MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — September 19, 2020
Prof. Alan Guth

PROBLEM SET 3

DUE DATE: Friday, September 25, 2020, 5:00 pm.

READING ASSIGNMENT: Steven Weinberg, *The First Three Minutes*, Chapter 3.

SHORT-TERM CALENDAR:

SEPTEMBER/OCTOBER				
MON	TUES	WED	THURS	FRI
September 14 Class 3	15	16 Class 4	17	18
21 Class 5 PS 2 due	22	23 Class 6	24	25 PS 3 due
28 Class 7	29	30 Quiz 1 — "in class"	October 1	2

QUIZ DATES FOR THE TERM:

Quiz 1: Wednesday, September 30, 2020
Quiz 2: Wednesday, October 28, 2020
Quiz 3: Wednesday, December 2, 2020

FIRST QUIZ: The first of three quizzes for the term will be given on Wednesday, September 30, 2020.

Coverage: Lecture Notes 1, 2, and 3; Problem Sets 1, 2, and 3; Weinberg, Chapters 1, 2, and 3; Ryden, Chapters 1, 2, and 3. While all of Ryden's Chapter 3 has been assigned, questions on the quiz will be limited to Sections 3.1 (*The Way of Newton*) and 3.3 (*The General Way of Einstein*). Section 3.2 (*The Special Way of Einstein*) describes special relativity. Ryden's approach is somewhat different from our Lecture Notes 1 — for the quiz, you will be responsible only for the issues discussed in Lecture Notes 1. The material in Sections 3.4–3.6 will be discussed in lecture later in the course, and you will not be responsible for it until then.

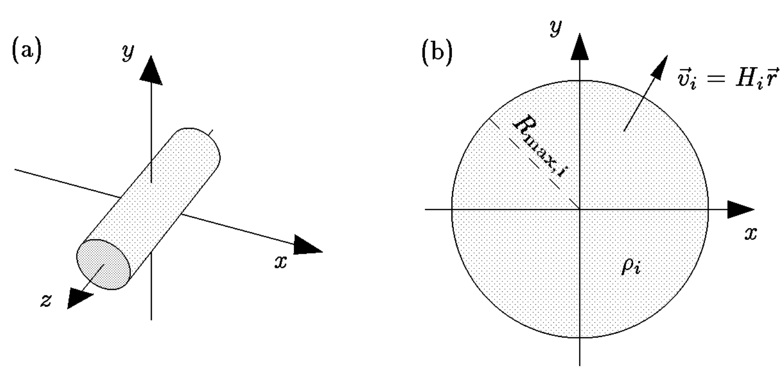
Quiz Logistics: The quiz will be closed book, no calculators, no internet, and 85 minutes long. I assume that most of you will take it during our regular class time on September 30, but you will have the option of starting it any time during a 24-hour window from

11:05 am EDT on September 30 to 11:05 am EDT on Thursday, October 1. If you want to start later than 11:05 am 9/30/2020, you should email me your choice of starting time by 11:59 pm on 9/29/2020. The quiz will be contained in a PDF file, which I am planning to distribute by email. You will each be expected to spend up to 85 minutes working on it, and then you will upload your answers to Canvas as a PDF file. I won't place any precise time limit on scanning or photographing and uploading, because the time needed for that can vary. If you have questions about the meaning of the questions, I will be available on Zoom during the September 30 class time, and we will arrange for either Bruno or me to be available by email as much as possible during the other quiz times. If you have any special circumstances that might make this procedure difficult, or if you need a postponement beyond the 24-hour window, please let me (guth@ctp.mit.edu) know.

PROBLEM 1: A CYLINDRICAL UNIVERSE *(25 points)*

The following problem originated on Quiz 2 of 1994, where it counted 30 points.

The lecture notes showed a construction of a Newtonian model of the universe that was based on a uniform, expanding, sphere of matter. In this problem we will construct a model of a cylindrical universe, one which is expanding in the x and y directions but which has no motion in the z direction. Instead of a sphere, we will describe an infinitely long cylinder of radius $R_{\rm max,i}$, with an axis coinciding with the z-axis of the coordinate system:



3D Perspective

2D Cross Section (Different scale from (a))

We will use cylindrical coordinates, so

$$r = \sqrt{x^2 + y^2}$$

and

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} \; ; \qquad \hat{r} = \frac{\vec{r}}{r} \; ,$$

where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the usual unit vectors along the x, y, and z axes. We will assume that at the initial time t_i, the initial density of the cylinder is ρ_i, and the initial velocity of a particle at position $\vec{r}$ is given by the Hubble relation

$$\vec{v}_i = H_i \vec{r} \ .$$

(a) *(5 points)* By using Gauss' law of gravity, it is possible to show that the gravitational acceleration at any point is given by

$$\vec{g} = -\frac{A\mu}{r}\hat{r} \ ,$$

where A is a constant and μ is the total mass per length contained within the radius r. Evaluate the constant A.

(b) *(5 points)* As in the lecture notes, we let $r(r_i, t)$ denote the trajectory of a particle that starts at radius r_i at the initial time t_i. Find an expression for $\ddot{r}(r_i, t)$, expressing the result in terms of r, r_i, ρ_i, and any relevant constants. (Here an overdot denotes a time derivative.)

(c) *(5 points)* Defining

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} \ ,$$

show that $u(r_i, t)$ is in fact independent of r_i. This implies that the cylinder will undergo uniform expansion, just as the sphere did in the case discussed in the lecture notes. As before, we define the scale factor $a(t) \equiv u(r_i, t)$.

(d) *(5 points)* Express the mass density $\rho(t)$ in terms of the initial mass density ρ_i and the scale factor $a(t)$. Use this expression to obtain an expression for $\ddot{a}$ in terms of a, ρ, and any relevant constants.

(e) *(5 points)* Find an expression for a conserved quantity of the form

$$E = \frac{1}{2}\dot{a}^2 + V(a) \ .$$

What is $V(a)$? Will this universe expand forever, or will it collapse?

PROBLEM 2: A FLAT UNIVERSE WITH UNUSUAL TIME EVOLUTION *(10 points)*

Consider a ***flat*** universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$a(t) = bt^{3/4} ,$$

where b is a constant.

(a) *(5 points)* For this universe, find the value of the Hubble expansion rate $H(t)$.

(b) *(5 points)* What is the mass density of the universe, $\rho(t)$? (In answering this question, you will need to know that the equation for $\dot{a}/a$ in Lecture Notes 3,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} ,$$

holds for all forms of matter, while the equation for $\ddot{a}$,

$$\ddot{a} = -\frac{4\pi}{3}G\rho(t)a ,$$

requires modification if the matter has a significant pressure. The $\ddot{a}$ equation is therefore not applicable to this problem.)

PROBLEM 3: ENERGY AND THE FRIEDMANN EQUATION *(30 points)*

The Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} , \tag{1}$$

was derived in Lecture Notes 3 as a first integral of the equations of motion. The equation was first derived in a different form,

$$E = \frac{1}{2}\dot{a}^2 - \frac{4\pi}{3}\frac{G\rho_i}{a} = \text{constant}, \tag{2}$$

where $k = -2E/c^2$. In this form the equation looks more like a conservation of energy relation, although the constant E does not have the dimensions of energy. There are two ways, however, in which the quantity E can be connected to the conservation of energy. It is related the energy of a test particle that moves with the Hubble expansion, and it is also related to the total energy of the entire expanding sphere of radius R_{max}, which was discussed in Lecture Notes 3 as a method of deriving the Friedmann equations. In this problem you will derive these relations.

First, to see the relation with the energy of a test particle moving with the Hubble expansion, define a physical energy $E_{\rm phys}$ by

$$E_{\rm phys} \equiv m r_i^2 E \ , \tag{3}$$

where m is the mass of the test particle and r_i is its initial radius. Note that the gravitational force on this particle is given by

$$\vec{F} = -\frac{GmM(r_i)}{r^2}\hat{r} = -\vec{\nabla}V_{\rm eff}(r) \ , \tag{4}$$

where $M(r_i)$ is the total mass initially contained within a radius r_i of the origin, r is the present distance of the test particle from the origin, and the "effective" potential energy $V_{\rm eff}(r)$ is given by

$$V_{\rm eff}(r) = -\frac{GmM(r_i)}{r} \ . \tag{5}$$

The motivation for calling this quantity the "effective" potential energy will be explained below.

(a) *(10 points)* Show that $E_{\rm phys}$ is equal to the "effective" energy of the test particle, defined by

$$E_{\rm eff} = \frac{1}{2}mv^2 + V_{\rm eff}(r) \ . \tag{6}$$

We understand that $E_{\rm eff}$ is conserved because it is the energy in an analogue problem in which the test particle moves in the gravitational field of a point particle of mass $M(r_i)$, located at the origin, with potential energy function $V_{\rm eff}(r)$. In this analogue problem the force on the test particle is exactly the same as in the real problem, but in the analogue problem the energy of the test particle is conserved.

We call (6) the "effective" energy because it is really the energy of the analogue problem, and not the real problem. The true potential energy $V(r,t)$ of the test particle is defined to be the amount of work we must supply to move the particle to its present location from some fixed reference point, which we might take to be $r = \infty$. We will not bother to write $V(r,t)$ explicitly, since we will not need it, but we point out that it depends on the time t and on $R_{\rm max}$, and when differentiated gives the correct gravitational force at any radius. By contrast, $V_{\rm eff}(r)$ gives the correct force only at the radius of the test particle, $r = a(t)r_i$. The true potential energy function $V(r,t)$ gives no conservation law, since it is explicitly time-dependent, which is why the quantity $V_{\rm eff}(r)$ is useful.

To relate E to the total energy of the expanding sphere, we need to integrate over the sphere to determine its total energy. These integrals are most easily carried out by dividing the sphere into shells of radius r, and thickness dr, so that each shell has a volume

$$dV = 4\pi r^2 \, dr \ . \tag{7}$$

(b) *(10 points)* Show that the total kinetic energy K of the sphere is given by

$$K = c_K M R^2_{\text{max},i} \left\{ \frac{1}{2} \dot{a}^2(t) \right\} , \tag{8}$$

where c_K is a numerical constant, M is the total mass of the sphere, and $R_{\text{max},i}$ is the initial radius of the sphere. Evaluate the numerical constant c_K.

(c) *(10 points)* Show that the total potential energy of the sphere can similarly be written as

$$U = c_U M R^2_{\text{max},i} \left\{ -\frac{4\pi}{3} G \frac{\rho_i}{a} \right\} . \tag{9}$$

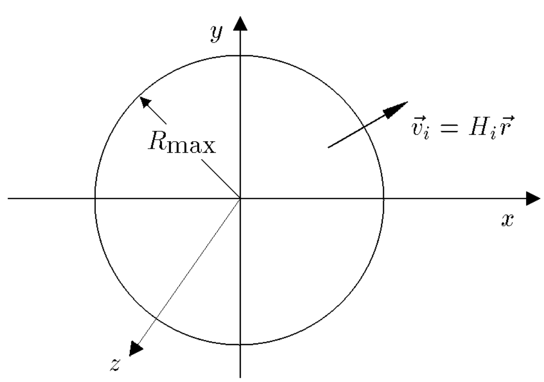
(*Suggestion:* calculate the total energy needed to assemble the sphere by bringing in one shell of mass at a time from infinity.) Show that $c_U = c_K$, so that the total energy of the sphere is given by

$$E_{\text{total}} = c_K M R^2_{\text{max},i} E . \tag{10}$$

PROBLEM 4: A POSSIBLE MODIFICATION OF NEWTON'S LAW OF GRAVITY *(20 points)*

READ THIS: *This problem was Problem 2 of Quiz 1 of 2011, and the solution is posted as http://web.mit.edu/8.286/www/quiz11/ecqs1-1.pdf. Unlike the situation with other problems, in this case you are encouraged to look at these solutions and benefit from them. When you write your solution, you can even copy it verbatim from these solutions if you wish, although obviously you will learn more if you think about the solution and write your own version.*

In Lecture Notes 3 we developed a Newtonian model of cosmology, by considering a uniform sphere of mass, centered at the origin, with initial mass density ρ_i and an initial pattern of velocities corresponding to Hubble expansion: $\vec{v}_i = H_i \vec{r}$:



We denoted the radius at time t of a particle which started at radius r_i by the function $r(r_i, t)$. Assuming Newton's law of gravity, we concluded that each particle would experience an acceleration given by

$$\vec{g} = -\frac{GM(r_i)}{r^2(r_i, t)}\,\hat{r} \ ,$$

where $M(r_i)$ denotes the total mass contained initially in the region $r < r_i$, given by

$$M(r_i) = \frac{4\pi}{3} r_i^3 \rho_i \ .$$

Suppose that the law of gravity is modified to contain a new, repulsive term, producing an acceleration which grows as the nth power of the distance, with a strength that is independent of the mass. That is, suppose $\vec{g}$ is given by

$$\vec{g} = -\frac{GM(r_i)}{r^2(r_i, t)}\,\hat{r} + \gamma r^n(r_i, t)\,\hat{r} \ ,$$

where γ is a constant. The function $r(r_i, t)$ then obeys the differential equation

$$\ddot{r} = -\frac{GM(r_i)}{r^2(r_i, t)} + \gamma r^n(r_i, t) \ .$$

(a) *(4 points)* As done in the lecture notes, we define

$$u(r_i, t) \equiv r(r_i, t)/r_i \ .$$

Write the differential equation obeyed by u. *(Hint: be sure that u is the only time-dependent quantity in your equation; r, ρ, etc. must be rewritten in terms of u, ρ_i, etc.)*

(b) *(4 points)* For what value of the power n is the differential equation found in part (a) independent of r_i?

(c) *(4 points)* Write the initial conditions for u which, when combined with the differential equation found in (a), uniquely determine the function u.

(d) *(8 points)* If all is going well, then you have learned that for a certain value of n, the function $u(r_i, t)$ will in fact not depend on r_i, so we can define

$$a(t) \equiv u(r_i, t) \ .$$

Show, for this value of n, that the differential equation for a can be integrated once to obtain an equation related to the conservation of energy. The desired equation should include terms depending on a and $\dot{a}$, but not $\ddot{a}$ or any higher derivatives.

Total points for Problem Set 3: 85.