.5}$$

For the scale factor of the zero mass density universe as found in the problem, one can see that this distance is infinite for any t— for the zero mass density model there is ***no*** horizon. For a radiation-dominated model, however, there is a finite horizon distance given by $2ct$.

Finally, in the zero mass density model the big bang occurs at a single point in spacetime, but for a nonzero mass density model it seems better to think of the big bang as occurring everywhere at once. In terms of the Robertson-Walker coordinates, the singularity occurs at $t = 0$, for all values of r, θ, and ϕ. There is a subtle issue, however, because with $a(t = 0) = 0$, all of these points have zero distance from each other. Mathematically the locus $t = 0$ in a nonzero mass density model is too singular to even be considered part of the space, which consists of all values of $t > 0$. Thus, the question of whether the singularity is a single point is not well defined. For any $t > 0$ the issue is of course clear— the space is homogeneous and infinite (for the case of the open universe). If one wishes to ignore the mathematical subtleties and call the singularity at $t = 0$ a single point, then one certainly must remember that the singularity makes it a very unusual point. Objects emanating from this "point" can achieve an infinite separation in an arbitrarily short length of time.

Total points for Problem Set 9: 130, plus an optional 40 points of extra credit.