

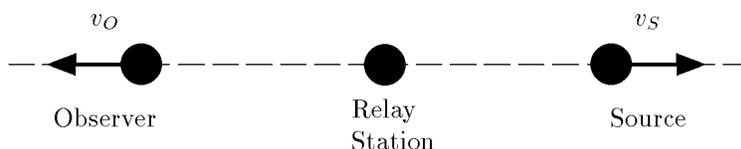
## PROBLEM SET 1 SOLUTIONS

### PROBLEM SET GRADING:

Reminder: The total number of points will vary some from problem set to problem set. Your final problem set grade will be the sum of the individual grades, so the problem sets with more points will count a little more than the others.

### PROBLEM 1: NONRELATIVISTIC DOPPLER SHIFT, SOURCE AND OBSERVER IN MOTION (15 points)

The easiest way to do this problem, given the results described in the lecture notes, is to introduce an intermediate relay station which is at rest relative to the air:



The relay station simply rebroadcasts the signal it receives from the source, at exactly the instant that it receives it. The relay station therefore has no effect on the signal received by the observer, but it serves as a crutch for our thinking—it allows us to divide the problem into two parts, each already solved. Let  $\Delta t$  denote the time interval between wave crests, with subscripts to indicate which clock is making the measurement. Then, from Eqs. (1.3) and (1.7) of Lecture Notes 1,

$$\Delta t_{\text{relay}} = \left(1 + \frac{v_s}{u}\right) \Delta t_{\text{source}}$$

and

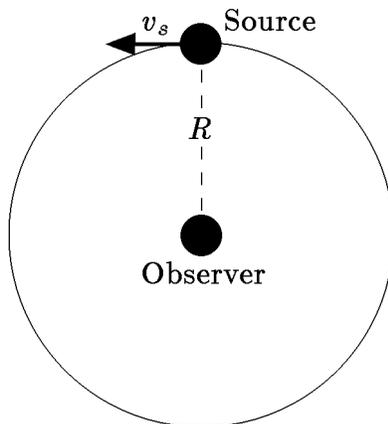
$$\Delta t_{\text{observer}} = \frac{1}{1 - v_o/u} \Delta t_{\text{relay}} .$$

Combining these equations,

$$\Delta t_{\text{observer}} = \frac{1 + v_s/u}{1 - v_o/u} \Delta t_{\text{source}} .$$

Thus

$$z \equiv \frac{\Delta t_{\text{observer}}}{\Delta t_{\text{source}}} - 1 = \frac{1 + v_s/u}{1 - v_o/u} - 1 .$$

**PROBLEM 2: THE TRANSVERSE DOPPLER SHIFT** (25 points)

- (a) Since this part of the problem is nonrelativistic, time can be considered universal, so all clocks run at the same speed. If wave crests are emitted by the source at a time interval  $\Delta t_S$  as measured by the source, the time interval between the emissions of the crests will also appear to the observer to be  $\Delta t_S$ . Furthermore, since the distance between the source and observer does not change with time, each crest must travel the same distance between the source and the observer, and hence the separation between their arrivals will also be  $\Delta t_S$ . Thus the time interval between crests as measured by the observer is  $\Delta t_O = \Delta t_S$ , and the redshift is

$$z \equiv \frac{\Delta t_O}{\Delta t_S} - 1 = 0 .$$

There is no Doppler shift.

- (b) Again the distance between the source and the observer remains constant, but in this case there is nonetheless a Doppler shift caused by time dilation, the relativistic effect of motion on the speed of the clocks. In the reference frame of the observer, which is also the reference frame of the diagram, the clock on the source is a moving clock which runs slowly by a factor of

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}} , \quad \beta \equiv v/c .$$

Since time dilation depends on the reference frame, it is important to be clear about what reference frame is being used — we will discuss this part and all subsequent parts in the reference frames of their respective diagrams, which we will call the “lab” frame. The clock in this case is accelerating, but unless we are told otherwise, we

assume that any accelerating clock is an ideal clock, which runs at the same speed as a clock with the same velocity which is not accelerating. If the time interval between crests is measured on the source clock as  $\Delta t_S$ , then the time interval in the lab frame will be  $\gamma \Delta t_S$ . That is, the slow-running clock takes longer for the reading to change from 0 to  $\Delta t_S$ . Since the observer is at rest, the observer's clock will agree with the lab frame clock, and hence will measure

$$\Delta t_O = \gamma \Delta t_S .$$

The redshift is then given by

$$z \equiv \frac{\Delta t_O}{\Delta t_S} - 1 = \gamma - 1 .$$

This phenomenon is known as the transverse Doppler effect.

- (c) Again the distance between the source and the observer remains constant, but in this case the time dilation factors are different from the previous case. Here the source is at rest relative to the lab frame, so the time interval between crests as measured by the source,  $\Delta t_S$ , is the time interval in the lab frame. Since the distance between the source and observer is constant, the observer will receive wave crests at a time interval  $\Delta t_S$ , as measured in the lab frame. But the observer's clock is moving, relative to the lab frame, so it will run slowly by a factor of  $\gamma$ . Since it is running slowly it will measure less time between crests, by a factor of  $\gamma$ , so

$$\Delta t_O = \frac{\Delta t_S}{\gamma} .$$

Then

$$z \equiv \frac{\Delta t_O}{\Delta t_S} - 1 = \frac{1}{\gamma} - 1 .$$

This situation is also sometimes called a transverse Doppler shift, but it is clearly different from the case in part (b): here  $z < 0$ , which means there is a blueshift (shift toward higher frequency, shorter wavelength), while  $z > 0$  in part (b), so there is a redshift. In part (b) the photon velocity  $\vec{v}_{\text{photon}}$  was orthogonal to  $\vec{v}_{\text{source}}$  in the frame of the observer, while here  $\vec{v}_{\text{photon}}$  is orthogonal to  $\vec{v}_{\text{observer}}$  in the frame of the source. Even in Newtonian physics, the orthogonality of two velocities depends on the reference frame.

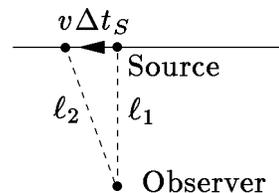
- (d) The situation here is almost identical to the situation in part (b). The only relevant difference is that in part (b) the source is accelerating in circular motion at the time

it emits the light ray in question, while here it is moving at a constant velocity. This could in principle lead to a difference between the path lengths of successive wave crests, but in this case it does not. In Doppler shift problems we always assume that the wavelength is very short compared to any other distance in the problem, or equivalently that the period at the source  $\Delta t_S$  or the period  $\Delta t_O$  as observed is short compared to any other time. (For visible light, for example, the period is of order  $10^{-15}$  second, which is ordinarily completely negligible.) Since the period is very short it is sufficient to calculate the change in the path length to first order in the period, which means that the change in path length is proportional to the first derivative of the path length with respect to time. In this case the derivative of the path length with respect to time is zero, since the path length is at its minimum when the light ray is emitted. Thus the answer is the same as part (b),

$$z \equiv \frac{\Delta t_O}{\Delta t_S} - 1 = \gamma - 1 .$$

Again we have the standard transverse Doppler shift.

You were not expected to do this, but suppose some skeptical questioner insisted that you give a more detailed justification for neglecting the difference in path lengths between successive wave crests. Then you might assume that one wave crest is emitted at precisely the moment of closest approach, with path length  $\ell_1$ , and then you can calculate the path length for the next crest,  $\ell_2$ . Then



$$\Delta \ell = \sqrt{\ell_1^2 + (v\Delta t_S)^2} - \ell_1 .$$

Provided that  $v\Delta t_S \ll \ell_1$ , we can use the Taylor expansion

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \dots$$

to approximate

$$\begin{aligned} \Delta \ell &= \ell_1 \sqrt{1 + \frac{(v\Delta t_S)^2}{\ell_1^2}} - \ell_1 \\ &= \ell_1 \left( 1 + \frac{(v\Delta t_S)^2}{2\ell_1^2} + \dots \right) - \ell_1 \approx \frac{(v\Delta t_S)^2}{2\ell_1} = \frac{1}{2} \left( \frac{v}{c} \right)^2 \left( \frac{\lambda}{\ell_1} \right) \lambda , \end{aligned}$$

where  $\lambda = c\Delta t_S$  is the wavelength measured at the source. Since  $\lambda \ll \ell_1$ , we find  $\Delta \ell \ll \lambda$ , so it can be neglected. In astronomical situations we might have  $\lambda \sim 10^{-6}$

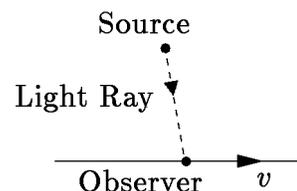
m, and  $\ell_1$  might be measured in light-years or even billions of light-years, so the inequality  $\lambda \ll \ell_1$  is easily satisfied.

- (e) As in part (d), the difference between path lengths for successive crests is completely negligible, so the situation is the same as in part (c). Thus

$$z \equiv \frac{\Delta t_O}{\Delta t_S} - 1 = \frac{1}{\gamma} - 1 .$$

This expression for  $z$  is negative, so the light is blue-shifted instead of redshifted.

While the situations in parts (d) and (e) look superficially like they might be the same situation, viewed in two different reference frames, they are really not. If we describe the situation in part (d) in the rest frame of the source, then the observer would have velocity  $v$  to the right, as in part (e). But in part (d), the photon velocity is vertical. If we transform the situation in (d) to the source rest frame, the light ray will acquire a component to the right, as shown at the right. So it would not look like part (e).



### PROBLEM 3: A HIGH-SPEED MERRY-GO-ROUND (15 points extra credit)\*

The following comments apply both for part (a) and for part (b). Number the cars from 1 to 4. Consider the signals emitted by car  $i$  and detected by car  $j$ , where  $j \neq i$  and  $i$  and  $j$  are any integers from one to four.

As seen in the stationary non-rotating laboratory frame the crests of the waves emitted by the source at  $i$  are always separated by the same time interval  $(\Delta t_S)_L$ ; here the subscript  $S$  means that we are defining the period at the source, and the subscript  $L$  indicates that it is the value measured in the laboratory frame. Each crest emitted by  $i$  always takes the same time to reach  $j$ , because the relative positions of  $i$  and  $j$  on the merry-go-round never change. It follows that in the laboratory frame the crests received by the observer at  $j$  are always separated by the time interval  $(\Delta t_O)_L$  that is exactly equal to  $(\Delta t_S)_L$ :

$$(\Delta t_O)_L = (\Delta t_S)_L . \quad (1)$$

This is true for any pair of cars on the merry-go-round. The angles that separate the various cars need not be multiples of  $\pi/2$ , as long as the relative angular positions are preserved in time.

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\* Solution by Barton Zwiebach, from 2007.

- (a) In the nonrelativistic approximation the time differences measured by the laboratory coincide with the time differences measured by either sources or receivers. We thus have that (1) gives

$$\Delta t_O = \Delta t_S, \quad (2)$$

where the absence of the  $L$  subscript indicates that  $\Delta t_O$  is time measured by the observer and  $\Delta t_S$  is time measured by the source. It follows from the definition of  $z$  that

$$1 + z \equiv \frac{\Delta t_O}{\Delta t_S} = 1, \quad \implies \quad \boxed{z = 0}. \quad (3)$$

- (b) Because the sources and receivers are moving their clocks lag — times measured at the laboratory are longer by the relativistic factor  $\gamma$  (this is true even though the motion is circular and only the speed is constant). We thus have

$$(\Delta t_O)_L = \gamma_O \Delta t_O, \quad (\Delta t_S)_L = \gamma_S \Delta t_S, \quad (4)$$

where  $\Delta t_O$  and  $\Delta t_S$  are the times measured by the clocks carried by the observer and the source, respectively. Moreover,

$$\gamma_O = \frac{1}{\sqrt{1 - (v_O/c)^2}}, \quad \gamma_S = \frac{1}{\sqrt{1 - (v_S/c)^2}}, \quad (5)$$

with  $v_O$  and  $v_S$  the speed of the observer and source, respectively. Since  $v_O = v_S$ , we have  $\gamma_O = \gamma_S \equiv \gamma$  and thus (4) becomes

$$(\Delta t_O)_L = \gamma \Delta t_O, \quad (\Delta t_S)_L = \gamma \Delta t_S, \quad (4')$$

Substituting this information in (1) we find

$$\Delta t_O = \Delta t_S. \quad (5')$$

Therefore, even relativistically there is no Doppler shift:  $\boxed{z = 0}$ . Intuitively, the equality of the lab intervals for emission and observation implies the equality of intervals at the level of the emitter and the observer because their  $\gamma$  factors are the same.