MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe   December 18, 2022
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PROBLEM SET 10 SOLUTIONS

PROBLEM 1: THE MAGNETIC MONOPOLE PROBLEM *(10 points)*

(a) If the universe is flat and radiation dominated, then the Friedmann equation becomes

$$H^2 = \frac{8\pi}{3} G\rho \ , \tag{S1.1}$$

where

$$\rho = \frac{u}{c^2} = g\frac{\pi^2}{30}\frac{(kT)^4}{\hbar^3 c^5} \ . \tag{S1.2}$$

We also know that for a radiation-dominated universe, $a(t) \propto t^{1/2}$, so

$$H = \frac{\dot{a}}{a} = \frac{1}{2t} \ , \tag{S1.3}$$

and then

$$t = \frac{1}{2H} \ . \tag{S1.4}$$

Using Eq. (S1.1) to replace H in the above equation, and then Eq. (S1.2) to replace ρ, we find an equation for t in terms of T. Taking T as the temperature of the phase transition, T_c, we find after some algebra that

$$\boxed{t = \frac{3}{4}\sqrt{\frac{5\hbar^3 c^5}{\pi^3 g_{\text{GUT}} G}} \frac{1}{(kT_c)^2} \ .} \tag{S1.5}$$

Inserting numbers,

$$t = \frac{3}{4}\sqrt{\frac{5\left(1.055 \times 10^{-34} \text{ J} \cdot \text{s}\right)^3 \left(2.998 \times 10^8 \text{ m} \cdot \text{s}^{-1}\right)^5}{\pi^3 \cdot 200 \cdot \left(6.674 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}\right)}} \frac{1}{\left(10^{16} \text{ GeV}\right)^2}$$

$$\times \left(\frac{1 \text{ GeV}}{1.602 \times 10^{-10} \text{ J}}\right)^2 \sqrt{\frac{\text{J}}{\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}}} \tag{S1.6}$$

$$= \boxed{1.713 \times 10^{-39} \text{ s} \ .}$$

(b) The horizon distance for a radiation-dominated universe is

$$\ell_{\text{hor,phys}} = a(t) \int_0^t \frac{c}{a(t')} \, dt' = t^{1/2} \int_0^t \frac{c}{t'^{1/2}} \, dt' = 2ct \ , \tag{S1.7}$$

so the monopole number density just after the phase transition can be written as

$$n_M \sim \frac{1}{\xi^3} \sim \frac{1}{(2ct)^3} \ . \tag{S1.8}$$

Using Eq. (S1.5) for t, this gives

$$\boxed{n_M \sim \left(\frac{4\pi^3 g_{\text{GUT}} G}{45 \hbar^3 c^7} \right)^{3/2} (kT_c)^6 \ .} \tag{S1.9}$$

Numerically,

$$n_M \sim \frac{1}{\left(2(2.998 \times 10^8 \ \text{m} \cdot \text{s}^{-1}) \, (1.713 \times 10^{-39} \ \text{s}) \right)^3}$$

$$\tag{S1.10}$$

$$= \boxed{9.23 \times 10^{89} \ \text{m}^{-3} \ .}$$

(c) From Lecture Notes 6, Eq. (6.52), the number density of photons at temperature T_c is given by

$$n_\gamma = 2 \frac{\zeta(3)}{\pi^2} \frac{(kT_c)^3}{(\hbar c)^3} \ . \tag{S1.11}$$

By combining Eq. (S1.9) with Eq. (S1.11), we find

$$\boxed{\frac{n_M}{n_\gamma} = \frac{4\pi^5}{27\zeta(3)} \left(\frac{\pi g_{\text{GUT}} G}{5 \hbar c^5} \right)^{3/2} (kT_c)^3 \ .} \tag{S1.12}$$

Inserting numbers,

$$\frac{n_M}{n_\gamma} = \frac{4\pi^5}{27 \times 1.202} \left(\frac{\pi \cdot 200 \cdot (6.674 \times 10^{-11} \ \text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2})}{5 \, (1.055 \times 10^{-34} \ \text{J} \cdot \text{s}) \, (2.998 \times 10^8 \ \text{m} \cdot \text{s}^{-1})^5} \right)^{3/2}$$

$$\times \left(10^{16} \ \text{GeV} \right)^3 \times \left(\frac{1.602 \times 10^{-10} \ \text{J}}{1 \ \text{GeV}} \right)^3 \times \left(\frac{\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}}{\text{J}} \right)^{3/2} \tag{S1.13}$$

$$= \boxed{2.92 \times 10^{-5} \ .}$$

Recall that the WMAP collaboration estimated that the ratio of baryons to photons is only

$$\eta \equiv \frac{n_b}{n_\gamma} = (6.1 \pm 0.2) \times 10^{-10} , \qquad (S1.14)$$

so this calculation is saying that the number of monopoles is more than four orders of magnitude larger than the number of baryons!

(d) To find the contribution of monopoles to the value of Ω today, we recall that the critical density today is given by

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} , \qquad (S1.15)$$

where the WMAP 7-year value of H_0 is 70.4 ± 2.5 km $\cdot$ s$^{-1} \cdot$ Mpc^{-1}. (Recall that the subscript "0" denotes the present time.) The mass density in monopoles can be expressed as

$$\rho_{M,0} = \frac{n_M}{n_\gamma} n_{\gamma,0} M_M , \qquad (S1.16)$$

where

$$n_{\gamma,0} = 2 \frac{\zeta(3)}{\pi^2} \frac{(kT_0)^3}{(\hbar c)^3} , \qquad (S1.17)$$

where the COBE experiment determined that $T_0 = 2.725 \pm 0.002$ K. Using Eq. (S1.12) with Eqs. (S1.15), (S1.16), and (S1.17), we find after some algebra that

$$\Omega_{M,0} \equiv \frac{\rho_{M,0}}{\rho_{c,0}} = \frac{\sqrt{3}\pi^4 G M_M}{H_0^2} \left(\frac{16\pi g_{\mathrm{GUT}} G}{135 \hbar^3 c^7} \right)^{3/2} (kT_0)^3 (kT_c)^3 . \qquad (S1.18)$$

Numerically, we could simply insert numbers into the above formula, but it is more informative to evaluate the pieces separately and then put them together. Using the numbers that have been stated,

$$\rho_{c,0} = \frac{3 \cdot (70.4 \text{ km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1})^2}{8\pi \left(6.674 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2} \right)}$$

$$\times \left(\frac{1 \text{ Mpc}}{3.086 \times 10^{22} \text{ m}} \right)^2 \times \left(\frac{1000 \text{ m}}{1 \text{ km}} \right)^2 \qquad (S1.19)$$

$$= \boxed{9.31 \times 10^{-27} \text{ kg} \cdot \text{m}^{-3} = 9.31 \times 10^{-30} \text{ g} \cdot \text{cm}^{-3} .}$$

$$n_{\gamma,0} = 2\frac{1.202}{\pi^2}\frac{\left(\left(1.380 \times 10^{-23} \text{ J} \cdot \text{K}^{-1}\right)(2.725 \text{ K})\right)^3}{\left(\left(1.055 \times 10^{-34} \text{ J} \cdot \text{s}\right)\left(2.998 \times 10^8 \text{ m} \cdot \text{s}^{-1}\right)\right)^3} \tag{S1.20}$$

$$= \boxed{4.09 \times 10^8 \text{ m}^{-3} = 409 \text{ cm}^{-3} .}$$

$$\rho_{M,0} = (2.92 \times 10^{-5})(4.09 \times 10^8 \text{ m}^{-3})\frac{10^{18} \text{ GeV}}{(2.998 \times 10^8 \text{ m} \cdot \text{s}^{-1})^2}$$

$$\times \left(\frac{1.602 \times 10^{-10} \text{ J}}{1 \text{ GeV}}\right) \times \left(\frac{\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}}{\text{J}}\right) \tag{S1.21}$$

$$= \boxed{2.13 \times 10^{-5} \text{ kg} \cdot \text{m}^{-3} = 2.13 \times 10^{-8} \text{ g} \cdot \text{cm}^{-3} .}$$

Finally,

$$\boxed{\Omega_{M,0} = \frac{2.13 \times 10^{-5} \text{ kg} \cdot \text{m}^{-3}}{9.31 \times 10^{-27} \text{ kg} \cdot \text{m}^{-3}} = 2.29 \times 10^{21} .} \tag{S1.22}$$

Thus, if conventional big bang cosmology is combined with grand unified theories, there is a prediction that vastly too many magnetic monopoles would be produced.

PROBLEM 2: EXPONENTIAL EXPANSION OF THE INFLATIONARY UNIVERSE *(15 points)*

Suppose $k < 0$. For convenience of notation, let $\kappa \equiv -k > 0$, and $\chi \equiv \sqrt{8\pi G \rho_f / 3}$. Then the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \chi^2 + \frac{\kappa c^2}{a^2}.$$

Taking the square root,

$$\frac{da}{dt} = \sqrt{\chi^2 a^2 + \kappa c^2}.$$

Rearranging, we can integrate both sides to find

$$\int \frac{da}{\sqrt{\chi^2 a^2 + \kappa c^2}} = \int dt .$$

The integral can be put in the form given in the problem set by defining $x = \chi a / \sqrt{\kappa} c$. Then

$$t = \chi^{-1} \int \frac{dx}{\sqrt{x^2 + 1}} = \chi^{-1} \sinh^{-1} x + \text{const} .$$

The constant of integration is conventionally chosen so that $a = 0$ at $t = 0$, which corresponds here to const $= 0$. Then we can take the sinh of both sides of the above equation to find

$$x = \sinh \chi t$$

or

$$\boxed{a = \frac{\sqrt{\kappa c}}{\chi} \sinh \chi t \ .}$$

For $k > 0$ the calculation proceeds identically, leading to

$$t = \chi^{-1} \int \frac{dx}{\sqrt{x^2 - 1}} \ ,$$

where $x = \chi a / \sqrt{kc}$. So

$$t = \chi^{-1} \cosh^{-1} x + \text{const} \ .$$

In this case a is never equal to zero, so the usual convention to choose the origin of time does not apply. The usual choice in this case is to set the constant in the above formula equal to zero, so that $t = 0$ is the time when $a(t)$ has its minimal value. This gives

$$\boxed{a = \frac{\sqrt{kc}}{\chi} \cosh \chi t \ .}$$

This describes a universe which begins infinitely large at $t = -\infty$, contracts to a minimum size at $t = 0$, and then starts to grow without limit as $t \to \infty$.

For large times one has

$$\sinh \chi t \to \frac{1}{2} e^{\chi t}$$

$$\cosh \chi t \to \frac{1}{2} e^{\chi t} \ ,$$

so

$$a(t) \propto e^{\chi t}$$

for all values of k, at large enough time. (One way to understand this result is by looking at the relative importance of the terms on the right hand side of the Friedmann equation. The curvature term scales like $1/a^2$, while the contribution from the vacuum energy is constant. Therefore, for large a or equivalently for large t, we can neglect the curvature term compared to the vacuum energy, and our solution must reduce to the solution for a flat universe.)

PROBLEM 3:   THE HORIZON DISTANCE FOR THE PRESENT UNIVERSE* (25 points)

(a) From the metric

$$ds^2 = -c^2 dt^2 + \frac{a^2(t)}{k} \left\{ d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2) \right\} ,$$

we deduce that for a light ray moving radially, that is, along ψ,

$$cdt = \frac{a(t)}{\sqrt{k}} d\psi \quad \rightarrow \quad \psi(t) = \sqrt{k} \int_0^t \frac{cdt'}{a(t')} .$$

In here we assumed that the light ray leaves the "origin" $\psi = 0$ at $t = 0$. $\psi(t)$ is the coordinate value of the horizon at time t for an observer at $\psi = 0$. The physical horizon distance $\ell_{\mathrm{p,hor}}(t)$ is

$$\boxed{\ell_{\mathrm{p,hor}}(t) = \frac{a(t)}{\sqrt{k}} \, \psi(t) = a(t) \int_0^t \frac{cdt'}{a(t')} .} \qquad (S3.1)$$

This is the formula we were asked to justify.

(b) To find the horizon distance now we must evaluate (S3.1) for $t = t_0$:

$$\ell_{\mathrm{p,hor}}(t_0) = a(t_0) \int_0^{t_0} \frac{cdt'}{a(t')} = \int_0^{t_0} \frac{ca(t_0)dt'}{a(t')} . \qquad (S3.2)$$

To evaluate this integral we recall the Friedmann equation in the form given in Eq. (7.22) of Lecture Notes 7:

$$dt = \frac{1}{H_0} \frac{x \, dx}{\sqrt{\Omega_{\mathrm{rad},0} + \Omega_{\mathrm{m},0}x + \Omega_{k,0}x^2 + \Omega_{\mathrm{vac},0}x^4}} .$$

Since $x(t) = a(t)/a(t_0)$ this equation can be rewritten as

$$\frac{a(t_0)dt}{a(t)} = \frac{1}{H_0} \frac{dx}{\sqrt{\Omega_{\mathrm{rad},0} + \Omega_{\mathrm{m},0}x + \Omega_{k,0}x^2 + \Omega_{\mathrm{vac},0}x^4}} . \qquad (S3.3)$$

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* Solution by Barton Zwiebach from 2007, with numerical calculations updated by Alan Guth.

Using (S3.3) to evaluate (S3.2), x becomes the variable of integration and runs from zero (for $t = 0$) to one (for $t = t_0$):

$$\ell_{\text{p,hor}}(t_0) = \frac{c}{H_0} \int_0^1 \frac{dx}{\sqrt{\Omega_{\text{rad},0} + \Omega_{\text{m},0}x + \Omega_{k,0}x^2 + \Omega_{\text{vac},0}x^4}} \, . \tag{S3.4}$$

This is a general formula for the horizon length. The numerical part of the evaluation requires finding the constant β_0 defined by

$$\beta_0 \equiv \int_0^1 \frac{dx}{\sqrt{\Omega_{\text{rad},0} + \Omega_{\text{m},0}x + \Omega_{k,0}x^2 + \Omega_{\text{vac},0}x^4}} \, , \tag{S3.5}$$

so that the horizon distance is

$$\ell_{\text{p,hor}}(t_0) = \frac{c}{H_0} \beta_0 \, . \tag{S3.6}$$

We are told that

$$\Omega_{\text{m},0} = 0.311 \, , \quad \Omega_{\text{rad},0} = 9.05 \times 10^{-5} \, , \quad \Omega_{\text{vac},0} = 1 - \Omega_{\text{m},0} - \Omega_{\text{r},0} = 0.688910 \, . \tag{S3.7}$$

The Planck team actually gives their best fit value as $\Omega_{\text{vac},0} = 0.689$, but their model also assumes that the universe is flat. Here we are imposing exact flatness, so that our numerics are exactly consistent with both $\Omega_{k,0} = 0$ and

$$1 = \Omega_{\text{rad},0} + \Omega_{\text{m},0} + \Omega_{k,0} + \Omega_{\text{vac},0} \, . \tag{S3.8}$$

With these values we find:

$$\beta_0 = \int_0^1 \frac{dx}{\sqrt{0.0000905 + 0.311\,x + 0.688910\,x^4}} = 3.19662 \, , \quad \ell_{\text{p,hor}}(t_0) = \frac{c}{H_0} \beta_0 \, . \tag{S3.9}$$

<u>Numerical analysis</u> Here we chose to maintain exact consistency with $\Omega_{k,0} = 0$ and Eq. (S3.8), which seemed like a safe way to proceed. Eq. (S3.8) is an exact equation of the theoretical formalism, and the Planck 2018 fit that we are using was derived under the assumption that $\Omega_{k,0} \equiv 0$. (The Planck 2018 estimate of $\Omega_0 = 0.9993 \pm 0.0037$ given as Eq. (8.8) of Lecture Notes 8 was obtained in a completely separate analysis

by the Planck team.) However, it was not really necessary for us to enforce these relations exactly. If we allowed a small error in Eq. (S3.8), and calculated

$$\beta_0' = \int_0^1 \frac{dx}{\sqrt{0.0000905 + 0.311\,x + 0.689\,x^4}} \; ,$$

we would have found $\beta_0' = 3.19659$, which is identical to the 3-figure accuracy that we can reasonably expect. If we had ignored radiation altogether, however, and calculated

$$\beta_0'' = \int_0^1 \frac{dx}{\sqrt{0.309\,x + 0.691\,x^4}} \; ,$$

we would have found $\beta_0'' = 3.25707$, which differs significantly, by about 2%.

We are asked to give the horizon distance both in light-years and Mpc's. For this we recall that

$$\frac{1}{H_0} = \frac{9.778 \times 10^9 \,\text{yr}}{h} \quad \text{with} \quad h = 0.677 \qquad \rightarrow \qquad \frac{1}{H_0} \simeq 14.44 \times 10^9 \,\text{years} \,.$$

Thus we get

$$\boxed{\ell_{\text{p,hor}}(t_0) = 3.20\,\frac{c}{H_0} = 46.3 \times 10^9 \,\text{light-years} \,.} \qquad \text{(S3.11)}$$

Roughly a horizon distance of 50 billion light years! For the computation in Mpc we recall that

$$H_0 = 100h\,\frac{\text{km}}{\text{s} \cdot \text{Mpc}} \,, \quad \text{with} \quad h = 0.677 \quad H_0 = 67.7\,\frac{\text{km}}{\text{s} \cdot \text{Mpc}} \,.$$

We thus find

$$\frac{c}{H_0} = \frac{2.998 \times 10^5}{67.7}\text{Mpc} = 4428 \,\text{Mpc} \,.$$

As a result,

$$\boxed{\ell_{\text{p,hor}}(t_0) = 3.20 \times 4428 = 14170 \,\text{Mpc} \,.} \qquad \text{(S3.12)}$$

This can also be checked with the relation 1 Mpc$= 3.262 \times 10^6$ ly.

PROBLEM 4: HUBBLE CROSSING FOR GALAXY-SIZE PERTURBA-TIONS *(25 points)*

(a) We are assuming that the galaxy consists of matter that collapsed from a bigger radius, and we are trying to figure out how big that radius is. So we ask, if the universe were still homogeneous, how large would a sphere have to be to contain enough matter to build a galaxy of mass $M_{\text{gal}} = 10^{12}\,\text{M}_\odot$? By "matter," here we mean baryons and dark matter. Neither radiation nor vacuum energy collapse in the formation of galaxies, so they do not contribute. Electrons can be considered part of "baryonic matter," but since an electron is about 2000 times less massive than a proton, their contribution to the mass density of matter is small compared to the uncertainty in the mass density of matter.

If we imagine that the matter was still uniformly distributed, its mass density $\rho_m(t_0)$ would be equal to the average mass density of matter today,

$$\rho_m(t_0) = \Omega_{m,0}\,\rho_c(t_0) = \Omega_{m,0}\frac{3H_0^2}{8\pi G}\ , \tag{S4.1}$$

were we used the facts that $\Omega_{m,0} \equiv \rho_{m,0}/\rho_c(t_0)$, and $\rho_c(t_0) = 3H_0^2/(8\pi G)$. The total mass of the matter in a sphere of radius R_0 is then

$$M = \frac{4\pi}{3}R_0^3\,\rho_m(t_0) = \frac{\Omega_{m,0}\,R_0^3\,H_0^2}{2G}\ . \tag{S4.2}$$

Setting $M = M_{\text{gal}}$ and solving for R_0,

$$\boxed{R_0 = \left(\frac{2GM_{\text{gal}}}{\Omega_{m,0}\,H_0^2}\right)^{1/3}\ .} \tag{S4.3}$$

Ryden's Benchmark Model takes $H_0 = 68$ km-s^{-1}-Mpc^{-1}, which can be converted to inverse seconds by

$$H_0 = 68\ \text{km-s}^{-1}\text{-Mpc}^{-1} = 68\ \frac{\text{km}}{\text{s}\cdot\text{Mpc}}\ \frac{10^3\ \text{m}}{\text{km}}\ \frac{\text{Mpc}}{10^6\ \text{pc}}\ \frac{1\ \text{pc}}{3.085678\times 10^{16}\ \text{m}} \tag{S4.4}$$

$$= 2.204\times 10^{-18}\ \text{s}^{-1}\ ,$$

where I am using the Particle Data Group's Review of Astrophysical Constants (https://pdg.lbl.gov/2020/reviews/rpp2020-rev-astrophysical-constants.pdf) for the conversion factors. Then R_0 can be found, in meters, as

$$R_0 = \left(\frac{2\cdot 6.674\times 10^{-11}\ \text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}\cdot 10^{12}\cdot 1.9884\times 10^{30}\ \text{kg}}{0.31\cdot(2.204\times 10^{-18})^2\ \text{s}^{-2}}\right)^{1/3}$$

$$\tag{S4.5}$$

$$= \boxed{5.607\times 10^{22}\ \text{m}\ .}$$

Converting to light-years,

$$R_0 = 5.607 \times 10^{22} \text{ m} \frac{\text{ly}}{0.946073 \times 10^{16} \text{ m}} = \boxed{5.927 \times 10^6 \text{ ly}} \,. \tag{S4.6}$$

Converting to Mpc,

$$R_0 = 5.607 \times 10^{22} \text{ m} \frac{\text{Mpc}}{3.085678 \times 10^{22} \text{ m}} = \boxed{1.817 \text{ Mpc}} \,. \tag{S4.7}$$

All of the above answers could appropriately have been rounded to two significant figures, since H_0 and $\Omega_{m,0}$ are only known to that accuracy. Here I kept 4 significant figures, to make it easier to check for consistency.

(b) For a flat universe, the first-order Friedmann equation reduces to

$$H^2 = \frac{8\pi}{3} G \rho \,. \tag{S4.8}$$

During the radiation-dominated era we can take $\rho = \rho_{\text{rad}}$ and $a(t) \propto t^{1/2}$, so $H = \dot{a}/a = 1/(2t)$. Thus,

$$\boxed{\rho_{\text{rad}}(t) = \frac{3}{32\pi G t^2}} \,. \tag{S4.9}$$

(c) The mass density in radiation today is given by

$$\boxed{\rho_{\text{rad},0} = \Omega_{r,0}\, \rho_c(t_0) = \Omega_{r,0} \frac{3H_0^2}{8\pi G}} \,. \tag{S4.10}$$

The energy density in radiation is proportional to $1/a^4(t)$, so

$$1 + z(t) = \frac{a(t_0)}{a(t)} = \left[\frac{\rho(t)}{\rho(t_0)}\right]^{1/4} = \boxed{\frac{1}{\Omega_{r,0}^{1/4}} \frac{1}{\sqrt{2H_0 t}}} \,. \tag{S4.11}$$

(d) Taking $\lambda_0 = R_0$ as the problem asks, we are looking for the time when

$$\lambda(t) = \frac{a(t)}{a(t_0)} R_0 = \Omega_{r,0}^{1/4} \sqrt{2H_0 t}\, R_0 \tag{S4.12}$$

is equal to the Hubble length,

$$cH^{-1}(t) = 2ct \ . \tag{S4.13}$$

So

$$\Omega_{r,0}^{1/4}\sqrt{2H_0t}R_0 = 2ct$$

$$\implies \boxed{\, t = \frac{\Omega_{r,0}^{1/2}H_0R_0^2}{2c^2} = \frac{\Omega_{r,0}^{1/2}}{2H_0^{1/3}c^2}\left(\frac{2GM_{\mathrm{gal}}}{\Omega_{m,0}}\right)^{2/3}\, .} \tag{S4.14}$$

Numerically, it is easiest to use the first of these expressions,

$$t = \frac{(9 \times 10^{-5})^{1/2} \cdot 2.204 \times 10^{-18}\ \mathrm{s}^{-1} \cdot (5.607 \times 10^{22}\ \mathrm{m})^2}{2 \cdot (2.998 \times 10^8\ \mathrm{m\ s}^{-1})^2}$$

$$= 3.657 \times 10^8\ \mathrm{s} \tag{S4.15}$$

$$= 3.657 \times 10^8\ \mathrm{s}\ \frac{\mathrm{yr}}{3.156 \times 10^7\ \mathrm{s}} = \boxed{\, 11.59\ \mathrm{yr}\, .}$$

Thus, rounded to the nearest year, we agree with Ryden's estimate of 12 years.

PROBLEM 5: A ZERO MASS DENSITY UNIVERSE— GENERAL RELATIVITY DESCRIPTION *(20 points extra credit)*

(a) To find the behavior of $a(t)$ with time in a zero mass density universe set $\rho = 0$ and $k = -1$ in the expression governing the evolution of the scale factor. The equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{c^2}{a^2} \implies$$

$$\dot{a}(t)^2 = c^2 \ .$$

After taking the square root of the above equation we choose the positive sign, since we believe the universe is expanding and not contracting. Then

$$da = c\,dt.$$

Integrating both sides of this equation, we have

$$\boxed{\, a(t) = ct\, .}$$

(b) We know the expression for the cosmological redshift is just

$$1 + z = \frac{a(t_o)}{a(t_e)} \; .$$

Using our result from part (a) we can rewrite this in terms of the time coordinate,

$$1 + z = \frac{t_o}{t_e} \quad \Longrightarrow$$

$$\boxed{z = \frac{t_o}{t_e} - 1 \; .}$$

(c) We find the trajectory of the light pulse by solving

$$c \, dt = a(t) \frac{dr}{\sqrt{1 - kr^2}}$$

for r as a fuction of t. Using the result that $a(t) = ct$, we rearrange the above expression to get

$$\frac{dt}{t} = \frac{dr}{\sqrt{1 - kr^2}}$$

We can now integrate this from the time of emission t_e to the time of observation t_o, finding

$$\int_{t_e}^{t_o} \frac{dt'}{t'} = \int_0^r \frac{dr'}{\sqrt{1 - kr'^2}}$$

$$\ln(t_o) - \ln(t_e) = \ln(t_o/t_e) = \sinh^{-1} r \; .$$

Solving this for r gives

$$r = \sinh\left(\ln(t_o/t_e)\right) \; .$$

To simplify this expression, remember that

$$\sinh \theta = \frac{e^\theta - e^{-\theta}}{2} \; .$$

Defining $x \equiv t_o/t_r$, we have $\theta = \ln x$ and $e^\theta = e^{\ln x} = x$. Then

$$r = \frac{t_o/t_e - t_e/t_o}{2} \; .$$

Alternatively we can multiply the top and bottom of the right hand side by t_o/t_e and write

$$r = \frac{(t_o/t_e)^2 - 1}{2(t_o/t_e)} \ .$$

(d) We found in part (b) that

$$1 + z = \frac{t_o}{t_e} \ .$$

We can reexpress z in terms of r by solving for t_o/t_e in terms of r. Again using the definition $x \equiv t_o/t_e$, the result from part (c) becomes

$$r = \frac{x^2 - 1}{2x} \qquad \Longrightarrow$$

$$x^2 - 2xr - 1 = 0 \ .$$

Using the quadratic formula we can solve this for x:

$$x = \frac{t_o}{t_e} = \frac{2r \pm \sqrt{4r^2 + 4}}{2} = r \pm \sqrt{r^2 + 1} \ .$$

Only the positive root is valid, since the negative root would give a physically meaningless negative value for t_o/t_e. Substituting this into the expression for z we get

$$1 + z = r + \sqrt{r^2 + 1} \ .$$

Since the mass density of the universe is zero there is no gravity and hence no force acting on the comoving observers. We know that such observers must then be moving with constant velocity. In the absence of gravity it is the relative velocity of two observers that determines the redshift, so it follows that z is independent of the time t_e.

PROBLEM 6: A ZERO MASS DENSITY UNIVERSE— SPECIAL RELATIVITY DESCRIPTION *(20 points extra credit)*

(a) Since there is no gravitational field, the comoving observers move at a constant velocity in the inertial frame of reference (described by coordinates t', r', θ', and ϕ'). Since the comoving observers all start at the origin of the coordinate system, each comoving observer travels on a trajectory $r' = vt'$, where $v = r'/t'$ will have a different value for different comoving observers. The cosmic time t is defined to be

the proper time as measured by comoving observers, so from the point of view of the inertial frame t is measured on clocks that are running slowly by a factor of $\gamma(v)$:

$$t = t'/\gamma(v) = t'\sqrt{1 - \frac{v^2}{c^2}} = t'\sqrt{1 - \frac{r'^2}{c^2 t'^3}} \ ,$$

or

$$\boxed{t = \sqrt{t'^2 - \frac{r'^2}{c^2}} \ .}$$

Notice that since v is constant the comoving observers are also inertial observers in the special relativistic sense.

(b) We are assuming that $\theta = \theta'$ and $\phi = \phi'$, and we also know that r is a rotationally invariant coordinate, so it depends only on r' and t', but not θ' or ϕ'. Thus, if an infinitesimal line segment has the property that θ is the only coordinate that changes in the Robertson-Walker (unprimed) coordinates, then θ' is the only coordinate that varies in the inertial (primed) coordinates. The special relativistic metric (inertial, primed coordinates) reduces to

$$ds^2 = r'^2 \, d\theta'^2 \ ,$$

while the general relativistic metric (Robertson-Walker, unprimed coordinates) becomes

$$ds^2 = a^2(t) \, r^2 \, d\theta^2.$$

The physical length of the line segment must be independent of the coordinate system used to describe it, so the two expressions for ds must be equal:

$$r' \, d\theta' = a(t) \, r \, d\theta \ .$$

We know from part (a) of the previous problem that $a(t) = ct$, and since we are assuming that $\theta' = \theta$, we have

$$\boxed{r = \frac{r'}{ct} = \frac{r'}{\sqrt{c^2 t'^2 - r'^2}} = \frac{v/c}{\sqrt{1 - \frac{v^2}{c^2}}} \ ,}$$

where $v = r'/t'$.

To sketch lines of constant t in the r'-t' plane, note that the answer to part (a) can be rewritten as

$$t' = \sqrt{t^2 + \frac{r'^2}{c^2}} \ ,$$

which for a fixed value of t describes a hyperbola. Each value of t gives a different hyperbola, and $t = 0$ gives the degenerate hyperbola $t' = |r'/c|$. To sketch lines of constant r, we can first solve the boxed equation above for v, finding

$$\frac{v}{c} = \frac{r}{\sqrt{1 + r^2}} .$$

Since $v = r'/t'$, this equation becomes

$$t' = \frac{\sqrt{1 + r^2}}{cr} r' ,$$

so the lines of constant r are straight lines in the r'-t' plane. Note that as $r \to \pm\infty$, the slope approaches $\pm 1/c$:
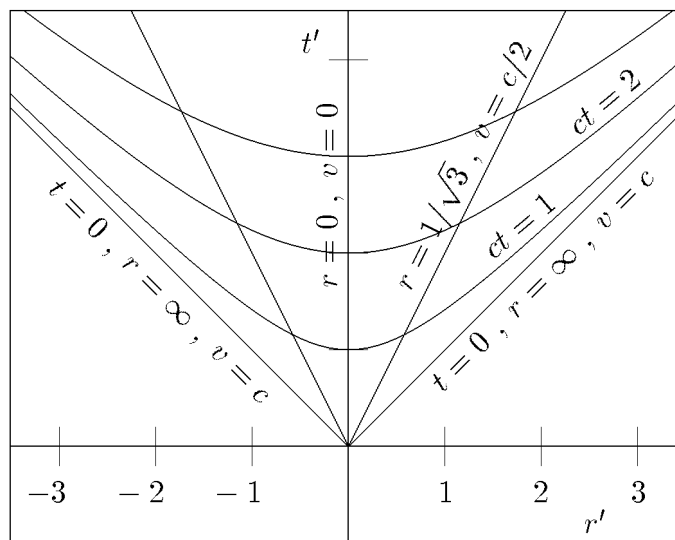
(c) We have shown in the previous part that

$$r = \frac{v/c}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad \frac{v}{c} = \frac{r}{\sqrt{1 + r^2}} ,$$

so all that remains is to calculate the redshift. The redshift in special relativity is given by

$$1 + z = \sqrt{\frac{1 + v/c}{1 - v/c}} .$$

Substituting the previous expression for v/c, one finds

$$1 + z = \sqrt{\frac{1 + \frac{r}{\sqrt{1+r^2}}}{1 - \frac{r}{\sqrt{1+r^2}}}} = \sqrt{\frac{\sqrt{1 + r^2} + r}{\sqrt{1 + r^2} - r}} .$$

The expression simplifies dramatically if one multiplies the numerator and denominator by $\sqrt{\sqrt{1 + r^2} + r}$, yielding

$$1 + z = \sqrt{\frac{(\sqrt{r^2 + 1} + r)(\sqrt{r^2 + 1} + r)}{(\sqrt{1 + r^2} - r)(\sqrt{1 + r^2} + r)}}$$

$$= \boxed{r + \sqrt{1 + r^2} \ .}$$

As expected, this agrees with the redshift found in part (d) of the previous problem.

(d) We have the following transformation equations:

$$ct = \sqrt{c^2 t'^2 - r'^2}$$

$$r = \frac{r'}{\sqrt{c^2 t'^2 - r'2}}$$

$$\theta = \theta'$$

$$\phi = \phi' \ .$$

We want to invert these equations in order to express t', r', θ' and ϕ' in terms of t, r, θ and ϕ. Immediately we know $\theta' = \theta$ and $\phi' = \phi$. Note that we get a simple relation by using the first two equations to calculate the product of ct and r: the r' transformation equation multiply ct with r:

$$ctr = \sqrt{c^2 t'^2 - r'^2} \frac{r'}{\sqrt{c^2 t'^2 - r'^2}} = r' \ ,$$

so

$$\boxed{r' = ctr \ .}$$

Substituting this result into the expression for t above yields an equation that can be solved for t', yielding

$$\boxed{t' = t\sqrt{1 + r^2} \ .}$$

We thus find for an infinitesimal change in the coordinates:

$$dt' = \sqrt{1 + r^2}\,dt + \frac{rt}{\sqrt{1 + r^2}}dr$$

$$dr' = ct\,dr + cr\,dt$$

$$d\theta' = d\theta$$

$$d\phi' = d\phi \ .$$

Finally, we substitute these expressions into the special relativistic expression for the invariant interval ds_{ST}^2, finding

$$
\begin{aligned}
ds_{\mathrm{ST}}^2 &= -c^2\,dt'^2 + dr'^2 + r'^2(d\theta'^2 + \sin^2\theta'\,d\phi'^2) \\
&= -c^2\left[dt^2(1+r^2) + \frac{r^2 t^2}{1+r^2}dr^2 + 2rt\,dr\,dt\right] \\
&\quad + c^2\left[t^2 dr^2 + r^2 dt^2 + 2rt\,dr\,dt\right] + c^2 t^2 r^2[d\theta^2 + \sin^2\theta\,d\phi^2] \\
&= -c^2\,dt^2 + \frac{c^2 t^2\,dr^2}{1+r^2} + c^2 t^2 r^2[d\theta^2 + \sin^2\theta\,d\phi^2] \ .
\end{aligned}
$$

Since $a(t) = ct$ we find that the spacetime metric can be written as

$$
ds_{\mathrm{ST}}^2 = -c^2 dt^2 + a(t)^2\left[\frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] ,
$$

which is identical to the general relativistic Robertson-Walker open-universe expression for the invariant spacetime interval.