MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — December 8, 2022
Prof. Alan Guth

PROBLEM SET 10 (The Last!)

DUE DATE: Wednesday, December 14, 2022, at 11:00 am.

READING ASSIGNMENT: None.

PROBLEM 1: THE MAGNETIC MONOPOLE PROBLEM *(20 points)*

In Lecture Notes 9, we learned that Grand Unified Theories (GUTs) imply the existence of magnetic monopoles, which form as "topological defects" (topologically stable knots) in the configuration of the Higgs fields that are responsible for breaking the grand unified symmetry to the SU(3)×SU(2)×U(1) symmetry of the standard model of particle physics. It was stated that if grand unified theories and the conventional (non-inflationary) cosmological model were both correct, then far too many magnetic monopoles would have been produced in the big bang. In this problem we will fill in the mathematical steps of that argument.

At very high temperatures the Higgs fields oscillate wildly, so the fields average to zero. As the temperature T falls, however, the system undergoes a phase transition. The phase transition occurs at a temperature T_c, called the critical temperature, where $kT_c \approx 10^{16}$ GeV. At this phase transition the Higgs fields acquire nonzero expectation values, and the grand unified symmetry is thereby spontaneously broken. The monopoles are twists in the Higgs field expectation values, so the monopoles form at the phase transition. Each monopole is expected to have a mass $M_M c^2 \approx 10^{18}$ GeV, where the subscript "M" stands for "monopole." According to an estimate first proposed by T.W.B. Kibble, the number density n_M of monopoles formed at the phase transition is of order

$$n_M \sim 1/\xi^3 \ , \tag{P1.1}$$

where ξ is the correlation length of the field, defined roughly as the maximum distance over which the field at one point in space is correlated with the field at another point in space. The correlation length is certainly no larger than the physical horizon distance at the time of the phase transition, and it is believed to typically be comparable to this upper limit. Note that an upper limit on ξ is a lower limit on n_M — there must be at least of order one monopole per horizon-sized volume.

Assume that the particles of the grand unified theory form a thermal gas of blackbody radiation, as described by Eq. (6.48) of Lecture Notes 6,

$$u = g\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3} \ , \tag{P1.2}$$

with $g_{\rm GUT} \sim 200$. Further assume that the universe is flat and radiation-dominated from its beginning to the time of the GUT phase transition, $t_{\rm GUT}$.

For each of the following questions, first write the answer in terms of physical constants and the parameters T_c, M_M, and $g_{\rm GUT}$, and then evaluate the answers numerically.

(a) *(5 points)* Under the assumptions described above, at what time $t_{\rm GUT}$ does the phase transition occur? Express your answer first in terms of symbols, and then evaluate it in seconds.

(b) *(5 points)* Using Eq. (P1.1) and setting ξ equal to the horizon distance, estimate the number density n_M of magnetic monopoles just after the phase transition.

(c) *(5 points)* Calculate the ratio n_M/n_γ of the number of monopoles to the number of photons immediately after the phase transition. Refer to Lecture Notes 6 to remind yourself about the number density of photons. You may assume that the temperature after the phase transition is still approximately T_c.

(d) *(5 points)* For topological reasons monopoles cannot disappear, but they form with an equal number of monopoles and antimonopoles, where the antimonopoles correspond to twists in the Higgs field in the opposite sense. Monopoles and antimonopoles can annihilate each other, but estimates of the rate for this process show that it is negligible. Thus, in the context of the conventional (non-inflationary) hot big bang model, the ratio of monopoles to photons would be about the same today as it was just after the phase transition. Use this assumption to estimate the contribution that these monopoles would make to the value of Ω today.

PROBLEM 2: EXPONENTIAL EXPANSION OF THE INFLATIONARY UNIVERSE *(15 points)*

Recall that the evolution of a Robertson-Walker universe is described by the equation

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} . \tag{P2.1}$$

Suppose that the mass density ρ is given by the constant mass density ρ_f of the false vacuum. For the case $k = 0$, the growing solution is given simply by

$$a(t) = \text{const}\ e^{\chi t} , \tag{P2.2}$$

where

$$\chi = \sqrt{\frac{8\pi}{3}G\rho_f} \tag{P2.3}$$

and *const* is an arbitrary constant. Find the growing solution to this equation for an arbitrary value of k. Be sure to consider both possibilities for the sign of k. You may find the following integrals useful:

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x \ . \tag{P2.4a}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \ . \tag{P2.4b}$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x \ . \tag{P2.4c}$$

Show that for large times one has

$$a(t) \propto e^{\chi t} \tag{P2.5}$$

for all choices of k.

PROBLEM 3: THE HORIZON DISTANCE FOR THE PRESENT UNIVERSE *(25 points)*

We have not discussed horizon distances since the beginning of Lecture Notes 4, when we found that

$$\ell_{p,\text{horizon}}(t) = a(t) \int_0^t \frac{c}{a(t')} dt' \ . \tag{P3.1}$$

This formula was derived before we discussed curved spacetimes, but the formula is valid for any Robertson-Walker universe, whether it is open, closed, or flat.

(a) *(5 points)* Show that the formula above is valid for closed universes. Hint: write the closed universe metric as it was written in Eq. (7.27):

$$ds^2 = -c^2\,dt^2 + \tilde{a}^2(t)\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\} \ , \tag{P3.2}$$

where

$$\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{k}} \tag{P3.3}$$

and ψ is related to the usual Robertson-Walker coordinate r by

$$\sin\psi \equiv \sqrt{k}\,r \ . \tag{P3.4}$$

Use the fact that the physical speed of light is c, or equivalently the fact that $ds^2 = 0$ for any segment of the light ray's trajectory.

(b) *(20 points)* The evaluation of the formula depends of course on the form of the function $a(t)$, which is governed by the Friedmann equations. For the Planck 2018 best fit to the parameters (see Table 7.1 of Lecture Notes 7, and Eq. (6.23) of Lecture Notes 6),

$$H_0 = 67.7\ \text{km}\cdot\text{s}^{-1}\cdot\text{Mpc}^{-1} \tag{P3.5a}$$

$$\Omega_{m,0} = 0.311 \tag{P3.5b}$$

$$\begin{aligned}\Omega_{r,0} &= 4.15\times10^{-5}h_0^{-2} \quad (T_{\gamma,0} = 2.725\,\text{K}) \\ &= 9.05\times10^{-5}\end{aligned} \tag{P3.5c}$$

$$\Omega_{\text{vac},0} = 1 - \Omega_{m,0} - \Omega_{r,0} \ , \tag{P3.5d}$$

find the current horizon distance, expressed both in light-years and in Mpc. Hint: find an integral expression for the horizon distance, similar to Eq. (7.23a) for the age of the universe. Then do the integral numerically.

Note that the model for which you are calculating does not explicitly include inflation. If it did, the horizon distance would turn out to be vastly larger. By ignoring the inflationary era in calculating the integral of Eq. (P3.1), we are finding an effective horizon distance, defined as the present distance of the most distant objects that we can in principle observe by using only photons that have left their sources after the end of inflation. Photons that left their sources earlier than the end of inflation have undergone incredibly large redshifts, so it is reasonable to consider them to be completely unobservable in practice.

PROBLEM 4: HUBBLE CROSSING FOR GALAXY-SIZE PERTURBATIONS *(25 points)*

This problem is a follow-up to the discussion in Section 11.5 (*Structure Formation: Gravitational Instability — Hot versus Cold*) in Ryden's **Introduction to Cosmology**, Second Edition. There is further discussion of this issue in the Appendix of *8.286: Notes on Ryden's Chapter 11.* But the discussion here is intended to be self-contained.

Density perturbations are usually discussed in terms of a Fourier transform of the mass density function. Using a notation that is slightly more explict than Ryden's, we define

$$\delta_m(\vec{r},t) \equiv \frac{\rho_m(\vec{r},t) - \bar{\rho}_m(t)}{\bar{\rho}_m(t)} \ , \qquad \text{(P4.1))}$$

where $\rho_m(\vec{r},t)$ is the mass density of "matter" (cold dark matter and baryons), and $\bar{\rho}_m(t)$ is the mass density of matter as a function of time in a homogeneous isotropic model of the universe. (Here $\vec{r}$ is the coordinate vector in the comoving coordinate system, so physical distances are found by multiplying by the scale factor $a(t)$.) Hence $\delta_m(\vec{r},t)$ measures departures in the matter density from the homogeneous isotropic model. The perturbations are most easily described in terms of their Fourier transform, since individual Fourier modes — i.e., components with a definite comoving wavelength — behave rather simply. We can choose to discuss the perturbations inside a cube of size $V = L \times L \times L$ in comoving coordinates, with periodic boundary conditions. (The boundary conditions are not realistic, but they will be irrelevant at the end when we consider the $L \to \infty$ limit.) Then define the Fourier amplitude

$$\tilde{\delta}_m(\vec{k},t) \equiv \frac{1}{V}\int_V \delta_m(\vec{r},t)\, e^{-i\vec{k}\cdot\vec{r}}\, d^3r \ , \qquad \text{(P4.2)}$$

where $\vec{k} = (2\pi/L)\vec{n}$, where $\vec{n}$ is a vector of integers (n_x, n_y, n_z), each running from $-\infty$ to ∞. The original function $\delta_m(\vec{r},t)$ can be reconstructed as

$$\delta_m(\vec{r},t) = \sum_{\vec{n}} \tilde{\delta}_m(\vec{k},t) e^{i\vec{k}\cdot\vec{r}} \ , \qquad \text{(P4.3)}$$

where again $\vec{k} = (2\pi/L)\vec{n}$, and $\vec{n}$ is summed over all integer values (positive, negative, or zero) of n_x, n_y, and n_z. Eqs. (P4.2) and (P4.3) are the correct form of Ryden's somewhat impressionistic Eqs. (11.61) and (11.62).*

At the end of inflation, the Fourier modes that are relevant for galaxy formation (or the cosmic microwave background) have physical wavelengths that are vastly larger than the Hubble length.

But this changes. This all happens in the radiation-dominated era, when $H = 1/2t$, so the Hubble length $cH^{-1} = 2ct$. By constrast, the physical wavelength corresponding to a Fourier mode $\vec{k}$ is given by

$$\lambda = a(t)\frac{2\pi}{|\vec{k}|} \propto t^{1/2} \ . \tag{P4.4}$$

Thus the Hubble length grows faster, so it can catch up and exceed the physical wavelength. The time at which the lengths are equal is called the Hubble crossing. In inflationary cosmology it is usually called the second Hubble crossing, because there is also a time during inflation when the two lengths are equal.

The (second) Hubble crossing is important in describing the evolution, because before this time the wave is essentially frozen. Assuming that t is some arbitrary time in the radiation-dominated era, and using the fact that nothing travels faster than light, the maximum distance that anything can travel from the end of inflation until the time t is the Hubble length, $2ct$. So if the Hubble length is much smaller than the wavelength, then any transfer of energy must be over a distance small compared to the wavelength, so the wave cannot be much affected. Thus physical effects that could modify the mode, such as free streaming, must wait to start until after the Hubble crossing.

In this problem we will calculate the time of Hubble crossing for a mode with a wavelength relevant to galaxy formation, using the same estimates as Ryden. We consider the mass of a typical large galaxy, $M_{\rm gal} \approx 10^{12}\,{\rm M_{sun}}$. We assume that the matter in the universe was very uniform during the radiation-dominated era, with gravitational clumping occuring later.

---

* It is also possible to avoid using a finite sized box, describing the infinite space with a Fourier integral, as opposed to a Fourier sum:

$$\tilde{\delta}_m(\vec{k},t) \equiv \frac{1}{(2\pi)^3}\int_{\text{all space}} \delta_m(\vec{r},t)\, e^{-i\vec{k}\cdot\vec{r}}\, d^3r$$

$$\delta_m(\vec{r},t) = \int_{\text{all }\vec{\text{k}}} \tilde{\delta}(\vec{k},t)\, e^{i\vec{k}\cdot\vec{r}}\, d^3k.$$

For this problem, it will not matter which Fourier expansion we use. All we need to know for this problem is that we can calculate the evolution of one Fourier mode at a time.

(a) *(5 points)* We model the galaxy as having collapsed from a spherical region with a current radius R_0 that is large enough so that if the matter were uniformly distributed, the mass of the matter in the sphere would be equal to $M_{\rm gal}$. Using the parameter of Ryden's Benchmark Model — $\Omega_{r,0} = 9.0 \times 10^{-5}$, $\Omega_{m,0} = 0.31$, and $\Omega_{{\rm vac},0} = 1 - \Omega_{r,0} - \Omega_{m,0}$ — calculate the value of R_0. Express your answer in terms of symbols, in meters, in light-years, and in Megaparsecs.

We assume that perturbations of wavelength $\lambda_0 \approx R_0$ are the perturbations most relevant for galaxy formation. Since each mode has a fixed comoving wave vector $\vec{k}$,

$$\lambda(t) = \frac{a(t)}{a(t_0)}\lambda_0 \ . \tag{P4.5}$$

(b) *(5 points)* For some arbitrary time t during the radiation-dominated era, what is the total mass density $\rho(t)$?

(c) *(5 points)* Find the mass density in radiation today in the Benchmark Model. Use this and the answer to part (b) to derive an expression for

$$1 + z(t) = \frac{a(t_0)}{a(t)} \ . \tag{P4.6}$$

Express your answer in terms of symbols only, since no value for t has been specified.

(d) *(10 points)* Using your answers to the previous parts, find the answer to the key question. At what time $t_{\rm gal}$ was the physical wavelength $\lambda(t)$ equal to the Hubble length $cH^{-1}(t)$? Express your answer in symbols, and also in years. You should find an answer close to Ryden's estimate of 12 years.

PROBLEM 5: A ZERO MASS DENSITY UNIVERSE— GENERAL RELATIVITY DESCRIPTION

(This problem is not required, but can be done for 20 points extra credit.)

In this problem and the next we will explore the connections between special relativity and the standard cosmological model which we have been discussing. Although we have not studied general relativity in detail, the description of the cosmological model that we have been using is precisely that of general relativity. In the limit of zero mass density the effects of gravity will become negligible, and the formulas must then be compatible with the special relativity which we discussed at the beginning of the course. The goal of these two problems is to see exactly how this happens.

These two problems will emphasize the notion that a coordinate system is nothing more than an arbitrary system of designating points in spacetime. A physical object might therefore look very different in two different coordinate systems, but the answer to any well-defined physical question must turn out the same regardless of which coordinate system is used in the calculation.

From the general relativity point of view, the model universe is described by the Robertson-Walker spacetime metric:

$$ds^2 = -c^2 dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right\} . \tag{P5.1}$$

This formula describes the analogue of the "invariant interval" of special relativity, measured between the spacetime points (t, r, θ, ϕ) and $(t + dt, r + dr, \theta + d\theta, \phi + d\phi)$.

The evolution of the model universe is governed by the general relation

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} , \tag{P5.2}$$

except in this case the mass density term is to be set equal to zero.

(a) *(5 points)* Since the mass density is zero, it is certainly less than the critical mass density, so the universe is open. We can then choose $k = -1$. Derive an explicit expression for the scale factor $a(t)$.

(b) *(5 points)* Suppose that a light pulse is emitted by a comoving source at time t_e, and is received by a comoving observer at time t_o. Find the Doppler shift ratio z.

(c) *(5 points)* Consider a light pulse that leaves the origin at time t_e. In an infinitesimal time interval dt the pulse will travel a physical distance $ds = cdt$. Since the pulse is traveling in the radial direction (i.e., with $d\theta = d\phi = 0$), one has

$$cdt = a(t) \frac{dr}{\sqrt{1 - kr^2}} . \tag{P5.3}$$

Note that this is a slight generalization of Eq. (2.9), which applies for the case of a Euclidean geometry ($k = 0$). Derive a formula for the trajectory $r(t)$ of the light pulse. You may find the following integral useful:

$$\int \frac{dr}{\sqrt{1 + r^2}} = \sinh^{-1} r . \tag{P5.4}$$

(d) *(5 points)* Use these results to express the redshift z in terms of the coordinate r of the observer. If you have done it right, your answer will be independent of t_e. (In the special relativity description that will follow, it will be obvious why the redshift must be independent of t_e. Can you see the reason now?)

PROBLEM 6: A ZERO MASS DENSITY UNIVERSE— SPECIAL RELATIVITY DESCRIPTION

(This problem is also not required, but can be done for 20 points extra credit.)

In this problem we will describe the same model universe as in the previous problem, but we will use the standard formulation of special relativity. We will therefore use an inertial coordinate system, rather than the comoving system of the previous problem. Please note, however, that in the usual case in which gravity is significant, there is no inertial coordinate system. Only when gravity is absent does such a coordinate system exist.

To distinguish the two systems, we will use primes to denote the inertial coordinates: (t', x', y', z'). Since the problem is spherically symmetric, we will also introduce "polar inertial coordinates" (r', θ', ϕ') which are related to the Cartesian inertial coordinates by the usual relations:

$$\begin{aligned} x' &= r' \sin\theta' \cos\phi' \\ y' &= r' \sin\theta' \sin\phi' \\ z' &= r' \cos\theta' \ . \end{aligned} \tag{P6.1}$$

In terms of these polar inertial coordinates, the invariant spacetime interval of special relativity can be written as

$$ds^2 = -c^2 dt'^2 + dr'^2 + r'^2 \left(d\theta'^2 + \sin^2\theta' d\phi'^2 \right) \ . \tag{P6.2}$$

For purposes of discussion we will introduce a set of comoving observers which travel along with the matter in the universe, following the Hubble expansion pattern. (Although the matter has a negligible mass density, I will assume that enough of it exists to define a velocity at any point in space.) These trajectories must all meet at some spacetime point corresponding to the instant of the big bang, and we will take that spacetime point to be the origin of the coordinate system. Since there are no forces acting in this model universe, the comoving observers travel on lines of constant velocity (all emanating from the origin). The model universe is then confined to the future light-cone of the origin.

(a) *(5 points)* The cosmic time variable t used in the previous problem can be defined as the time measured on the clocks of the comoving observers, starting at the instant of the big bang. Using this definition and your knowledge of special relativity, find the value of the cosmic time t for given values of the inertial coordinates— i.e., find $t(t', r')$. [Hint: first find the velocity of a comoving observer who starts at the origin and reaches the spacetime point (t', r', θ', ϕ'). Note that the rotational symmetry makes θ' and ϕ' irrelevant, so one can examine motion along a single axis.]

(b) *(5 points)* Let us assume that angular coordinates have the same meaning in the two coordinate systems, so that $\theta = \theta'$ and $\phi = \phi'$. We will verify in part (d) below that this assumption is correct. Using this assumption, find the value of the comoving radial coordinate r in terms of the inertial coordinates— i.e., find $r(t', r')$. [Hint: consider an infinitesimal line segment which extends in the θ-direction, with constant values of t, r, and ϕ. Use the fact that this line segment must have the same physical length, regardless of which coordinate system is used to describe it.] Draw a graph of the t'-r' plane, and sketch in lines of constant t and lines of constant r.

(c) *(5 points)* Show that the radial coordinate r of the comoving system is related to the magnitude of the velocity in the inertial system by

$$r = \frac{v/c}{\sqrt{1 - v^2/c^2}} \ . \tag{P6.3}$$

Suppose that a light pulse is emitted at the spatial origin ($r' = 0$, $t' = \textit{anything}$) and is received by another comoving observer who is traveling at speed v. With what redshift z is the pulse received? Express z as a function of r, and compare your answer to part (d) of the previous problem.

(d) *(5 points)* In this part we will show that the metric of the comoving coordinate system can be derived from the metric of special relativity, a fact which completely establishes the consistency of the two descriptions. To do this, first write out the equations of transformation in the form:

$$\begin{aligned} t' &= ? \\ r' &= ? \\ \theta' &= ? \\ \phi' &= ? \ , \end{aligned} \tag{P6.4}$$

where the question marks denote expressions in t, r, θ, and ϕ. Now consider an infinitesimal spacetime line segment described in the comoving system by its two endpoints: (t, r, θ, ϕ) and $(t + dt, r + dr, \theta + d\theta, \phi + d\phi)$. Calculating to first order in the infinitesimal quantities, find the separation between the coordinates of the two endpoints in the inertial coordinate system— i.e., find dt', dr', $d\theta'$, and $d\phi'$. Now insert these expressions into the special relativity expression for the invariant interval ds^2 , and if you have made no mistakes you will recover the Robertson-Walker metric used in the previous problem.

DISCUSSION OF THE ZERO MASS DENSITY UNIVERSE:

The two problems above demonstrate how the general relativistic description of cosmology can reduce to special relativity when gravity is unimportant, but it provides a misleading picture of the big-bang singularity which I would like to clear up.

First, let me point out that the mass density of the universe increases as one looks backward in time. So, if we imagine a model universe with $\Omega = 0.01$ at a given time, it could be well-approximated by the zero mass density universe at this time. However, no matter how small Ω is at a given time, the mass density will increase as one follows the model to earlier times, and the behavior of the model near $t = 0$ will be very different from the zero mass density model.

In the zero mass density model, the big-bang "singularity" is a single spacetime point which is in fact not singular at all. In the comoving description the scale factor $a(t)$ equals zero at this time, but in the inertial system one sees that the spacetime metric is really just the usual smooth metric of special relativity, expressed in a peculiar set of coordinates. In this model it is unnatural to think of $t = 0$ as really defining the beginning of anything, since the the future light-cone of the origin connects smoothly to the rest of the spacetime.

In the standard model of the universe with a nonzero mass density, the behavior of the singularity is very different. First of all, it really is singular— one can mathematically prove that there is no coordinate system in which the singularity disappears. Thus, the spacetime cannot be joined smoothly onto anything that may have happened earlier.

The differences between the singularities in the two models can also be seen by looking at the horizon distance. We learned in Lecture Notes 4 that light can travel only a finite distance from the time of the big bang to some arbitrary time t, and that this "horizon distance" is given by

$$\ell_p(t) = a(t) \int_0^t \frac{c}{a(t')} dt' \ . \tag{P6.5}$$

For the scale factor of the zero mass density universe as found in the problem, one can see that this distance is infinite for any t— for the zero mass density model there is ***no*** horizon. For a radiation-dominated model, however, there is a finite horizon distance given by $2ct$.

Finally, in the zero mass density model the big bang occurs at a single point in spacetime, but for a nonzero mass density model it seems better to think of the big bang as occurring everywhere at once. In terms of the Robertson-Walker coordinates, the singularity occurs at $t = 0$, for all values of r, θ, and ϕ. There is a subtle issue, however, because with $a(t = 0) = 0$, all of these points have zero distance from each other. Mathematically the locus $t = 0$ in a nonzero mass density model is too singular to even be considered part of the space, which consists of all values of $t > 0$. Thus, the question of whether the singularity is a single point is not well defined. For any $t > 0$ the issue is of course clear— the space is homogeneous and infinite (for the case of the open universe). If one wishes to ignore the mathematical subtleties and call the singularity at $t = 0$ a single point, then one certainly must remember that the singularity makes it a very unusual point. Objects emanating from this "point" can achieve an infinite separation in an arbitrarily short length of time.

Total points for Problem Set 10: 85, plus an optional 40 points of extra credit.